On virtual quotients for actions of semigroups

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Abstract

We explain how to construct (virtual) quotients in the context of semigroups and how to construct categories of equivariant vector bundles and their K-theory on such quotients. The usual induction functor for vector bundles gives a characteristic homomorphism, which is an isomorphism in the case of monoids.

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1 Virtual quotients for actions of semigroups

A *semigroup* is a set equipped with an internal law which is *associative*. If the law admits a (necessary unique) identity element then the semigroup is a *monoid*, and if furthermore every element is invertible then it is a group. These set theoretic notions induce corresponding notions for set-valued functors on a given category, in particular on the category of schemes. Using the Yoneda embedding, we get the notions of a semigroup scheme, monoid scheme and group scheme (over a fixed base scheme).

We explain how to construct (virtual) quotients in the context of semigroups and how to construct categories of equivariant vector bundles and their K-theory on such quotients. The usual induction functor for vector bundles gives a characteristic homomorphism, which is an isomorphism in the case of monoids. Although maybe well-known, we could not find this material in the literature.

The main application we have in mind is the construction of the characteristic homomorphism in the equivariant K-theory over the Vinberg monoid of a given connected split reductive group, cf. [PS, Thm. D] for the case of the group GL_2.

Notation: We fix a base scheme S and let (Sch/S) be the category of schemes over S. We fix a semigroup scheme G over S and a subsemigroup scheme $B \subset G$ (i.e. a subsemigroup functor which is representable by a scheme). We denote by $\alpha_{G,G} : G \times G \to G$ the law of G (resp. $\alpha_{B,B} : B \times B \to B$ the law of B). If G is a monoid we denote by e_G its identity section and then we suppose that $B \subset G$ is a submonoid: $e_B := e_G \in B$. If G is a group then we suppose that $B \subset G$ is a subgroup, and denote by $i_G : G \to G$ the inverse map of G (resp. $i_B : B \to B$ the inverse map of B).

1.1 Virtual quotients

Recall that an *S-space in groupoids* is a pair of sheaves of sets (R, U) on (Sch/S) with five morphisms s, t, e, c, i (source, target, identity, composition, inversion)

$$R \underset{t}{\overset{s}{\rightrightarrows}} U \xrightarrow{e} R \qquad R \times_{s,U,t} R \xrightarrow{c} R \qquad R \xrightarrow{i} R$$

satifying certain natural compatibilities. Given a groupoid space, one defines the fibered groupoid over (Sch/S) to be the category $[R, U]'$ over (Sch/S) whose objects resp. morphisms over a scheme T are the elements of the set $U(T)$ resp. $R(T)$. Given a morphisms $f : T' \to T$ in (Sch/S) one defines the pull-back functor $f^* : [R, U]'(T) \to [R, U]'(T')$ using the maps $U(T) \to U(T')$ and $R(T) \to R(T')$. An equivalent terminology for 'fibered groupoid over (Sch/S)' is 'prestack over S', and given a Grothendieck topology on (Sch/S), one can associate a stack to a prestack; in the case of the prestack $[R, U]'$, the associated stack is denoted by $[R, U]$.

If X is a scheme equipped with a (right) action of a group scheme B, one takes $U = X$, $R = X \times B$, and let s be the action of the group and $t = p_1$ be the first projection. Then c is the product in the group and e, i are defined by means of the identity and the inverse of B. By definition, the quotient stack $[X/B]$ is the stack $[X \times B, X]$. For all of this, we refer to [LM00, (2.4.3)].

In the context of semigroups, we adopt the same point of view, however, the maps e and i are missing. This leads to the following definition.

