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Splitting of exact sequences of Fréchet spaces in the absence of continuous norms

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Abstract

In the present paper we study the splitting of a short exact sequence

$$0 \longrightarrow G \longrightarrow E \stackrel{S}{\longrightarrow} F \longrightarrow 0$$

of nuclear Fréchet spaces, where F need not admit a continuous norm, i.e., E or F may be spaces of continuous functions on an open subset of $\mathbb{R}^n$ or on a σ-compact C^∞-manifold. If E is such a space, then we give a necessary and sufficient condition for the splitting of the sequence, in terms of a condition on S and of linear topological invariants (Ω_{loc}) and $(\mathrm{DN}_{\mathrm{loc}})$. This is used to give a structure theory of the space $s^{\mathbb{N}}$, a space which is isomorphic to many spaces of smooth functions, and to study the splitting of differential complexes. The paper offers a different approach to problems which have been studied in Domański–Vogt [J. Funct. Anal. 153 (1998) 203].

Keywords: Splitting of exact sequences; Linear topological invariants; Structure theory; Differential complexes

For nuclear Fréchet spaces F and G there are well-known necessary and sufficient conditions for $\mathrm{Ext}^1(F,G)=0$ (see [4,19,22]), where $\mathrm{Ext}^1(F,G)=0$ means that every exact sequence

$$0 \longrightarrow G \longrightarrow E \longrightarrow F \longrightarrow 0$$

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of Fréchet spaces splits. The most well-known case is that of $G \in (\Omega)$ and $F \in (\mathrm{DN})$ (see [19]). It goes back to [16] and [21]. The conditions for $\mathrm{Ext}^1(F, G) = 0$ imply, under very general assumptions, that F must have a continuous norm (see [18]). E.g., in the case of $F = C^\infty(\Omega)$ the only infinite-dimensional nuclear Fréchet space G with $\mathrm{Ext}^1(F, G) = 0$ is $\omega = \mathbb{K}^{\mathbb{N}}$.

On the other hand, many important exact sequences of spaces of smooth functions (i.e., $C^\infty(\Omega)$ or similar), have a stronger form of exactness, they are graded exact, i.e., the spaces are in a natural way projective limits of Fréchet spaces with continuous norm and the exact sequence can be written as a projective limit of exact sequences involving these "local" Fréchet spaces. This applies, in particular, to differential complexes.

In Domański–Vogt [1] conditions for the splitting of such graded exact sequences were studied and it was shown that graded exact complexes of spaces of smooth functions split automatically from a certain point on.

In the present paper we study the same problem with a different approach. We do not impose conditions on the type of exactness but on the maps which are admitted in our exact sequences. We call them SK-homomorphisms and they will be defined below. This concept has proved its usefulness also in other connections (see [22]). The condition is natural in the sense that the maps must be SK-homomorphisms if the above exact sequence splits.

We hope that the present approach will give new insights and also will help to simplify and clarify the theory and improve its accessibility. It also leads to a structure theory of the space $s^{\mathbb{N}}$ which is very similar in spirit to the structure theory of s as developed in [16,21] (see also [9, Section 31] or [20]).

Throughout the paper we will use common notation for locally convex spaces and Fréchet spaces. The scalar field is always $\mathbb{K}$, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. For all this we refer to [5] and [9]. This holds, in particular, for the concepts of local Banach spaces, invariants (DN) and (Ω), where we refer to [9, Part IV]. For homological concepts we refer to [22].

1. SK-homomorphisms

Let E, F be locally convex spaces. Seminorm will always mean a continuous seminorm. Let $A : E \to F$ be a continuous linear map.

Lemma 1.1. *For every seminorm q on F there is a seminorm p on E, so that $A(\ker p) \subset \ker q$.*

The proof is an easy consequence of the continuity estimates.

We may call a map with the property in the lemma *seminorm-kernel-continuous* or SK-*continuous*, because it is continuous for the additive group topologies in E and F, whose neighbourhoods of zero are the kernels of continuous seminorms.

Definition 1.2. A is called an SK-*homomorphism* if for every seminorm p on E there is a seminorm q on F, so that

$$\ker q \cap \operatorname{im} A \subset A(\ker p).$$

Equivalently we might say

$$A^{-1}(\ker q) = \ker(q \circ A) \subset \ker p + \ker A. \tag{1}$$

Clearly A is an SK-homomorphism iff it is an homomorphism with respect to the topologies mentioned above.

Notice that not every homomorphism, even not every surjective homomorphism of Fréchet spaces needs to be an SK-homomorphism. As an example we might take the map $A: C^\infty[-1,+1] \to \omega$ with $Af = (f^{(p)}(0))_{p\in\mathbb{N}_0}$.

We want to describe, when a quotient map is an SK-homomorphism. We use the following notation: for an absolutely convex set M we put $\ker M = \ker p_M$, where p_M is its Minkowski functional. Then for every linear subspace G, every continuous seminorm and $V = \{x\colon\ q(x) \leqslant 1\}$ we have $\ker(V+G) = \bar{G}^q$ where the closure is taken in E with respect to the seminorm q.

Lemma 1.3. *Let $G \subset E$ be a closed linear subspace. Then the quotient map $E \to E/G$ is an* SK-*homomorphism if and only if the following holds*: *for every seminorm p in E there is a seminorm q in E, so that*

$$\bar{G}^q \subset \ker p + G,$$

or, equivalently, for every absolutely convex neighbourhood of zero U in E there is an absolutely convex neighbourhood V in E, so that

$$\ker(V+G) \subset \ker U + G.$$

Definition 1.4. A subspace $G \subset E$ with the property described in the previous lemma is called an SK-subspace.

If E admits a continuous norm p, then this means that there is a continuous (semi-)norm q on E, so that G is closed with respect to q.

Clearly, the subspace

$$C^\infty\big([-1,+1],\{0\}\big) = \big\{f \in C^\infty[-1,+1]\colon\ f^{(p)}(0) = 0 \text{ for all } p\big\}$$

of $C^\infty([-1,+1])$ does not fulfill this condition.

Lemma 1.5. *If A admits a continuous linear right inverse $R: \operatorname{im} A \to E$, then it is an* SK-*homomorphism.*

Proof. If R is the right inverse, then for a seminorm p on E we choose a seminorm q on F, so that $q \geqslant p \circ R$ on $\operatorname{im} A$.

We obtain

$$\ker q \cap \operatorname{im} A \subset \big\{x \in \operatorname{im} A\colon\ p(Rx) = 0\big\}.$$

Since $A(Rx) = x$, we have the result. □

The following is an obvious consequence of Lemma 1.5.

Lemma 1.6. *Every continuous projection is an* SK-*homomorphism.*

2. The invariants ($\mathrm{DN_{loc}}$) and ($\boldsymbol{\Omega}_{\mathbf{loc}}$)

We will define in this section two linear topological invariants, which we will use later. They are closely related to properties (DN) and (Ω). We will also study their basic properties so to make it easy to handle them.

We recall that a Fréchet space with a fundamental system $\| \|_1 \leqslant \| \|_2 \leqslant \cdots$ has property (DN) if there is a continuous norm $\| \|$ so that for every k we have K and $C > 0$ with

$$\| \|_k^2 \leqslant C \| \| \cdot \| \|_K.$$

$\| \|$ is called a dominating norm. For the definition, the role and the basic properties of (DN) see [9, p. 359 ff.].

Definition 2.1. F has property ($\mathrm{DN_{loc}}$) if for every seminorm p the quotient map $F \to F/\ker p$ factorizes through a Fréchet space with property (DN).

Some of the basic properties of ($\mathrm{DN_{loc}}$) are collected in the following lemma.

Lemma 2.2.

(1) *F has property* (DN) *iff it has property* ($\mathrm{DN_{loc}}$) *and a continuous norm.*
(2) *Every countable product of Fréchet spaces with property* (DN) *has property* ($\mathrm{DN_{loc}}$).
(3) $s^{\mathbb{N}}$, $C^\infty(\Omega)$, $\Omega \subset \mathbb{R}^n$ *open, have property* ($\mathrm{DN_{loc}}$).

