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A DIVISION THEOREM FOR REAL ANALYTIC FUNCTIONS

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Abstract

We characterize those homogeneous polynomials $P \in \mathbb{C}[z_1, \ldots, z_d]$ for which the principal ideal $(P) = P \cdot A(\mathbb{R}^d)$ is complemented in $A(\mathbb{R}^d)$ or, equivalently, those which admit a continuous linear division operator. The condition is the same as that which characterizes, among the homogeneous polynomials, those which are non-elliptic and for which $P(D)$ is surjective in $A(\mathbb{R}^d)$, and those for which $P(D)$ admits a continuous linear right inverse in $C^\infty(\mathbb{R}^d)$. It depends only on the type of real singularities.

In this note we study the problem of when, for a polynomial $P \in \mathbb{C}[z_1, \ldots, z_d]$, the principal ideal $(P) = P \cdot A(\mathbb{R}^d)$ is complemented in the space $A(\mathbb{R}^d)$ of real analytic functions or, equivalently, when there exists a continuous linear operator T in $A(\mathbb{R}^d)$ such that $T(P \cdot f) = f$ for every $f \in A(\mathbb{R}^d)$. The operator T and its dual, which divides analytic functionals through P, are called division operators, so we study when P admits *continuous linear division*. We give a necessary condition which is also sufficient if either P is homogeneous or its real zero set is compact. So we get a complete characterization for the case of homogeneous polynomials.

In this case, the principal ideal (P) is complemented if and only if the variety $V = \{z \in \mathbb{C}^d : P(z) = 0\}$ satisfies the so-called *local Phragmén–Lindelöf condition* (denoted here, the $\mathrm{PL}_{\mathrm{loc}}$ condition) at every real point. This condition appeared first in Hörmander [**9**], and it characterizes those non-elliptic homogeneous polynomials for which $P(D)$ is surjective in $A(\mathbb{R}^d)$. It also characterizes those homogeneous polynomials for which $P(D)$ admits a continuous linear right inverse in $C^\infty(\mathbb{R}^d)$ (see Meise, Taylor and Vogt [**14**]). It has been intensely studied in the meantime; see, for example, [**3**–**5**, **15**, **19**].

It is also interesting to compare our result with the classical results in the C^∞-case, and in the case of entire functions. In the C^∞-case, this comparison will be made in Section 4. The connection to the ideal property (propriété des zéros) and the analytic ideal property will also be discussed there. By a result of Zahariuta [**24**] and Djakov and Mitiagin [**6**], the principal ideal $P \cdot H(\mathbb{C}^d)$ is always complemented in the space $H(\mathbb{C}^d)$ of entire functions.

1. *Preliminaries*

Throughout the paper, $A(\mathbb{R}^d)$ denotes the space of complex-valued real analytic functions with its natural locally convex topology (see Martineau [**13**]). For compact $K \subset \mathbb{C}^d$, we denote by $H(K)$ the space of germs of holomorphic functions on K with its natural (LB)-topology; that is, it is the inductive limit of the spaces $H^\infty(U_n)$ where the U_n runs through an open neighborhood basis of K. This understood, $A(\mathbb{R}^d)$ is the projective limit of the spaces $H(K_n)$ where K_n is any compact exhaustion of $\mathbb{R}^d$. For open $\omega \subset \mathbb{C}^d$ we denote by $H(\omega)$ the Fréchet space of holomorphic functions on ω with the compact open topology.

We use common notation for locally convex spaces. In particular, for locally convex spaces E and F we denote by $L(E, F)$ the space of all continuous linear maps from E to F, and we set $L(E) := L(E, E)$. The dual space E' is always assumed to carry its strong topology. For all

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unexplained concepts of functional analysis, we refer to [16]. For homological concepts, we refer to [21], for real analytic spaces to [8] and [17], and for concepts of pluripotential theory to [11].

Let V_a be the germ of a complex variety at the real point a. We always assume that it is given by a relatively compact open connected neighborhood Ω of a in $\mathbb{C}^d$ and finitely many holomorphic functions $f_1,\dots,f_m$ on Ω so that $V_a = \{z \in \Omega : f_1(z) = \ldots = f_m(z) = 0\}$ and $f_1,\dots,f_m$ generate the ideal of V_a in $\mathscr{O}_a$. We set $X_a = V_a \cap \mathbb{R}^d$.

If the ideal of X_a in $\mathscr{O}_a$ is generated by $f_1,\dots,f_m$, we call V_a the *complexification* of X_a. We define

$$\omega_{a,V}(z) = \limsup_{\zeta\to z} \sup\{u(\zeta) : u \text{ plurisubharmonic on } V_a,\ u \leqslant 1,\ u \leqslant 0 \text{ on } X_a\}.$$

For two germs ω_a and $\tilde{\omega}_a$ we set $\omega_a \prec \tilde{\omega}_a$ if there is a constant $C > 0$ such that $\omega_a \leqslant C\tilde{\omega}_a$ in a neighborhood of a in $V_a \cap \widetilde{V}_a$. If $\omega_a \prec \tilde{\omega}_a$ and $\tilde{\omega}_a \prec \omega_a$, we write $\omega_a \sim \tilde{\omega}_a$, and call such germs *equivalent*. Then we find that: up to equivalence, the germ of $\omega_{a,V}$ depends only on the germ of V_a. If V_a is the complexification of X_a, then, of course, it depends only on the germ of X_a. In this case we also write $\omega_{a,X}$.