Proof. To prove (1) we assume that F has ($\mathrm{DN_{loc}}$) and a continuous norm. We apply Definition 2.1 to a continuous norm p. This means that the identity of F factorizes through a space H with (DN), hence F is isomorphic to a (even complemented) subspace of H. Since (DN) is clearly inherited by subspaces, F has (DN). The reverse implication is obvious, and the same holds for (2). Since s has (DN) (see [9, Lemma 29.2]), the first part of (3) follows from (2). The second part follows from $C^\infty(\Omega) \cong s^{\mathbb{N}}$ (see [15, p. 383] or [17, Theorem 5.3]) or, simpler, from the fact that for any p the quotient map $C^\infty(\Omega) \to C^\infty(\Omega)/\ker p$ factorizes through $\mathscr{D}(K)$ for a suitable compact $K \subset \Omega$ and $\mathscr{D}(K)$ has (DN) by [9, Lemma 31.10]. □

Let H be a Fréchet space with property (DN), $\| \|$ a dominating norm on H, and

$$F \xrightarrow{A} H \xrightarrow{B} F/\ker p \tag{2}$$

a factorization. We put $P(x) = \|Ax\|$, $x \in F$. Then $\ker A = \ker P$ and we obtain a factorization

$$F/\ker P \overset{\tilde{A}}{\hookrightarrow} H \xrightarrow{B} F/\ker p. \tag{3}$$

We might use this for an equivalent description of ($\mathrm{DN_{loc}}$): F has property ($\mathrm{DN_{loc}}$) iff for every seminorm p there is a seminorm $P \geqslant p$ and a space H with (DN), so that (3) holds.

The relevant permanence property of ($\mathrm{DN_{loc}}$) for our theory is contained in the following lemma. It is useful to remark in advance that property ($\mathrm{DN_{loc}}$) is in general not inherited by subspaces as we see from $H(\Omega) \subset C^\infty(\Omega)$ where $\Omega = \{z \in \mathbb{C}\colon |z| < 1\}$ and $H(\Omega)$ denotes the holomorphic functions. By Lemma 2.2(3), $C^\infty(\Omega)$ has property $(\mathrm{DN}_{\mathrm{loc}})$. By Lemma 2.2(1), property $(\mathrm{DN}_{\mathrm{loc}})$ for $H(\Omega)$ would imply property (DN), which $H(\Omega)$ does not have by [9, 29.4, 29.12, 29.22].

Lemma 2.3. *If $G \subset F$ is an* SK-*subspace and F has property* $(\mathrm{DN}_{\mathrm{loc}})$, *then also G.*

Proof. For a given seminorm p on F we choose $p_1 \geqslant p$, so that

$$\bar{G}^{p_1} \subset \ker p + G.$$

Due to the closed graph theorem the embedding is continuous, even if we equip $\ker p + G$ with the quotient topology of $\ker p \times G \to \ker p + G$.

We obtain the following chain of maps:

$$\bar{G}^{p_1}/\ker p_1 \subset (\ker p + G)/\ker p_1 \twoheadrightarrow (\ker p + G)/\ker p \cong G/G \cap \ker p.$$

The composition of these maps yields a continuous linear map

$$\bar{G}^{p_1}/\ker p_1 \xrightarrow{\psi} G/G \cap \ker p,$$

which is obviously surjective.

If we consider the factorization (2) for $\ker p_1$,

$$F \xrightarrow{A} H \xrightarrow{B} F/\ker p_1,$$

then we obtain with $H_0 = B^{-1}(\bar{G}^{p_1}/\ker p_1)$ the factorization

$$G \xrightarrow{A} H_0 \xrightarrow{B} \bar{G}^{p_1}/\ker p_1 \xrightarrow{\psi} G/G \cap \ker p$$

and H_0, as a subspace of H, has property (DN). □

We recall now that a Fréchet space with a fundamental system $\|\ \|_1 \leqslant \|\ \|_2 \leqslant \cdots$ has property (Ω) if for every k there is m so that for every n there are $0 < \theta < 1$ and $C > 0$ with

$$\|\ \|_m^* \leqslant C \|\ \|_k^{*\,1-\theta} \|\ \|_n^{*\,\theta}.$$

For the definition, the role and the basic properties of (Ω) see [9, p. 367 ff.]. For the following definition cf. the concept of s-friendliness in [1, p. 226] and, in particular, condition (6) in [1, Theorem 4.1].

Definition 2.4. G has property (Ω_{loc}) if G has property (Ω) and for every seminorm p there is a seminorm $P \geqslant p$ so that $\ker P \subset X \subset \ker p$ where X is a Fréchet space with property (Ω) and the embeddings are continuous.

It should be remarked that (Ω_{loc}) is stronger than (Ω) while $(\mathrm{DN}_{\mathrm{loc}})$ is weaker than (DN). Again we collect some of the obvious properties in a lemma.

Lemma 2.5.

(1) *If G has property (Ω) and a fundamental system $\| \, \|_1 \leqslant \| \, \|_2 \leqslant \cdots$ of seminorms so that* $\ker \| \, \|_k$ *is complemented in G for every k, then G has property (Ω_{loc}).*
(2) *Every Köthe space with property (Ω) has property (Ω_{loc}).*
(3) *$s^{\mathbb{N}}$, $C^\infty(\Omega)$, $\Omega \subset \mathbb{R}^n$ open, have property (Ω_{loc}).*

Proof. (1) is obvious since (Ω) is inherited by the complemented subspace $\ker \| \, \|_k$ (see [9, Lemma 29.11]). (2) follows from (1) since the kernels of the standard Köthe seminorms are complemented. Since $s^{\mathbb{N}}$ has (Ω) (see [9, Lemma 29.3 and Exercise 1, p. 376]), the first part of (3) follows from (2). The second part follows from $C^\infty(\Omega) \cong s^{\mathbb{N}}$ (see [15, p. 383] or [17, Theorem 5.3]) or, simpler, from the fact that $C^\infty(\Omega)$ has property (Ω) (see [9, Corollary 31.13]) and for any seminorm of the form

$$\|f\|_{K,k} = \sup\{ |f^{(\alpha)}(x)| \colon x \in K, \ |\alpha| \leqslant k \},$$

where $K \subset \Omega$ is a compact with smooth boundary $\ker \| \, \|_{K,k}$ is complemented (see [13, VI, 3.1, Theorem 5] or [14, Satz 4.6]). □

The relevant permanence property is contained in the following lemma and, in contrast to the case of $(\mathrm{DN}_{\text{loc}})$, quite easily seen.

Lemma 2.6. *If G has property (Ω_{loc}) and $A : G \to Y$ is an* SK-*homomorphism onto Y, then Y has property (Ω_{loc}).*

By use of Lemma 1.3, we obtain the following corollary.

Corollary 2.7. *If G has property (Ω_{loc}) and $H \subset G$ is an* SK-*subspace, then G/H has property (Ω_{loc}).*

And, using Lemma 1.6 and the previous remark, we have:

Corollary 2.8. *Every complemented subspace of a Köthe space with property (Ω) has property (Ω_{loc}).*

For Corollary 2.8 we have even a stronger version:

Proposition 2.9. *If K is a Köthe space, $G \subset K$ a complemented subspace with property (Ω), then G has property (Ω_{loc}).*

Proof. Let π be the projection and p a seminorm on K. Then there is a seminorm P on K, so that $p(\pi x) \leqslant P(x)$ for all $x \in K$, and $\ker P$ complemented in K. We may, e.g., choose P to be one of the canonical seminorms in K. Let π_P be a projection onto $\ker P$.