DEFINITION 1.1. *V_a satisfies* $\mathrm{PL}_{\mathrm{loc}}$ *if* $\omega_{a,V} \prec |\mathrm{Im}\, z|$.

A complex variety V satisfies $\mathrm{PL}_{\mathrm{loc}}$ at the real point a if its germ in a satisfies $\mathrm{PL}_{\mathrm{loc}}$. The germ X_a of a real analytic variety at the real point a is of type PL if its complexification satisfies $\mathrm{PL}_{\mathrm{loc}}$. Notice that for a complex variety V and $X = V \cap \mathbb{R}^d$ the germ X_a can be of type PL, while V_a does not satisfy $\mathrm{PL}_{\mathrm{loc}}$ at a.

EXAMPLE. Let $V = \{z : \sum_{j=1}^d z_j^2 = 0\}$ and $d > 2$; then $X_0 = \{0\}$ and X_0 coincides with its complexification. Therefore X_0 is of type PL, while V_0 does *not* satisfy $\mathrm{PL}_{\mathrm{loc}}$.

DEFINITION 1.2. *For a germ W_a of a subvariety of V_a, we set $W_a \overset{\mathrm{PL}}{\subset} V_a$ if $\omega_{a,W} \prec \omega_{a,V}$ on W_a.*

Notice that with this notation we have $V_a \overset{\mathrm{PL}}{\subset} \mathbb{C}^d_a$ if and only if V_a satisfies $\mathrm{PL}_{\mathrm{loc}}$.

LEMMA 1.3. *We have the following simple facts.*

(i) *If $W_a \overset{\mathrm{PL}}{\subset} \widetilde{W_a} \overset{\mathrm{PL}}{\subset} V_a$, then $W_a \overset{\mathrm{PL}}{\subset} V_a$.*

(ii) *If V_a is in $\mathbb{C}^{d_1}$ and W_b is in $\mathbb{C}^{d_2}$, $d_1 + d_2 = d$, then $V_a \times \{b\} \overset{\mathrm{PL}}{\subset} V_a \times W_b$.*

It should be pointed out that these relations remain unchanged, up to the base point, under biholomorphic maps defined in a neighborhood of a, mapping reals to reals. This is used, in particular, in connection with Lemma 1.3(ii).

We consider now a homogeneous complex variety V. We set

$$S = \left\{ z \in \mathbb{C}^d : \sum_{j=1}^d z_j^2 = 1 \right\}$$

and $V^0 = S \cap V$. Then V^0 is a complex variety and $V^0 \cap \mathbb{R}^d = X \cap S^{d-1} =: X^0$ is a compact real analytic variety.

In a neighborhood of $a \in X^0$ we identify by polar coordinates V_a with $(1 + \varepsilon\mathbb{D}) \times V_a^0$, where $\mathbb{D}$ is the open unit disc. Therefore $V_a^0 \overset{\mathrm{PL}}{\subset} V_a$. If V_a satisfies $\mathrm{PL}_{\mathrm{loc}}$, then so also does V_a^0.

LEMMA 1.4. *If V is homogeneous and satisfies* $\mathrm{PL}_{\mathrm{loc}}$ *at any real point, then there exists a continuous linear extension operator $A(X^0) \longrightarrow A(\mathbb{R}^d)$. In particular, there exists a continuous linear extension operator $A(X^0) \longrightarrow A(S^{d-1})$.*

Proof. According to the previous discussion, V_0 satisfies $\mathrm{PL}_{\mathrm{loc}}$ at any point of X^0. That means that X^0 is a compact subvariety of R^d, of type PL. By [**19**, Theorem 2.2], there exists a continuous linear extension operator $A(X^0) \longrightarrow A(\mathbb{R}^d)$. If we compose this with the restriction map to S^{d-1}, we get the second assertion. $\square$

From there, we obtain the following theorem (see [**19**, Theorem 7.2]). We set $\mathbb{R}^d_* = \mathbb{R}^d \setminus \{0\}$, $X_* = \mathbb{R}^d_* \cap X$.

THEOREM 1.5. *If the homogeneous real analytic variety V satisfies* $\mathrm{PL}_{\mathrm{loc}}$ *at any real point, then there exists a continuous linear extension operator $A(X_*) \longrightarrow A(\mathbb{R}^d_*)$.*

Proof. The map

$$z \mapsto \left(\log \|z\|, \frac{z_1}{\|z\|}, \ldots, \frac{z_d}{\|z\|} \right)$$

is a real analytic diffeomorphism from R^d_* to $\mathbb{R} \times S^{d-1}$, and also from X_* to $\mathbb{R} \times X^0$. So we may identify

$$A(\mathbb{R}^d_*) \cong A(\mathbb{R}) \hat{\bigotimes}_\pi A(S^{d-1}) \qquad \text{and} \qquad A(X_*) \cong A(\mathbb{R}) \hat{\bigotimes}_\pi A(X^0).$$

If $\varphi : A(X^0) \longrightarrow A(S^{d-1})$ is the extension map of Lemma 1.4, then $\mathrm{id} \hat{\bigotimes}_\pi \varphi$ leads to the extension map, as claimed. $\square$

2. *Description of the problem*

Let $P \in \mathbb{C}[x_1, \ldots, x_d]$ be a polynomial. We denote by $(P) = P \cdot A(\mathbb{R}^d)$ the principal ideal of P in the real analytic functions. The following lemma is then quite obvious.