We obtain

$$G \cap \ker P \subset \pi(\pi_P K) \subset G \cap \ker p.$$

$\pi(\pi_P G)$ has property (Ω) in its quotient topology. The second imbedding is obviously continuous, the first one is continuous due to the closed graph theorem. $\square$

3. A lifting theorem for continuous linear maps

We consider now an exact sequence

$$0 \longrightarrow G \hookrightarrow E \xrightarrow{A} F \longrightarrow 0 \tag{4}$$

of Fréchet spaces and assume A to be an SK-homomorphism.

Lemma 3.1. *There exist fundamental systems of seminorms $\|\ \|_1 \leqslant \|\ \|_2 \leqslant \cdots$ on E, F, and G, respectively, so that for every n* (4) *induces an exact sequence*

$$0 \longrightarrow G/\ker\|\ \|_n \xrightarrow{j_n} E/\ker\|\ \|_n \xrightarrow{A_n} F/\ker\|\ \|_n \longrightarrow 0. \tag{5}$$

Proof. If for seminorms p on E and q on F we have $A(\ker p) \supset \ker q$, then we have for the seminorm $P = p + q \circ A$ on E that $A(\ker P) = \ker q$. We may use this procedure to choose fundamental systems $\|\ \|_1 \leqslant \|\ \|_1 \leqslant \cdots$ in E and F, respectively, so that $A(\ker\|\ \|_n) = \ker\|\ \|_n$ for all n. Then A induces a map $A_n : E/\ker\|\ \|_n \to F/\ker\|\ \|_n$ and we have an exact sequence

$$G \xrightarrow{q_n} E/\ker\|\ \|_n \xrightarrow{A_n} F/\ker\|\ \|_n \longrightarrow 0, \tag{6}$$

where q_n is the quotient map. As $\ker q_n = G \cap \ker\|\ \|_n = \ker\|\ \|_n^G$ if the latter denotes the restriction of $\|\ \|_n$ to G, we obtain the exact sequence (5). $\square$

Lemma 3.2. *If Y_1 is a nuclear Fréchet space, Y_2 a Fréchet space, $A \in L(Y_2, Y_1)$, and A factorizes through a Fréchet space with property* (DN)*, then A factorizes through an* SK*-subspace X of s. If, moreover, Y_1 has property (Ω), then X can be chosen as s.*

Proof. Let $A = A_1 \circ A_2$ where $A_1 \in L(H, Y_1)$, $A_2 \in L(Y_2, H)$. By [9, Lemma 31.4] there exists an exact sequence

$$0 \longrightarrow s \longrightarrow X \xrightarrow{q} Y_1 \longrightarrow 0, \tag{7}$$

where X is a subspace of s, and by [19, Theorems 4.3 and 1.8] A_1 has a lifting $B \in L(H, X)$, i.e., $A_1 = q \circ B$. Hence $A = q \circ (B \circ A_2)$, i.e., A factorizes through X. Since, by [1, Proposition 1.3], X can be imbedded into s so that the quotient is isomorphic to s, it can be considered as an SK-subspace of s.

If Y_1 has property (Ω) then, by [1, Proposition 1.3], X in (7) can be chosen as s. $\square$

Lemma 3.3. *Let $0 \to X \to Y \xrightarrow{\chi} Z \to 0$ be an exact sequence of Fréchet spaces. Let X have property (Ω), Z be nuclear and have a continuous norm and let H be a Fréchet space with property* $(\mathrm{DN}_{\mathrm{loc}})$*, then for every $\varphi \in L(H, Z)$ there exists $\psi \in L(H, Y)$, so that $\chi \circ \psi = \varphi$.*

Proof. Let $\| \ \|$ be a continuous norm on Z and p a seminorm on F so that $\|\chi(x)\| \leqslant p(x)$ for all $x \in H$. Then $\varphi = \tilde{\varphi} \circ \tau$ where $\tau : H \to H / \ker p$ is the quotient map. τ can be factorized as $\tau_1 \circ \tau_2$ with $\tau_2 \in L(H, H_0)$, $\tau_1 \in L(H_0, H/\ker p)$ and H_0 has property (DN). Therefore φ factorizes through a Fréchet space with property (DN). By Lemma 3.2, φ factorizes through a subspace H_1 of s, i.e., $\varphi = \varphi_1 \circ \varphi_2$ where $\varphi_2 \in L(H, H_1)$ and $\varphi_1 \in L(H_1, Z)$. By [19, Theorem 5.1] and [19, Theorem 1.8] there is $\psi_1 \in L(H_1, Y)$ so that $\chi \circ \psi_1 = \varphi_1$. Putting $\psi = \psi_1 \circ \varphi_2$ proves the result. □

We want to prove the following theorem.

Theorem 3.4. *Let $0 \to G \hookrightarrow E \xrightarrow{A} F \to 0$ be an exact sequence of nuclear Fréchet spaces, A an* SK*-homomorphism, and H a Fréchet space. If H has property* ($\mathrm{DN}_{\mathrm{loc}}$) *and G has property (Ω_{loc}), then for every $\varphi \in L(H, F)$ there exists $\psi \in L(H, E)$, so that $A \circ \psi = \varphi$.*

Proof. We choose a fundamental system of seminorms $\| \ \|_1 \leqslant \| \ \|_2 \leqslant \cdots$ according to Lemma 3.1. By going to a subsequence, if necessary, we may assume that for every n there is a Fréchet space X_n with property (Ω) so that $\ker \| \ \|_{n+1} \subset X_n \subset \ker \| \ \|_n$ with continuous embeddings.

For every n we set $E_n = E / \ker \| \ \|_n$ and by $\chi_n : E \to E_n$ we denote the quotient map, likewise for F and G. We apply Lemma 3.3 to the exact sequence

$$0 \longrightarrow G_n \xrightarrow{j_n} E_n \xrightarrow{A_n} F_n \longrightarrow 0$$

and the map $\varphi_n = \chi_n \circ \varphi$ and we obtain $\psi_n \in L(H, E_n)$ so that $A_n \circ \psi_n = \varphi_n$.

By $\chi_n^m : E_m \to E_n$ for $m \geqslant n$ we denote the natural quotient maps, likewise for F and G. We set $u_n = \chi_n^{n+1} \circ \psi_{n+1} - \psi_n$. Since $A_n \circ u_n = 0$, we obtain $v_n \in L(H, G_n)$ so that $j_n \circ v_n = u_n$.

We set up the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \ker \| \ \|_n \arrow[r] & G \arrow[r, "\chi_n"] & G_n \arrow[r] & 0 \\
0 \arrow[r] & X_n \arrow[r] \arrow[u] & Y_n \arrow[r, "\sigma"] \arrow[u, "\tau"'] & G_{n+1} \arrow[r] \arrow[u, "\chi_n^{n+1}"'] & 0 \\
0 \arrow[r] & \ker \| \ \|_{n+1} \arrow[r] \arrow[u] & G \arrow[r, "\chi_{n+1}"] \arrow[u] & G_{n+1} \arrow[r] \arrow[u, "\mathrm{id}"'] & 0 \\
 & & & H. \arrow[u, "v_{n+1}"'] &
\end{tikzcd}$$

Here the first and the third row are clear, the second one is constructed by the standard push-out construction (see, e.g., [22, Definition 5.1.2]).

By Lemma 3.3, we can find $\tilde{w}_n \in L(H, Y_n)$ so that $\sigma \circ \tilde{w}_n = v_{n+1}$. We set $w_n = \tau \circ \tilde{w}_n$ and obtain $w_n \in L(H, G)$ so that $\chi_n \circ w_n = \chi_n^{n+1} \circ v_{n+1}$.