LEMMA 2.1. *The following statements are equivalent.*

(i) *There is a continuous linear operator $T = T_P \in L(A(\mathbb{R}^d))$ ('division operator') such that $T(Pf) = f$ for all $f \in A(\mathbb{R}^d)$.*

(ii) *There is a continuous linear operator $S = S_P \in L(A'(\mathbb{R}^d))$ such that $P \cdot S(\mu) = \mu$ for all $\mu \in A'(\mathbb{R}^d)$.*

(iii) *(P) is complemented in $A(\mathbb{R}^d)$.*

Proof. Since (i) and (ii) are just dual to each other, it suffices to show the equivalence of (i) and (iii). If (i) is given, then $f \mapsto P \cdot T(f)$ defines a continuous linear projection onto (P). If (iii) is given and π is a continuous linear projection onto (P), then (P) is a quotient of $A(\mathbb{R}^d)$, and hence is ultrabornological. By the Grothendieck–de Wilde theorem, the map $T_0 : Pf \mapsto f$ is continuous linear from (P) onto $A(\mathbb{R}^d)$. We set $T := T_0 \circ \pi$. $\square$

DEFINITION. *We say that P* admits continuous linear division *if the equivalent conditions of Lemma 2.1 are fulfilled.*

Since P obviously admits continuous linear division if and only if each of its irreducible factors does so, we may assume, for our investigation, that P is irreducible. In this case, we can describe (P) by the zeros of its functions.

We use the following notation:

$$V = \{z \in \mathbb{C}^d : P(z) = 0\},$$
$$X = V \cap \mathbb{R}^d = \{x \in \mathbb{R}^d : P(x) = 0\},$$

and we obtain the following lemma.

LEMMA 2.2. *(P) is the set of all $f \in A(\mathbb{R}^d)$ for which there exists an open neighborhood ω of $\mathbb{R}^d$ and a function $F \in H(\omega)$ such that $F|_{\mathbb{R}^d} = f$ and $F|_{V\cap\omega} = 0$.*

Proof. If $f \in (P)$, then there is $g \in A(\mathbb{R}^d)$ such that $f = Pg$. There is an open neighborhood ω of $\mathbb{R}^d$ and $G \in H(\omega)$ such that $G|_{\mathbb{R}^d} = g$. We put $F = PG$.

To show the converse we assume that ω and F are as described in the lemma. Then P divides F; that is, there is $G \in H(\omega)$ with $F = PG$. □

We set

$$H_V(X) = \{(f, \Omega) : \Omega \text{ is an open neighborhood of } X \text{ in } V, f \text{ holomorphic on } \Omega\}$$

with $(f_1, \Omega_1) = (f_2, \Omega_2)$ if there exists an open set $\Omega \subset V$ with $X \subset \Omega \subset \Omega_1 \cap \Omega_2$ and $f_1|_\Omega = f_2|_\Omega$.

We obtain a natural restriction map $\rho : A(\mathbb{R}^d) \longrightarrow H_V(X)$ by setting $\rho(f) = F|_V$, where F is an extension of f to a holomorphic function on an open neighborhood of $\mathbb{R}^d$.

LEMMA 2.3. *The sequence*

$$0 \longrightarrow (P) \hookrightarrow A(\mathbb{R}^d) \xrightarrow{\rho} H_V(X) \longrightarrow 0 \tag{1}$$

is exact.

Proof. Due to Lemma 2.2, we have to show only the surjectivity of ρ. For given (f, Ω) we find, by use of the Cartan–Grauert theorem, an open pseudoconvex set $\omega \subset \mathbb{C}^d$ such that $\mathbb{R}^d \subset \omega$ and $\omega \cap V \subset \Omega$. By the Cartan–Oka theory there exists an $F \in H(\omega)$ such that $F|_{\omega\cap V} = f$. □

3. *Necessity of* $\mathrm{PL}_{\mathrm{loc}}$

Using $|z| = \max_j |z_j|$, we put $D_r = \{x \in \mathbb{R}^d : |x| \leqslant r\}$. By V_r we denote the pluricomplex Green function of D_r (see [11, p. 207]), and we set

$$D_{r,\alpha} = \{z \in \mathbb{C}^d : V_r(z) < \alpha\}, \qquad W_{r,\alpha} = D_{r,\alpha} \cap V.$$

By $|\ |_{r,\alpha}$ we denote the norm of $H^\infty(D_{r,\alpha})$, and by $\|\ \|_{r,\alpha}$ the norm in $H^\infty(W_{r,\alpha})$.

In complete analogy to [19], we obtain the following lemmas from [22, 23, 25].

LEMMA 3.1. *For $0 < \alpha_1 < \alpha_2' < \alpha_2 < \alpha_3$ we have $C > 0$ such that*

$$|\eta|_{r,\alpha_2}^{*\,\alpha_3-\alpha_1} \leqslant C\, |\eta|_{r,\alpha_1}^{*\,\alpha_3-\alpha_2'}\, |\eta|_{r,\alpha_3}^{*\,\alpha_2'-\alpha_1}$$

for all $\eta \in H^\infty(D_{r,\alpha_1})'$.

LEMMA 3.2. *For $0 < \alpha_1 < \alpha_2 < \alpha_3$ and $f \in H^\infty(D_{r,\alpha_3})$, we have*

$$|f|_{r,\alpha_2}^{\alpha_3-\alpha_1} \leqslant |f|_{r,\alpha_1}^{\alpha_3-\alpha_2} |f|_{r,\alpha_3}^{\alpha_2-\alpha_1}.$$

Since the restriction of $H^\infty(W_{r,\alpha})$ to $W_{r,\alpha'}$ is contained in the range of the restriction of $H^\infty(D_{r,\alpha})$ to $W_{r,\alpha'}$, we obtain the next statement from Lemma 3.1.

LEMMA 3.3. *For $0<\alpha_1<\alpha_2'<\alpha_2<\alpha_3$, we have $C>0$ such that*

$$\|\eta\|_{r,\alpha_2}^{*\,\alpha_3-\alpha_1} \leqslant C\,\|\eta\|_{r,\alpha_1}^{*\,\alpha_3-\alpha_2'}\|\eta\|_{r,\alpha_3}^{*\,\alpha_2'-\alpha_1}$$

for all $\eta\in H^\infty(W_{r,\alpha_1})'$.

We set

$$A(\mathbb{R}^d)=\operatorname*{proj}_r\operatorname*{ind}_\alpha H^\infty(D_{r,\alpha})$$

with the locally convex limit topologies, and likewise

$$H_V(X)=\operatorname*{proj}_r\operatorname*{ind}_\alpha H^\infty(W_{r,\alpha}),$$

and it is easy to show that $H_V(X)$, equipped with this topology, carries the quotient topology of $A(\mathbb{R}^d)$ under ρ. Both spaces are (PLB) spaces, that is, countable projective limits of unions of Banach spaces. From the exact sequence (1) we see that (P) is complemented if and only if ρ has a continuous linear right inverse.

PROPOSITION 3.4. *If (P) is complemented in $A(\mathbb{R}^d)$, then V satisfies $\mathrm{PL}_{\mathrm{loc}}$ at every $x\in X$.*

Proof. Let φ be a right inverse for ρ. From the theory of (PLB)-spaces (see, for example, [7, p. 63]) we know that for every r there is an R such that we have the factorization expressed in the left-hand square of the following commutative diagram.

$$\begin{array}{ccccc}
A(\mathbb{R}^d) & \longrightarrow & A(D_r) & \longrightarrow & H_V(X\cap D_r)\\
\uparrow{\scriptstyle\varphi} & & \uparrow{\scriptstyle\widetilde{\varphi}} & & \uparrow\\
H_V(X) & \longrightarrow & H_V(X\cap D_R) & \xrightarrow{\ \mathrm{id}\ } & H_V(X\cap D_R)
\end{array}$$

The unnamed maps are the natural restrictions.

Now, arguing precisely as in the proof of [19, Lemma 5.3], we obtain $\varepsilon>0$ and C_α such that

$$|\widetilde{\varphi}f|_{r,\varepsilon\alpha}\leqslant C_\alpha\|f\|_{R,\alpha}$$

for all $f\in H^\infty(V\cap D_{R,\alpha})$.

We get the following chain of inequalities for $0<\alpha_1<\alpha_2<\alpha_3$ and $f\in H^\infty(W_{r,\alpha_3})$:

$$\begin{aligned}
\|f\|_{r,\varepsilon\alpha_2} &\leqslant |\widetilde{\varphi}f|_{r,\varepsilon\alpha_2}\\
&\leqslant |\widetilde{\varphi}f|_{r,\varepsilon\alpha_1}^{(\alpha_3-\alpha_2)/(\alpha_3-\alpha_1)}|\widetilde{\varphi}f|_{r,\varepsilon\alpha_3}^{(\alpha_2-\alpha_1)/(\alpha_3-\alpha_1)}\\
&\leqslant C_{\alpha_1}^{(\alpha_3-\alpha_2)/(\alpha_3-\alpha_1)}C_{\alpha_3}^{(\alpha_2-\alpha_1)/(\alpha_3-\alpha_1)}\|f\|_{R,\alpha_1}^{(\alpha_3-\alpha_2)/(\alpha_3-\alpha_1)}\|f\|_{R,\alpha_3}^{(\alpha_2-\alpha_1)/(\alpha_3-\alpha_1)}.
\end{aligned}$$

By applying this to f^n, taking nth roots and letting n go to infinity we obtain:

$$\|f\|_{r,\varepsilon\alpha_2}\leqslant\|f\|_{R,\alpha_1}^{(\alpha_3-\alpha_2)/(\alpha_3-\alpha_1)}\|f\|_{R,\alpha_3}^{(\alpha_2-\alpha_1)/(\alpha_3-\alpha_1)}.$$

We set $\|f\|_{R,0}=\sup\{|f(x)|:x\in X\cap D_R\}$. Letting α_1 tend to 0 and replacing α_2 by α and α_3 by γ, we get, for $0<\alpha<\gamma$ and $f\in H^\infty(W_{R,\gamma})$,

$$\|f\|_{r,\varepsilon\alpha}\leqslant\|f\|_{R,0}^{(\gamma-\alpha)/\gamma}\|f\|_{R,\gamma}^{\alpha/\gamma}.$$

This means that for any function $u(z)=c\log|f(z)|$ where f is holomorphic on $W_{R,\gamma}$, $c>0$, such that $u(z)<0$ on $\mathbb{R}^d\cap W_{R,\gamma}$ and $u(z)<1$ on $W_{R,\gamma}$, we have

$$u(z)\leqslant\frac{\alpha}{\gamma},\qquad z\in W_{r,\varepsilon\alpha}.$$

We now fix $0 < \rho < r$, $\gamma > 0$. From the explicit form of the level sets of V_{D_r} (see [11, p. 207]) we see that there is $\delta > 0$ such that for any $z \in \partial D_{r,\varepsilon\alpha}$, $\alpha < \gamma$, with $|x| \leqslant \rho$, we have $|y| \geqslant \delta\alpha$.