We put

$$W_n = \sum_{k=1}^{n-1} \chi_n \circ w_k - v_n.$$

Then we have $W_n \in L(H, G_n)$ and

$$\chi_n^{n+1} \circ W_{n+1} - W_n = \chi_n \circ w_n - \chi_n^{n+1} \circ v_{n+1} + v_n = v_n.$$

Finally we set $\Psi_n = \psi_n - j_n \circ W_n$ and obtain $\Psi_n \in L(H, E_n)$ and

$$\begin{aligned}\chi_n^{n+1} \circ \Psi_{n+1} - \Psi_n &= \left(\chi_n^{n+1} \circ \psi_{n+1} - \psi_n\right) - j_n \circ \left(\chi_n^{n+1} \circ W_{n+1} - W_n\right) \\ &= u_n - j_n \circ v_n = 0.\end{aligned}$$

Since $E = \lim \operatorname{proj} E_n$, there is $\psi \in L(H, E)$ with $\chi_n \circ \psi = \Psi_n$ and we have

$$\chi_n \circ (A \circ \psi - \varphi) = A_n \circ \Psi_n - \chi_n \circ \varphi = A_n \circ \psi_n - \varphi_n = 0$$

for all n. This means that the range of $A \circ \psi - \varphi$ is in the intersection of the kernels of all seminorms on E, hence $A \circ \psi = \varphi$. $\square$

An immediate consequence of Theorem 3.4 is the following theorem.

Theorem 3.5. *Let $0 \to G \hookrightarrow E \xrightarrow{A} F \to 0$ be an exact sequence of nuclear Fréchet spaces, A an* SK*-homomorphism. If F has property* $(\mathrm{DN}_{\mathrm{loc}})$ *and G has property* (Ω_{loc})*, then the sequence splits.*

Using, moreover, the permanence properties we obtain the following.

Theorem 3.6. *An exact sequence*

$$0 \longrightarrow G \longrightarrow s^{\mathbb{N}} \xrightarrow{A} E \longrightarrow 0$$

splits if and only if A is an SK*-homomorphism, E has property* $(\mathrm{DN}_{\mathrm{loc}})$ *and G has property* (Ω_{loc}).

Proof. Sufficiency of the conditions follows from Theorem 3.4.

To prove necessity we see first from Lemma 1.5 that A is an SK-homomorphism. If $A \circ R = \mathrm{id}_E$ and $\pi = \mathrm{id}_{s^{\mathbb{N}}} - R \circ A$, then π is a projection from $s^{\mathbb{N}}$ onto G and therefore an SK-homomorphism, hence G has property (Ω_{loc}). $R(E) \cong E$ is the kernel of the SK-homomorphism π, hence an SK-subspace of $s^{\mathbb{N}}$. So E has property $(\mathrm{DN}_{\mathrm{loc}})$. $\square$

Finally we see that, as in the case of s, the only obstruction for an SK-subspace G of $s^{\mathbb{N}}$ to be complemented is that either G or $s^{\mathbb{N}}/G$ might not have the necessary structural property.

Corollary 3.7. *A closed subspace $G \subset s^{\mathbb{N}}$ is complemented if and only if G is an* SK*-subspace with property* (Ω_{loc})*, so that $s^{\mathbb{N}}/G$ has property* $(\mathrm{DN}_{\mathrm{loc}})$.

4. Structure theory of $s^{\mathbb{N}}$

The complemented subspaces of s are described by the conditions (DN) and (Ω) (see [9, 31.7]). In particular, E is isomorphic to a complemented subspace of s if and only if it is isomorphic to a subspace and also to a quotient space of s. The situation in the case of $s^{\mathbb{N}}$ is more complicated, as the following example shows.

Due to the Komura–Komura theorem (see [9, 29.8]), every nuclear Fréchet space is isomorphic to a subspace of $s^{\mathbb{N}}$ and due to the structure theory, a nuclear Fréchet space is isomorphic to a quotient space of $s^{\mathbb{N}}$ if and only if it has property (Ω). For the "if"-part see [9, 31.6], for the "only if"-part additionally [9, 31.3]. Therefore, every nuclear finite type power series space, since having property (Ω) by [9, 29.11], is isomorphic to a subspace and to a quotient space of $s^{\mathbb{N}}$. However, it is not isomorphic to a complemented subspace of $s^{\mathbb{N}}$. Because then, having a continuous norm, it would even be isomorphic to a complemented subspace of s, which cannot be by [9, 29.23].

The following gives a description of those Fréchet spaces which are isomorphic to complemented subspaces of $s^{\mathbb{N}}$.

Theorem 4.1. *A nuclear Fréchet space E is isomorphic to a complemented subspace of $s^{\mathbb{N}}$ iff it has properties* $(\mathrm{DN}_{\mathrm{loc}})$ *and* (Ω_{loc}).

Proof. Necessity comes from Corollary 3.7 applied to E and its complement.

To prove the converse let $\|\ \|_1 \leqslant \|\ \|_2 \leqslant \cdots$ be a fundamental system of seminorms in E. Then $E/\ker\|\ \|_p$ has property (Ω) for every p. We may choose the fundamental system so that

$$E_{k+1} := E/\ker\|\ \|_{k+1} \to E/\ker\|\ \|_k =: E_k$$

factorizes through a Fréchet space with property (DN) hence, by Lemma 3.2, through a space $X_k \cong s$.

Therefore $E = \lim\operatorname{proj} X_k$, $X_k \cong s$. It is easy to see that in the canonical resolution

$$0 \longrightarrow E \longrightarrow \prod_k E_k \xrightarrow{\sigma} \prod_k E_k \longrightarrow 0$$

the map σ is an SK-homomorphism and that therefore the same is true for σ in the canonical resolution

$$0 \longrightarrow E \longrightarrow \prod_k X_k \xrightarrow{\sigma} \prod_k X_k \longrightarrow 0. \tag{8}$$

Therefore, we have an exact sequence

$$0 \longrightarrow E \longrightarrow s^{\mathbb{N}} \xrightarrow{A} s^{\mathbb{N}} \longrightarrow 0, \tag{9}$$

where A is an SK-homomorphism. Due to Theorem 3.4 this exact sequence splits which yields the result. $\square$

Problem. It is unknown whether or not every complemented subspace of $s^{\mathbb{N}}$ is isomorphic to a product of complemented subspaces of s. A counterexample would have no basis and therefore be an example of a complemented subspace of a nuclear Köthe space without basis, so solving Pełczyński's problem (see [12]) in the negative.

As seen in the remarks at the beginning of this section, the structure theory of $s^{\mathbb{N}}$, i.e., the characterization of its subspaces and quotient spaces, is known and, as for the quotients, not very informative. More relevant information is contained in the SK-structure theory which we now present.

Theorem 4.2. *A nuclear Fréchet space E is isomorphic to an* SK-*subspace of $s^{\mathbb{N}}$ iff it has property* $(\mathrm{DN}_{\mathrm{loc}})$.

Proof. One direction follows from Lemmas 2.2 and 2.3. The other follows from the exact sequence (8) where, again by Lemma 3.2, we may choose the X_k as SK-subspaces of s. Since σ is an SK-homomorphism, E is imbedded as an SK-subspace into $\prod_k X_k \subset s^{\mathbb{N}}$ where the latter imbedding is factorwise. It is easily seen that then E is isomorphic to an SK-subspace of $s^{\mathbb{N}}$. □

Theorem 4.3. *A nuclear Fréchet space E is isomorphic to an* SK-*quotient of $s^{\mathbb{N}}$ iff it has property* (Ω_{loc}).

Proof. One direction follows from Lemmas 2.5 and 2.6.

To prove the other we choose a fundamental system of seminorms $\| \ \|_1 \leqslant \| \ \|_2 \leqslant \cdots$, and subspaces $X_k \in E$ with property (Ω) so that $\ker \| \ \|_{k+1} \subset X_k \subset \ker \| \ \|_k$ for all $k \in \mathbb{N}$ with continuous imbeddings. Due to Lemma 6.3, X_k can be chosen as a quotient of s. We set $X_0 = E$.