Therefore there is an A such that for any $u(z) = c \log |f(z)|$ where f is holomorphic on $W_{R,\gamma}$, $c > 0$, so that $u(z) < 0$ on $\mathbb{R}^d \cap W_{R,\gamma}$ and $u(z) < 1$ on $W_{R,\gamma}$, we have

$$u(z) \leqslant A|\mathrm{Im}\, z|, \qquad z \in W_{r,\varepsilon\gamma} \cap \{z : |\mathrm{Re}\, z| \leqslant \rho\}.$$

By standard arguments (cf. [19]) we conclude that V has $\mathrm{PL}_{\mathrm{loc}}$ at any $x \in \mathbb{R}^d \cap V \cap D_\rho$. As r and $\rho < r$ were arbitrary, this proves the result. □

4. *Consequences of the necessary condition*

Before we narrow our focus to homogeneous polynomials for which we are able to prove the converse of Proposition 3.4 and obtain a complete characterization, we show some consequences of the necessary condition proved so far.

If V satisfies $\mathrm{PL}_{\mathrm{loc}}$ at some point $a \in X$, then for any $f \in \mathscr{O}_a$ with $f|_{X_a} = 0$ we have $f|_{V_a} = 0$; that is, the ideal $\mathscr{I}_{X_a}$ of X_a in $\mathscr{O}_a$ coincides with the ideal $\mathscr{I}_{V_a}$ of V_a and, for irreducible P, is $P \cdot \mathscr{O}_a$. Since $\widetilde{P}(z) := \overline{P(\overline{z})}$ vanishes on X_a, the polynomial P must divide $\widetilde{P}$ in this case; that is, P has to be proportional to a real polynomial. For general P satisfying $\mathrm{PL}_{\mathrm{loc}}$ in a, we know that every irreducible factor satisfies $\mathrm{PL}_{\mathrm{loc}}$ in a. Therefore every irreducible factor which vanishes in a has to be proportional to a real polynomial. Finally, we conclude from the equality of ideals that $\dim_{\mathbb{R}} X_a = \dim_{\mathbb{C}} V_a = d - 1$.

If V satisfies $\mathrm{PL}_{\mathrm{loc}}$ at every $a \in X$, we immediately get the following corollary.

Corollary 4.1. *If (P) is complemented in $A(\mathbb{R}^d)$, then $P = P_1 P_2$, where P_1 has no real zeros and P_2 is a real polynomial, X is of pure dimension $d - 1$, and $H_V(X) = A(X)$.*

For all this, cf. [10, 14, 15, 19]. We may prove an even stronger local version. Let $a \in X$, and let $P_a = f_1 \cdots f_p$ be an irreducible decomposition of the germ P_a of P in $\mathscr{O}_a$. Put $V_{a,j}$ the germ of the complex zero variety of f_j in a, and $X_{a,j} = V_{a,j} \cap \mathbb{R}^d$. Then every $V_{a,j}$ satisfies $\mathrm{PL}_{\mathrm{loc}}$; hence $X_{a,j}$ is coherent, of pure dimension $d - 1$, and f_j is, up to a unit in $\mathscr{O}_a$, real on $\mathbb{R}^d$.

Proposition 4.2. *If P is a real polynomial and (P) is complemented in $A(\mathbb{R}^d)$, then for every $a \in X$ the germ X_a is coherent and the germ P_a has, in $\mathscr{O}_a$, an irreducible decomposition $P_a = f_1 \cdots f_p$, where all $f_j \in \mathscr{O}_a^{\mathbb{R}}$, and their real zero varieties are coherent and have pure dimension $d - 1$.*

We use the following notation. We set

$$\mathscr{I}_X = \{f \in A(\mathbb{R}^d) : f|_X = 0\} \qquad \text{and} \qquad \mathscr{I}_V = \{f \in H(\mathbb{C}^d) : f|_V = 0\}.$$

Moreover, we set

$$(P)^\infty = P \cdot C^\infty(\mathbb{R}^d) \qquad \text{and} \qquad \mathscr{I}_X^\infty = \{f \in C^\infty(\mathbb{R}^d) : f|_X = 0\}.$$

We say that P has the *ideal property* (or *propriété des zéros*) if $(P)^\infty = \mathscr{I}_X^\infty$, and that P has the *analytic ideal property* if $(P) = \mathscr{I}_X$.

Lemma 4.3. *If P has the ideal property, then P also has the analytic ideal property. If P has the analytic ideal property and X_a is coherent at every $a \in X$, then P has the ideal property.*

Proof. For the first part, see [2, Proposition 3]); the second follows from [12, Theorem 3.10] and a partition of unity argument. □

Since it is obvious from the previous discussion that the $\mathrm{PL}_{\mathrm{loc}}$ condition being fulfilled at every point of X implies the analytic ideal property, and it also implies coherence of X at every $a \in X$, we obtain our next theorem from Lemma 4.3.

THEOREM 4.4. *If (P) is complemented in $A(\mathbb{R}^d)$, then P has the ideal property and the analytic ideal property.*

Now, from results of Bierstone and Schwarz [1] and Langenbruch [10, Theorem 1.6] we know that $(P)^\infty$ is complemented in $C^\infty(\mathbb{R}^d)$ if and only if P has the ideal property, and we can conclude the following theorem.

COROLLARY 4.5. *If (P) is complemented in $A(\mathbb{R}^d)$, then $(P)^\infty$ is complemented in $C^\infty(\mathbb{R}^d)$.*

EXAMPLE. The real zero variety of the irreducible polynomial $P(x,y) = y^3 - x^3(1+x^2)$ is of pure dimension 1. In $\mathscr{O}_0$ it decomposes into one real and two complex factors, and hence, by Proposition 4.2, (P) is not complemented. In $A(\mathbb{R}^d)$ it decomposes into two real-valued factors,

$$f_1(x,y) = y - x\sqrt[3]{1+x^2}, \qquad f_2 = \left(y + \tfrac{x}{2}\sqrt[3]{1+x^2}\right)^2 + \tfrac{3}{4}x^2\left(1+x^2\right)^{2/3}.$$

Here, f_1 describes X; f_2 vanishes in 0 only. Hence P does not even have the analytic ideal property. On the other hand, there is a continuous linear projection in $A(\mathbb{R}^d)$ onto $\mathscr{J}_X$, namely

$$f(x,y) \mapsto f(x,y) - f\left(x, x\sqrt[3]{1+x^2}\right).$$

Let us remark that the description in Proposition 4.2 corresponds to the characterization of the ideal property contained in Bochnak [2, Corollaire 2].

Let P have the ideal property, and let π be a continuous linear projection in $C^\infty(\mathbb{R}^d)$ onto $(P)^\infty$. Assume now that (P) is not complemented; then π necessarily sends some $f \in A(\mathbb{R}^d)$ to $\pi(f) \in C^\infty(\mathbb{R}^d) \setminus A(\mathbb{R}^d)$, because $\pi A(\mathbb{R}^d) \subset A(\mathbb{R}^d)$ would, due to the closed graph theorem, imply that $\pi|_{A(\mathbb{R}^d)}$ is a continuous linear projection in $A(\mathbb{R}^d)$ onto (P).

5. *Sufficiency of* $\mathrm{PL}_{\mathrm{loc}}$, *local case*

From Proposition 3.4 and [19, Theorem 2.2] we get the following theorem. Notice that $\mathrm{PL}_{\mathrm{loc}}$ at any real point of V implies that X is coherent (see [19]); hence $A(X) = H_V(X)$.

THEOREM 5.1. *If X is compact, then (P) is complemented if and only if V satisfies $\mathrm{PL}_{\mathrm{loc}}$ at any $x \in X$.*

We will now study the case of a homogeneous P. As a first step we extend the proof of [19, Proposition 4.5] — that is, the sufficiency part of [19, Theorem 2.2] — from the case of compact X to a semiglobal result in the general case.

THEOREM 5.2. *If V satisfies $\mathrm{PL}_{\mathrm{loc}}$ at any real point, then for every compact $K \subset \mathbb{R}^d$ there is a continuous linear map $\varphi_K : A(X) \longrightarrow H(K)$ such that $\varphi_K f|_{K\cap X} = f|_{K\cap X}$ for all $f \in A(X)$.*

Proof. We choose $r > 0$ such that K is contained in the interior of D_r. We use the following neighborhoods of D_r:

$$U_{r,\alpha} = \{z : |x| < r + \alpha,\ |y| < \alpha\}.$$

They are analytic polyhedra.

By a compactness argument, the $\mathrm{PL}_{\mathrm{loc}}$-condition gives us a neighborhood

$$U = \{z : |x| < R,\ |y| < \gamma\}$$

of D_r in $\mathbb{C}^d$, and constants A and $\gamma_0 > 0$, such that for any plurisubharmonic function u on V with $u(z) < 1$ for $z \in U \cap V$ and $u(x) < 0$ for $x \in U \cap X$ we have $u(z) \leqslant A\,|\mathrm{Im}\, z|$ for $z \in U_{r,\gamma_0}$.

Let $\omega(z) = \omega(D_R \cap X, U_{R,\gamma} \cap V, z)$ be the relative extremal function on $U_{R,\gamma} \cap V$, that is, the upper regularization of

$$\sup\{u(z) : u \text{ plurisubharmonic on } V,\ u < 1 \text{ on } U_{R,\gamma} \cap V,\ u < 0 \text{ on } D_R \cap X\}.$$

Then the above condition says that there are a constant A and $\gamma_0 > 0$ such that

$$\omega(z) \leqslant A\,|\mathrm{Im}\, z|, \qquad z \in V \cap U_{r,\gamma_0}.$$

We set

$$\widetilde{V}_{r,\alpha} = V \cap U_{r,\alpha}, \qquad V_\beta = \{z \in V : \omega(z) < \beta\},$$

and we obtain, for $0 < \alpha < \gamma_0$,

$$\widetilde{V}_{r,\alpha} \subset V_{A\alpha}.$$

Changing our previous notation, we denote now by $|\ |_{r,\alpha}$ the norm in $H^\infty(U_{r,\alpha})$, and by $\|\ \|_\beta$ the norm in $H^\infty(V_\beta)$. In analogy to Lemmas 3.1 and 3.2, we see that for $0 < \alpha_1 < \alpha_2' < \alpha_2 < \alpha_3$ and $\eta \in H^\infty(U_{r,\alpha_1})'$, we have

$$\|\eta\|_{r,\alpha_2}^{*\,\alpha_3-\alpha_1} \leqslant C\,\|\eta\|_{r,\alpha_1}^{*\,\alpha_3-\alpha_2'}\,\|\eta\|_{r,\alpha_3}^{*\,\alpha_2'-\alpha_1}. \tag{2}$$