For $k \in \mathbb{N}_0$ let $\imath_k : X_k \hookrightarrow E$ be the imbedding and $q_k : s \to X_k$ a quotient map (see [9, 31.6]). We put $\varphi_k = \imath_k \circ q_k$. We define a map $\varphi : s^{\mathbb{N}_0} \to E$ by

$$\varphi(x) = \sum_{k=0}^{\infty} \varphi_k(x_k), \qquad x = (x_k)_{k \in \mathbb{N}_0} \in s^{\mathbb{N}_0}. \tag{10}$$

Since for any $m \in \mathbb{N}$ we have

$$\sum_{k=0}^{\infty} \|\varphi_k(x_k)\|_m = \sum_{k=0}^{m-1} \|\varphi_k(x_k)\|_m,$$

the series in (10) converges and defines $\varphi \in L(s^{\mathbb{N}_0}, E)$. Due to the summand φ_0, it is surjective and it is obviously an SK-homomorphism. □

So, returning to the remarks at the beginning of this section, we see that we have a similar situation as in the case of s if we replace *subspace* by SK-*subspace* and *quotient space* by SK-*quotient*. A Fréchet space is isomorphic to a complemented (= SK-complemented) subspace of $s^{\mathbb{N}}$ iff is isomorphic to an SK-subspace and to an SK-quotient of $s^{\mathbb{N}}$.

5. Complexes

We investigate now exact complexes of the form

$$0 \to G \hookrightarrow E_0 \xrightarrow{T_0} E_1 \xrightarrow{T_1} E_2 \longrightarrow \cdots, \tag{11}$$

where E_k are nuclear Fréchet spaces for all $k \in \mathbb{N}_0$.

Lemma 5.1. *Let $E_{k-1} \xrightarrow{T_{k-1}} E_k \xrightarrow{T_k} E_{k+1} \xrightarrow{T_{k+1}} E_{k+2}$ be exact and T_{k-1}, T_k, T_{k+1} be* SK-*homomorphisms. If E_{k-1} has property* (Ω_{loc}) *and E_{k+1} has property* $(\mathrm{DN}_{\mathrm{loc}})$, *then T_k has a right inverse $T_k^{-1} : \operatorname{im} T_k \to E_k$ and there is a projection P_k onto $\ker T_k$.*

Proof. Since T_{k-1} is an SK-homomorphism $\operatorname{im} T_{k-1}$ has, by Lemma 2.6, property (Ω_{loc}). Because T_{k+1} is an SK-homomorphism $\ker T_{k+1}$ is an SK-subspace, hence has property $(\mathrm{DN}_{\mathrm{loc}})$, by Lemma 2.3.

Since T_k is an SK-homomorphism Theorem 3.5 implies that T_k has a right inverse $T_k^{-1} : \operatorname{im} T_k \to E_k$. Setting $P_k = \mathrm{id} - T_k^{-1} \circ T_k$, we obtain the result. □

A complex

$$\cdots \longrightarrow E_{-2} \xrightarrow{T_{-2}} E_{-1} \xrightarrow{T_{-1}} E_0 \xrightarrow{T_0} E_1 \xrightarrow{T_1} E_2 \longrightarrow \cdots \tag{12}$$

of topological linear spaces with maps $T_k \in L(E_k, E_{k+1})$ is said to *split* if there exist maps $S_k \in L(E_k, E_{k-1})$ so that

$$T_{k-1} \circ S_k + S_{k+1} \circ T_k = \mathrm{id}$$

for all k. Equivalent is that the complex (12) is exact and for every k there is a right inverse $T_k^{-1} : \operatorname{im} T_k \to E_{k-1}$. For every k then there is a projection P_k in E_k onto $\ker T_k$ and S_k can be chosen as $S_k = T_k^{-k} \circ P_k$.

The complex is said to split for $k \geqslant k_0$ if

$$\cdots \longrightarrow 0 \longrightarrow \ker T_{k_0} \xrightarrow{\mathrm{id}} E_{k_0} \xrightarrow{T_{k_0}} E_{k_0+1} \xrightarrow{T_{k_0+1}} E_{k_0+2} \longrightarrow \cdots \tag{13}$$

splits.

As an immediate consequence of Lemma 5.1 we obtain the following theorem.

Theorem 5.2. *Let in the complex* (11) *all T_k be* SK-*homomorphisms and all E_k have properties* (Ω_{loc}) *and* $(\mathrm{DN}_{\mathrm{loc}})$. *Then we have*:

(1) *The complex splits for $k \geqslant 1$.*
(2) *The complex splits for $k \geqslant 0$ iff G has property* (Ω_{loc}).

Remark. By the previous sections, the assumptions on the space E_k are satisfied for $s^{\mathbb{N}}$, $C^\infty(\Omega)$, $\Omega \subset \mathbb{R}^n$ open, and all of their complemented subspaces. We may also replace C^∞ by certain ultradifferentiable classes of Beurling type (see [17, Corollary 7.8]), in particular by Gevrey classes of this type.

6. Relation to the graded theory

As mentioned in the introduction, the problem of splitting exact sequences or complexes of spaces isomorphic to $C^\infty(\Omega) \cong s^{\mathbb{N}}$ has been closely investigated in [1] in terms of graded exact sequences (cf. also [2,3]). We will now describe the connection between the two theories. Throughout this section we will refer to the notation of Domański–Vogt [1]. For a good description see also Karidopoulou [6].

A graded Fréchet space is a Fréchet space F together with an equivalence class of reduced projective spectra $\mathscr{F} = (F_n, i_n^k)_{k,n\in\mathbb{N},\ n<k}$ so that F is isomorphic to the projective limit of $\mathscr{F}$. A graded operator between graded Fréchet spaces is a continuous linear map which "respects the grading," which means that it arises from a morphism between the respective projective spectra.

A linear topologically exact sequence of Fréchet spaces

$$0 \longrightarrow G \xrightarrow{T} E \xrightarrow{S} F \longrightarrow 0 \tag{14}$$

is called graded exact, if we may choose projective spectra representing the gradings of E, F, and G so that the exact sequence (14) arises as the projective limit of exact sequences

$$0 \longrightarrow G_n \xrightarrow{T_n} E_n \xrightarrow{S_n} F_n \longrightarrow 0. \tag{15}$$

While [1] has a more general scope, in the present paper only the *strict norm grading* is of relevance. It can be described as follows. We choose a fundamental system $\|\ \|_1 \leqslant \|\ \|_2 \leqslant \cdots$ of seminorms and set $E_n = E/\ker\|\ \|_n$ with the quotient topology. As linking maps i_n^k we take the natural quotient maps. The spaces E_n then have continuous norms.

In the case of $E = C^\infty(\Omega)$, where Ω is an open subset of $\mathbb{R}^n$ and $K_1 \subset K_2 \subset \cdots$ a compact exhaustion of Ω, we have $E_n = \mathscr{E}(K_n)$ the space of Whitney jets on K_n, i.e., we separate the "grading in space" from the grading by order of differentiability.

We will now discuss the concept of graded exactness in the case where all spaces carry the strict norm grading. Since we are treating exact complexes as a sequence of short exact sequences, we need to discuss only the concept of a graded short exact sequence.

Notice that we did not adopt the definition of graded exactness given in [1, p. 214], since it involves the concept of a graded homomorphism. However, the definition of a graded homomorphism given in [1, p. 211] is erroneous and does not reflect the essence of what should be a homomorphism. An easy example (see [6, Bemerkung 2.23(b)]) shows that its inverse, if existent, needs not to be a graded homomorphism. For a correct definition see [6, Definition 2.25]. This error does not touch the correctness of the theory in [1], since the definition of a graded short exact sequence as we use it here is established as equivalent in [1, Proposition 3.1] and used throughout [1].

Proposition 6.1. *If E, F, G carry the strict norm grading and* (14) *is exact, then* (14) *is graded exact if and only if S is an* SK-*homomorphism.*

Proof. One side of the proof is Lemma 3.1. For the other side we may, by suitable numeration, arrange things so that obtain the following commutative diagram:

$$\begin{tikzcd}
 & G/\ker\|\ \|_n \arrow[r, "\tau_n"] & E/\ker\|\ \|_n & & \\
0 \arrow[r] & G_n \arrow[r, "T_n"] \arrow[u, "\rho_n"] & E_n \arrow[r, "S_n"] \arrow[u, "\rho_n"] & F_n \arrow[r] & 0 \\
 & & E \arrow[u, "i_n"] \arrow[r, "\sigma_l"] & F/\ker\|\ \|_l \arrow[u, "j_n^l"] \arrow[r] & 0.
\end{tikzcd}$$

The maps ρ_n and j_n^l are the respective maps arising from the equivalence of spectra. $\sigma_l = \chi_l \circ S$ and $\chi_l : F \to F/\ker\|\ \|_l$ is the quotient map, the latter likewise for E, G, and n. Then we have $i_n \circ \rho_n = \chi_n$.

An easy diagram chase yields that for every $x \in S^{-1}(\ker\|\ \|_l)$, i.e., with $\sigma_l(x) = 0$, we obtain $\xi \in G$ so that $\tau_n(\chi_n(\xi)) = \chi_n(x)$, which means that $x \in \operatorname{im} T + \ker\|\ \|_n$. Hence, we have proved that for any n there is l so that

$$S^{-1}\big(\ker\|\ \|_l\big) \subset \ker S + \ker\|\ \|_n,$$

which is formula (1) after Definition 1.2. □

To carry on the comparison we need an analogue to Lemma 3.2 for the case of condition (Ω). First, we have to provide an analogue to the exact sequence (7).

Lemma 6.2. *For every nuclear Fréchet space E there is an exact sequence*

$$0 \longrightarrow E \longrightarrow X \longrightarrow s \longrightarrow 0, \tag{16}$$

where X is a quotient of s.

Proof. From the proof of [9, Proposition 31.6] we get an exact sequence

$$0 \longrightarrow E \longrightarrow H \longrightarrow \tilde{Q} \longrightarrow 0, \tag{17}$$

where $\tilde{Q}$ is a subspace of s. From the diagram there we conclude with the help of [21, Lemma 1.7] that H and $\tilde{Q}$ have property (Ω). Therefore, H is isomorphic to a quotient of s and $\tilde{Q}$ is isomorphic to a complemented subspace of s. By [9, Lemma 31.2], we have $\tilde{Q} \oplus s \cong s$. Putting $X = H \oplus s$, we get the result. □

Lemma 6.3. *If Y_1 is a Fréchet space, Y_2 a nuclear Fréchet space, $A \in L(Y_2, Y_1)$, and A factorizes through a Fréchet space with property (Ω), then A factorizes through a quotient X of s. If, moreover, Y_2 has property* (DN)*, then X can be chosen as s.*

Proof. Let $A = A_1 \circ A_2$ where $A_1 \in L(H, Y_1)$, $A_2 \in L(Y_2, H)$. By Lemma 6.2, there exists an exact sequence

$$0 \longrightarrow Y_2 \overset{j}{\longrightarrow} X \longrightarrow s \longrightarrow 0, \tag{18}$$

where X is a quotient of s, and by [19, Theorems 4.1 and 1.8] A_2 has an extension $B \in L(X, H)$, i.e., $A_2 = B \circ j$. Hence $A = (A_1 \circ B) \circ j$, i.e., A factorizes through X.

If Y_2 has property (DN) then, by [1, Proposition 1.3] X in (18) can be chosen as s. $\square$

Finally, we want to relate the concept of *s-friendliness* from [1, p. 226] to our present work. We write the definition down for the special case of a strict norm graded space.

Definition 6.4. The strict norm graded space E is called s-friendly if it has property (Ω), and for every seminorm p there is a seminorm $P \geqslant p$ so that for every seminorm $Q \geqslant P$ the canonical map $\ker P \to \ker p / \ker Q$ factorizes through a space with property (Ω).

It is obvious from the definition that every Fréchet space with property (Ω_{loc}) is s-friendly. For the nuclear case also the converse is true.

Proposition 6.5. *A nuclear Fréchet space E is s-friendly in the strict norm grading if and only if it has property (Ω_{loc}).*

Proof. One direction is clear from the previous, so we assume that E is s-friendly in the strict norm grading. We show that E is an SK-quotient of $s^{\mathbb{N}}$.

We choose a fundamental system of seminorms $\| \ \|_1 \leqslant \| \ \|_2 \leqslant \cdots$ so that for every $k = 2, 3, \ldots$ we have a factorization

$$\ker \| \ \|_k \xrightarrow{u_k} X_k \xrightarrow{v_k} \ker \| \ \|_{k-1} / \ker \| \ \|_{k+2}.$$

By Lemma 6.3, we may assume that X_k is a quotient of s. Let $q_k : s \to X_k$ be the quotient map. For $k > n$ we set up the following commutative diagram putting $G = \ker \| \ \|_n$. As previously χ_n^m and χ_n denote the respective quotient maps.

$$\begin{tikzcd}
0 \arrow[r] & \ker \| \ \|_{k-1} / \ker \| \ \|_{k+1} \arrow[r] & G / \ker \| \ \|_{k+1} \arrow[r, "\chi_{k-1}^{k+1}"] & G / \ker \| \ \|_{k-1} \arrow[r] & 0 \\
0 \arrow[r] & X_k \arrow[u] \arrow[r] & Y_k \arrow[u, "\tau"] \arrow[r, "\sigma"] & G / \ker \| \ \|_k \arrow[u, "\chi_{k-1}^{k}"] \arrow[r] & 0 \\
0 \arrow[r] & \ker \| \ \|_k \arrow[u] \arrow[r] & G \arrow[u] \arrow[r, "\chi_{n+1}"] & G / \ker \| \ \|_k \arrow[u, "\mathrm{id}"] \arrow[r] & 0 \\
 & & & s. \arrow[u, "B_k"] &
\end{tikzcd}$$

The diagram is self-explaining, the middle row comes from the third row by the standard push-out construction (see, e.g., [22, Definition 5.1.2]). The map $B_k \in L(s, G / \ker \| \ \|_k)$ is assumed to be given. Then the same argument as in the proof of Theorem 3.4 yields a map $B_{k+1} \in L(s, G / \ker \| \ \|_{k+1})$ with $\chi_{k-1}^{k+1} \circ B_{k+1} = \chi_{k-1}^{k} \circ B_k$.

Keeping n fixed, we set up an inductive procedure for $k \geqslant n + 3$. We start with $B_{n+3} = v_{n+1} \circ q_{n+1}$ and obtain a sequence $B_k \in L(s, G / \ker \| \ \|_k)$ so that $\chi_{k-1}^{k} \circ (\chi_k^{k+1} \circ B_{k+1}) = \chi_{k-1}^{k} \circ B_k$ for all $k \geqslant n + 3$.

Since $G = \lim\operatorname{proj} G/\ker\| \ \|_k$, we get a map $\varphi_n \in L(s, G) = L(s, \ker\| \ \|_n)$ so that $\chi_{k-1} \circ \varphi_n = \chi_{k-1}^k \circ B_k$ for all $k \geqslant n+3$. In particular, we have

$$\chi_{n+2} \circ \varphi_n = \chi_{n+2}^{n+3} \circ B_{n+3} = \chi_{n+2}^{n+3} \circ v_{n+1} \circ q_{n+1}.$$

Since E has property (Ω), there is a quotient map $\varphi_0 : s \to E$. As in the proof of Theorem 4.3, we set

$$\varphi(x) = \sum_{k=0}^{\infty} \varphi_k(x_k), \qquad x = (x_k)_{k\in\mathbb{N}_0} \in s^{\mathbb{N}_0}. \tag{19}$$

Since for any $m \in \mathbb{N}$ we have

$$\sum_{k=0}^{\infty} \left\|\varphi_k(x_k)\right\|_m = \sum_{k=0}^{m-1} \left\|\varphi_k(x_k)\right\|_m,$$

the series in (19) converges and defines $\varphi \in L(s^{\mathbb{N}_0}, E)$. Due to the summand φ_0, it is surjective. We have to show that it is an SK-homomorphism.