For $0 < \alpha_1 < \alpha_2 < \alpha_3$ and $f \in H^\infty(V_{\alpha_3})$, we have

$$|f|_{r,\alpha_2}^{\alpha_3-\alpha_1} \leqslant |f|_{r,\alpha_1}^{\alpha_3-\alpha_2}|f|_{r,\alpha_3}^{\alpha_2-\alpha_1}. \tag{3}$$

Moreover, for any small α we have the following diagram with exact row:

$$\begin{tikzcd} 0 \arrow[r] & H(U_{r,\alpha}) \arrow[r, "M_P"] & H(U_{r,\alpha}) \arrow[r] & H(\widetilde{V}_{r,\alpha}) \arrow[r] & 0 \\ & & & H(V_{A\alpha}) \arrow[u] & \end{tikzcd}$$

where M_P is the multiplication with P and the unnamed arrows are the restriction maps.

As in the proof of [**19**, Proposition 4.5], we conclude that there is a continuous linear map $\varphi : A(X) \to H(D_r)$ with the desired property. □

If V satisfies $\mathrm{PL}_{\mathrm{loc}}$ at any real point, then we have, for every $n \in \mathbb{N}$, a map $\varphi_n \in L(A(X), H(D_n)$ with $\varphi_n(f)|_X = f|_X$, for all $f \in A(X)$. If there exists a sequence $\psi_n \in L(A(X), H(D_n)$, $n \in \mathbb{N}$, such that (omitting the restriction map) $\psi_{n+1} - \psi_n = \varphi_n$ for all n, then we obtain a right inverse φ for ρ. Therefore, for a proof that (P) is complemented, it would suffice to show that $\mathrm{Proj}^1_n L(A(X), H(D_n) = 0$. Unfortunately, this is not known in general (see [**20**]). However, in the homogeneous case we can use a simpler argument.

6. *Sufficiency of* $\mathrm{PL}_{\mathrm{loc}}$, *homogeneous case*

Let P now be a homogeneous polynomial of degree $m > 0$ such that V satisfies the $\mathrm{PL}_{\mathrm{loc}}$-condition at every real point. In consequence, P cannot be elliptic. Therefore $X \neq \{0\}$. We set, as previously, $\mathbb{R}^d_* = \mathbb{R}^d \setminus \{0\}$, $X_* = \mathbb{R}^d_* \cap X$.

From Theorem 1.5 we obtain our next lemma.

Lemma 6.1. *There exists a continuous linear extension operator* $A(X_*) \longrightarrow A(\mathbb{R}^d_*)$.

So we have an extension operator near 0, and one off 0. To patch them together, we need the following lemma. For $0 \leqslant r < \rho \leqslant \infty$, we put $D_r^\rho = \{x : r \leqslant \|x\| \leqslant \rho\}$. Here, $\| \ \|$ denotes the euclidian norm.

Lemma 6.2. *For any $0 < r < \rho$ there are σ_1, σ_2 with $0 < \sigma_1 < r < \rho < \sigma_2$, and continuous linear maps $\psi_0 : H(D_{\sigma_1}^{\sigma_2}) \longrightarrow H(D_0^\rho)$ and $\psi_\infty : H(D_{\sigma_1}^{\sigma_2}) \longrightarrow H(D_r^\infty)$ such that $\psi_0 f + \psi_\infty f = f$ on D_r^ρ.*

Proof. Due to the real analytic diffeomorphism $x \mapsto (\arctan x_1, \ldots, \arctan x_d)$ it suffices to replace ψ_∞ by $\psi_R : H(D_{\sigma_1}^{\sigma_2}) \longrightarrow H(D_r^R)$ for large $R > \sigma_2$ in the statement of the lemma. In fact, one would need only to consider the case of $0 < \sigma_1 < r < \rho < \sigma_2 < \pi/2 < R$.

We put

$$w(z) = \sqrt{\left|\sum_{j=1}^d z_j^2\right|},$$

and we set, for small $\alpha > 0$,

$$D_{r,\alpha}^\rho = \{z \in \mathbb{C}^d : r - \alpha < w(z) < \rho + \alpha;\ |\mathrm{Im}\, z| < \alpha\}.$$

The $D_{r,\alpha}^\rho$ are analytic polyhedra and $\bigcap_\alpha D_{r,\alpha}^\rho = D_r^\rho$. From Zahariuta [**22**, **23**, **25**] we learn that for the spaces $H^\infty(D_{r,\alpha}^\rho)$ we have inequalities like those in Lemma 3.1.

Since $D_{0,\alpha}^\rho \cap D_{r,\alpha}^R = D_{r,\alpha}^\rho$, we obtain by the Cartan–Oka theory the exactness of the row in the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H(D_{0,\alpha}^R) & \longrightarrow & H(D_{0,\alpha}^\rho) \oplus H(D_{r,\alpha}^R) & \longrightarrow & H(D_{r,\alpha}^\rho) & \longrightarrow & 0 \\ & & & & & & \uparrow & & \\ & & & & & & H(U_{A\alpha}) & & \end{array}$$

Here, $U_\alpha = \{z : \omega(z) < \alpha\}$, where $\gamma < \sigma_1$ and $\omega(z) = \omega(D_{\sigma_1}^{\sigma_2}, D_{\sigma_1,\gamma}^{\sigma_2}, z)$ is the relative extremal plurisubharmonic function. Comparison with the pluricomplex Green function of neighborhoods of points in $D_{\sigma_1}^{\sigma_2}$ shows that for small α and suitable $A > 0$ we have $\omega(z) \leqslant A|\mathrm{Im}\, z|$ on $D_{r,\alpha}^\rho$, and hence $D_{r,\alpha}^\rho \subset U_{A\alpha}$. The vertical arrow in the diagram now means the restriction map.

For the norms in $H^\infty(U_\alpha)$ we obtain, due to the maximality of ω, the inequalities as in Lemma 3.2. As previously, by small changes of the α we can set up the scheme for the application of [**18**] and obtain the result. □

Proposition 6.3. *If V satisfies $\mathrm{PL}_{\mathrm{loc}}$ at any point of X, then there is a continuous linear extension operator $\varphi : A(X) \longrightarrow A(\mathbb{R}^d)$.*

Proof. By Theorem 5.2 there is a continuous linear operator $\varphi_0 : A(X) \longrightarrow H(D_0^{\sigma_2})$ such that $\varphi_0 f = f$ on $X \cap D_0^{\sigma_2}$, and by Lemma 6.1 there is a continuous linear map $\varphi_\infty : A(X) \longrightarrow A(\mathbb{R}_*^d)$ such that $\varphi_\infty f = f$ on X_*.

We choose $\chi : A(X) \longrightarrow H(D_{\sigma_1}^{\sigma_2})$ such that $P \cdot \chi f = \varphi_0 f - \varphi_\infty f$ on $D_{\sigma_1}^{\sigma_2} \cap X$. With ψ_0 and ψ_∞ of Lemma 6.2 we put, for $f \in A(X)$,

$$\varphi f = \varphi_0 f - P \cdot \psi_0(\chi f) \quad \text{on } D_0^\rho; \qquad \varphi f = \varphi_\infty f + P \cdot \psi_\infty(\chi f) \quad \text{on } D_r^\infty.$$

On D_r^ρ we have

$$(\varphi_0 f - P \cdot \psi_0(\chi f)) - (\varphi_\infty + P \cdot \psi_\infty(\chi f)) = P \cdot \chi f - P \cdot \chi f = 0.$$

Therefore φ is well defined and the assertion is proved. □

7. *Main theorems*

From Propositions 3.4 and 6.3, and Theorem 5.1, we obtain the following characterization.

THEOREM 7.1. *If P is homogeneous or has a compact real zero set, then (P) is complemented if and only if V satisfies* $\mathrm{PL}_{\mathrm{loc}}$ *at every real point.*

If we restrict our attention to the homogeneous case and combine our result with the results in [**15**, Theorem 3.13], [**9**] and [**14**], then we have the following result.

THEOREM 7.2. *For homogeneous P, the following are equivalent.*
(i) *The principal ideal of P is complemented in $A(\mathbb{R}^d)$.*
(ii) *$P(D)$ is non-elliptic and $P(D) : A(\mathbb{R}^d) \longrightarrow A(\mathbb{R}^d)$ is surjective.*
(iii) *$P(D) : C^\infty(\mathbb{R}^d) \longrightarrow C^\infty(\mathbb{R}^d)$ has a continuous linear right inverse.*
(iv) *V satisfies* $\mathrm{PL}_{\mathrm{loc}}$ *at any real point of V.*

8. *Examples and further results*

Concrete examples of homogeneous polynomials to satisfy or not satisfy $\mathrm{PL}_{\mathrm{loc}}$ at every real point of its zero variety (which we refer to as *of type* PL from now on) can be found in [**3**–**5**, **9**, **14**, **15**, **19**]. We present some of them. We always assume, without restriction of generality, that $P \in \mathbb{R}[x_1, \ldots, x_n]$.

If $n = 2$, then P is a product of linear forms, and PL means that all of them must be real. General, not necessarily homogeneous, P is of type PL if and only if all real singularities of X are intersections of smooth lines (see [**5**]). A good example is $P_1(x, y) = y^2 - x^2 + x^4$ (lemniscate); a bad one is $P_2(x, y) = y^2 - x^3 + x^5$ (see [**19**]).

For $\deg P = 2$, the polynomial P is of type PL if and only if the underlying quadratic form is either indefinite or the product of two real linear forms (see [**9**]).

Similarly, if $n \geqslant 3$ and P has the form

$$P(x_1, \ldots, x_n) = \sum_{k=1}^{n} a_k x_k^m,$$

then P is of type PL if and only if m is odd, or the a_k have different signs, or only one $a_k \neq 0$ (see [**14**]).

We recall that a real analytic variety $X \subset \mathbb{R}^d$ is called *of type* PL if its complexification at every point satisfies $\mathrm{PL}_{\mathrm{loc}}$. Now, with exactly the same modifications of the proof of [**19**, Proposition 4.5] as were applied to show Theorem 5.2, we can prove our final theorem.

THEOREM 8.1. *If X is of type* PL, *then for every compact $K \subset \mathbb{R}^d$ there is a continuous linear map $\varphi_K : A(X) \longrightarrow H(K)$ such that $\varphi_K f|_{K\cap X} = f|_{K\cap X}$ for all $f \in A(X)$.*

This modification could serve to extend [**19**, Theorem 2.2] also to noncompact real analytic varieties. That would be immediate if $\mathrm{Proj}^1_n L(A(X), H(K_n)) = 0$ which, unfortunately, is not known in general (see [**20**]).

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