For $x \in \ker\| \ \|_{n+1}$ we have $u_{n+1}(x) \in X_{n+1}$, hence there is $\xi_n \in s$ so that $u_{n+1}(x) = q_{n+1}(\xi_n)$. We apply v_{n+1} to both sides of this equation and obtain $\chi_{n+3}(x) = B_{n+3}(\xi_n)$. Applying χ_{n+2}^{n+3} this yields $\chi_{n+2}(x) = \chi_{n+2}^{n+3}(B_{n+3}(\xi_n)) = \chi_{n+2}(\varphi_n(\xi_n))$. Therefore, we have $x - \varphi_n(\xi_n) \in \ker\| \ \|_{n+2}$.

Repeating this procedure, we construct inductively a sequence $\xi_n, \xi_{n+1}, \ldots$ in s so that

$$x = \sum_{k=n}^{\infty} \varphi_n(\xi_n).$$

This means that for $\xi = (0, \ldots, 0, \xi_n, \xi_{n+1}, \ldots) \in s^{\mathbb{N}_0}$ we have $\varphi(\xi) = x$.

The sets $N_n = \{x = (x_0, x_1, \ldots) \in s^{\mathbb{N}_0}\colon\ x_j = 0 \text{ for } j = 0, \ldots, n-1\}$ are a basis of the seminorm kernels in $s^{\mathbb{N}_0}$. We have shown that $\varphi(N_n) \supset \ker\| \ \|_n$ for all n. This shows that φ is an SK-homomorphism. $\square$

7. Remarks on differential complexes

The case of differential complexes has been studied carefully in [1, Section 5]. We do not want to duplicate this. We will make a few remarks how these results are related to the present theory.

A Fréchet sheaf on a topological space Ω, which we always assume locally compact and σ-compact, is a sheaf $\mathcal{G}$ with the following properties:

(1) for any open $\omega \subset X$ the space $G(\omega) = \Gamma(\omega, \mathcal{G})$ of sections is a Fréchet space,
(2) for $\omega_1 \subset \omega_2$ the restriction map $G(\omega_2) \to G(\omega_1)$ is continuous.

Notice that in this case $G(\Omega) = \lim\operatorname{proj} G(\omega_k)$ for any exhaustion $\omega_1 \subset\subset \omega_2 \subset\subset \cdots$ of Ω by open sets. Algebraically this is trivially true, topologically by the open mapping theorem.

We say that $\mathcal{G}$ *admits locally continuous norms* if

(3) for any open $\omega \subset\subset \Omega$ there is a seminorm p on $G(\Omega)$ so that $p(s) = 0$ implies $s|_\omega = 0$.

Notice that this is automatically the case if the elements $G(\Omega)$ are locally continuous functions with values in a finite-dimensional space (case of sections through a vector-bundle) and the topology is stronger than pointwise convergence.

Let $\mathscr{E}$ and $\mathscr{F}$ be Fréchet sheaves on Ω and $\Psi : \mathscr{E} \to \mathscr{F}$ a sheaf homomorphism so that $\psi_U : E(U) \to F(U)$ is continuous for every open $U \subset \omega$. Let $\mathscr{G}$ be the kernel sheaf, i.e., $G(U) = \ker \psi_U$ for open $U \subset \Omega$. We set $\psi = \psi_\Omega$ and for open sets $U \subset W \subset \Omega$,

$$E(W,U) = \{s \in E(W): s|_U = 0\}, \qquad E_{\mathscr{G}}(W,U) = \{s \in E(W): s|_U \in G(U)\}.$$

Lemma 7.1. *If $\mathscr{E}$ and $\mathscr{F}$ admit locally continuous norms, then ψ is an* SK-*homomorphism if and only if for any open set $\omega \subset\subset \Omega$ there is an open set $\omega' \subset\subset \Omega$ so that*

$$E_{\mathscr{G}}(\Omega, \omega') \subset E(\Omega, \omega) + G(\Omega). \tag{20}$$

Proof. Let the above condition be fulfilled and a seminorm p on $E(\Omega)$ be given. Then we choose ω so that $E(\Omega,\omega) \subset \ker p$. This can be done since $E(\Omega)$ is the projective limit of the $E(\omega)$, $\omega \subset\subset \Omega$. For ω we choose ω' according to our condition and then, using that $\mathscr{F}$ admits locally continuous norms, a seminorm q on $F(\Omega)$ so that $\ker q \subset F(\Omega, \omega')$. We obtain

$$\psi^{-1}(\ker q) \subset \psi^{-1}\big(F(\Omega,\omega')\big) = E_{\mathscr{G}}(\Omega,\omega') \subset E(\Omega,\omega) + G(\Omega) \subset \ker p + G(\Omega).$$

The reverse direction works along the same scheme, using that $\mathscr{E}$ admits locally continuous norms. □

A sheaf $\mathscr{E}$ is called *strict on Ω* if it satisfies the following condition: for any open set $\omega \subset\subset \Omega$ there is an open set ω' with $\omega \subset \omega' \subset\subset \Omega$ so that

$$\{s|_\omega: s \in E(\omega')\} = \{s|_\omega: s \in E(\Omega)\}. \tag{21}$$

Every soft sheaf (even every C-soft sheaf, C denoting the compact sets) fulfills this condition.

Proposition 7.2. *If the sheaf $\mathscr{E}$ is strict on Ω and $\mathscr{E}$ and $\mathscr{F}$ admit locally continuous norms, then the following are equivalent*:

(1) *ψ is an* SK-*homomorphism*;
(2) *the strict norm grading and the grading given by the projective limit $G(\Omega) = \operatorname{lim\,proj} G(\omega_k)$, $\omega_1 \subset\subset \omega_2 \subset\subset \cdots$ an exhaustion of Ω, coincide*;
(3) *$\mathscr{G}$ is strict on Ω.*

Proof. (1) $\Rightarrow$ (3). For ω we choose ω' according to (20) and for ω' an ω'' according to (21). For $s \in G(\omega'')$ there is $s_1 \in E(\Omega)$ so that $s_1|_{\omega'} = s|_{\omega'}$. This implies $s_1 \in E_{\mathscr{G}}(\Omega, \omega') \subset E(\Omega,\omega) + G(\Omega)$. Therefore there is $s_2 \in G(\Omega)$ so that $s_2|_\omega = s_1|_\omega = s|_\omega$.

(3) ⇒ (2). We may assume the exhaustion chosen so that $\{s|_{\omega_k}\colon s \in G(\omega_{k+1})\} = \{s|_{\omega_k}\colon s \in G(\Omega)\}$. Then we get in a natural way continuous linear maps

$$G(\Omega)/G(\Omega,\omega_{k+1}) \longrightarrow G(\omega_{k+1}) \longrightarrow G(\Omega)/G(\Omega,\omega_k) \longrightarrow G(\omega_k)$$

so that the composition of either two of them gives the quotient, respectively restriction map.

(2) ⇒ (1) follows from Proposition 6.1. □

We consider now a differential complex, i.e., an exact complex of the form

$$0 \to G \hookrightarrow E_0 \xrightarrow{T_0} E_1 \xrightarrow{T_1} E_2 \longrightarrow \cdots. \tag{22}$$

We assume that for every $k \in \mathbb{N}_0$ either $E_k = 0$ or $E_k = C^\infty(\Omega, B_k)$ where Ω is a given C^∞-manifold and B_k a vector bundle on Ω and we assume that T_k is a differential operator. For $k \in \mathbb{N}$ we assume that for every $\omega \subset\subset \Omega$ there is $\omega' \subset\subset \Omega$ so that for any $s \in C^\infty(\omega', B_k)$ with $T_k s = 0$ we have $\sigma \in C^\infty(\omega, B_{k-1})$ with $T_{k-1}\sigma = s|_\omega$.

The sheaves $C^\infty(\,\cdot\,, B_k)$ are soft. The last assumption implies that the kernel sheaf of T_k, $k \geqslant 1$, is strict on Ω. Hence all T_k, $k \geqslant 1$ are SK-homomorphisms. All E_k are nuclear and have properties (Ω_{loc}) and $(\mathrm{DN}_{\mathrm{loc}})$. The proof that even $E_k \cong s^{\mathbb{N}}$ works like the proof of $C^\infty(\Omega) \cong s^{\mathbb{N}}$ in [15] or [17]. One uses a partition of unity subordinate to a covering of Ω by coordinate patches, where B is trivial. Therefore, Theorem 5.2 yields the following result of [1, Section 5].

Theorem 7.3.

(1) *The complex splits for $k \geqslant 1$.*
(2) *The complex splits for $k \geqslant 0$ if and only if $\mathscr{G}$ is strict on Ω and G has property (Ω_{loc}).*
(3) *If $\Omega \subset \mathbb{R}^n$ open, $E_0 = C^\infty(\Omega)^s$ and T_0 has constant coefficients, then the complex splits for $k \geqslant 0$ if and only if $\mathscr{G}$ is strict on Ω.*

Here $\mathscr{G}$ denotes the kernel sheaf of T_0. (3) follows from Proposition 7.2 and [1, Theorem 5.5]. This theorem generalizes results of Meise, Taylor, and Vogt [7,8] and Palamodov [10,11].

In the case of Palamodov [10] this is immediately seen: if (Ω) is a Stein manifold and the complex under consideration is the Dolbeault complex, then Theorem 7.3(1) yields the result of Palamodov [10] that the Dolbeault complex splits for $k \geqslant 1$. Since the sheaf $\mathscr{O}$ is not strict it does not split for $k = 0$.

In the other cases the connection to Theorem 7.3 is established by a dual characterization for the strictness of $\mathscr{G}$ which is expressed in the following theorem.

Theorem 7.4. *If $\Omega \subset \mathbb{R}^n$ open, $E_j = C^\infty(\Omega)^{s_j}$ for $j = 0, 1$ and T_0 has constant coefficients, then the following are equivalent*:

(1) *The complex splits for $k \geqslant 0$.*
(2) *For every open set $\omega \subset\subset \Omega$ there is an open set ω' with $\omega \subset \omega' \subset\subset \Omega$ and so that for every bounded $B \subset E_0'(\Omega)$ there is a bounded $D \subset E_0'(\omega')$ such that for any $\mu \in$*

$E_0'(\omega)$ *condition* (a) *implies condition* (b) *where*
(a) $\mu + T_0'\nu \in B$ *for some* $\nu \in E_1'(\Omega)$;
(b) $\mu + T_0'\nu_1 \in D$ *for some* $\nu_1 \in E_1'(\omega')$.

Proof. First we prove necessity of the condition. We may, for this purpose, express strictness in the following way:

For every open $\omega \subset\subset \Omega$ there is an open $\omega \subset \omega' \subset\subset \Omega$ so that the map $\chi : G(\Omega) \to G(\omega')/G(\omega', \omega)$, defined by $\chi(s) = s|_{\omega'} + G(\omega', \omega)$, is surjective.

By use of the surjectivity criterion [9, 26.1] this is equivalent to the following: for every bounded $B \subset E_0'(\Omega)$ there is a bounded $D \subset E_0'(\omega')$ such that for any $\mu \in E_0'(\omega')$ conditions (a) and (b) imply (c) where

(a) $\mu \in G(\omega', \omega)^\perp$;
(b) $\mu + T_0'\nu \in B$ for some $\nu \in E_1'(\Omega)$;
(c) $\mu + T_0'\nu_1 \in D$ for some $\nu_1 \in E_1'(\omega')$.

This condition clearly implies (2).

To prove sufficiency of the condition we assume $f \in G(\omega')$ to be given. We may assume that ω' is chosen so large that $T_0'^{-1}\mathscr{D}(\omega)^{s_0} \subset E_1'(\omega') + T_1'E_2'(\Omega)$. We set

$$H := \mathscr{D}(\omega)^{s_0} + T_0'E_1'(\Omega)$$

and we define a linear form F on H in the following way: for $\lambda = \mu + T_0'\nu$ we set $\langle\lambda, F\rangle = \langle\mu, f\rangle$.

We have to show that F is well-defined. If $\lambda = \varphi_1 + T_0'\nu_1 = \varphi_2 + T_0'\nu_2$, then $\varphi_1 - \varphi_2 = T_0'(\nu_2 - \nu_1) \in \mathscr{D}(\omega)^{s_0}$. This implies the existence of $\gamma_1 \in E_1'(\omega')$ and $\gamma_2 \in E_2'(\Omega)$ so that $\nu_2 - \nu_1 = \gamma_1 + T_1'\gamma_2$. Therefore $\varphi_1 - \varphi_2 = T_0'\gamma_1$, hence $\langle\varphi_1, f\rangle - \langle\varphi_1, f\rangle = 0$.

For any $k \in \mathbb{Z}$ and open $W \subset\subset \Omega$ with smooth boundary we denote by $\|\ \|_{k,W}$ the norm of the Sobolev space $H^k(W)^{s_0}$, $U_{k,W}$ the unit ball of $H_0^k(W)^{s_0}$. For $k \in \mathbb{N}$, $W \supset \omega'$ we apply the assumption on $B = U_{-k,W}$ and obtain D which may be assumed to be of the form $CU_{-l,W'}$, $W' \subset\subset \omega'$ and $l \geqslant k$.

For $\lambda \in H \cap H_0^{-k}(W)^{s_0}$, $\|\lambda\|_{k,W} \leqslant 1$ we find $\lambda_1 \in CU_{-l,W'}$ of the form $\lambda_1 = \mu + T_0'\nu_1$ with $\nu_1 \in E_1'(\omega')$. Therefore we have

$$\left|\langle\lambda, F\rangle\right| = \left|\langle\mu, f\rangle\right| = \left|\langle\lambda_1, f\rangle\right| \leqslant C\|f\|_{l,W'}.$$

We extend F by means of the Hahn–Banach theorem to a linear form on $H_0^{-k}(W)^{s_0}$ which corresponds to an element of $H^k(W)^{s_0}$ denoted again by F. By definition $F|_\omega = f|_\omega$. For $\varphi \in \mathscr{D}(W)^{s_1}$ we have $\langle\varphi, T_0F\rangle = \langle T_0'\varphi, F\rangle = 0$.

Therefore, we have proved: for every $k \in \mathbb{N}$ and $W \subset\subset \Omega$ there are $l \in \mathbb{N}$, $W' \subset\subset \omega'$ and C so that for every $f \in G(\omega')$ there is $F \in H^k(W)^{s_0}$ with $T_0F = 0$, $F|_\omega = f|_\omega$ and $\|F\|_{k,W} \leqslant C\|f\|_{l,W'}$. By a straightforward density argument it follows that the map χ is almost open, hence open and surjective. □

For convex Ω the condition in Theorem 7.4(2) is precisely the condition which can be translated by Fourier transform to the respective Phragmén–Lindelöf conditions in the above mentioned papers.

For general open Ω and one single differential operator $P(D)$ with constant coefficients the condition translates into:

For every bounded $B \subset \mathscr{E}'(\Omega)$ there is a bounded $D \subset \mathscr{E}'(\omega')$ such that for any $\mu \in \mathscr{E}'(\omega)$ condition (a) implies condition (b) where

(a) $\mu + P(-D)\nu \in B$ for some $\nu \in \mathscr{E}'(\Omega)$;
(b) $\mu + P(-D)\nu_1 \in D$ for some $\nu_1 \in \mathscr{E}'(\omega')$.

This appears as condition $(*)$ in [8, Lemma 2.5] which is shown there to be equivalent to $P(D)$-convexity with bounds.

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