1.1.1. Definition. *The* virtual quotient *associated to the inclusion of semigroups $B \subset G$ is the semigroupoid consisting of the source and target maps $\alpha_{G,B} := \alpha_{G,G}|_{G\times B}$ and first projection p_1*

$$G \times B \underset{p_1}{\overset{\alpha_{G,B}}{\rightrightarrows}} G$$

together with the composition

$$\begin{array}{rcl} c : (G \times B)_{\alpha_{G,B}} \times_{G\ p_1} (G \times B) & \longrightarrow & G \times B \\ \big((g, b), (gb, b')\big) & \longmapsto & (g, bb'). \end{array}$$

We denote it by G/B.

1.1.2. Saying that these data define a semigroupoid means that they satisfy the following axioms:

(0) $\alpha_{G,B} \circ c = \alpha_{G,B} \circ p_2$ and $p_1 \circ c = p_1 \circ p_1$ where we have denoted the two projections $(G \times B)_{\alpha_{G,B}} \times_{G\ p_1} (G \times B) \to G \times B$ by p_1, p_2 ;

(i) (associativity) the two composed maps

$$(G \times B)_{\alpha_{G,B}} \times_{G\ p_1} (G \times B)_{\alpha_{G,B}} \times_{G\ p_1} (G \times B) \underset{\mathrm{id}_{G\times B} \times c}{\overset{c\times \mathrm{id}_{G\times B}}{\rightrightarrows}} (G \times B)_{\alpha_{G,B}} \times_{G\ p_1} (G \times B) \xrightarrow{c} (G \times B)$$

are equal.

1.1.3. If $B \subset G$ is an inclusion of monoids, then G/B becomes a monoidoid thanks to the additional datum of the identity map

$$\varepsilon : G \xrightarrow{\mathrm{id}_G \times e_B} G \times B.$$

This means that the following additional axioms are satisfied:

(0)' $\alpha_{G,B} \circ (\mathrm{id}_G \times e_B) = p_1 \circ (\mathrm{id}_G \times e_B) = \mathrm{id}_G$;

(ii) (identity element) the two composed maps

$$G \times B = (G \times B)_{\alpha_{G,B}} \times_G G = G \times_{G\ p_1} (G \times B) \underset{\mathrm{id}_{G\times B} \times \varepsilon}{\overset{\varepsilon \times \mathrm{id}_{G\times B}}{\rightrightarrows}} (G \times B)_{\alpha_{G,B}} \times_{G\ p_1} (G \times B) \xrightarrow{c} (G \times B)$$

are equal.

1.1.4. If $B \subset G$ is an inclusion of groups, then G/B becomes a groupoid thanks to the additional datum of the inverse map

$$i : G \times B \xrightarrow{\alpha_{G,B} \times i_B} G \times B.$$

This means that the following additional axioms are satisfied:

(0)" $\alpha_{G,B} \circ (\alpha_{G,B} \times i_B) = p_1$ and $p_1 \circ (\alpha_{G,B} \times i_B) = \alpha_{G,B}$;

(iii) (inverse) the two diagrams

$$\begin{array}{ccc} G \times B & \xrightarrow{(\alpha_{G,B} \times i_B) \times \mathrm{id}_{G \times B}} & (G \times B)_{\alpha_{G,B}} \times_{G\ p_1} (G \times B) \\ {\scriptstyle \alpha_{G,B}} \downarrow & & \downarrow {\scriptstyle c} \\ G & \xrightarrow{\mathrm{id}_G \times e_B} & G \times B \end{array}$$

$$\begin{array}{ccc} G \times B & \xrightarrow{\mathrm{id}_{G \times B} \times (\alpha_{G,B} \times i_B)} & (G \times B)_{\alpha_{G,B}} \times_{G\ p_1} (G \times B) \\ {\scriptstyle p_1} \downarrow & & \downarrow {\scriptstyle c} \\ G & \xrightarrow{\mathrm{id}_G \times e_B} & G \times B \end{array}$$

are commutative.

1.2 Categories on the virtual quotient

Let $\mathcal{C}$ be a category fibered over (Sch/S).

1.2.1. Definition. *The* (fiber of the) category $\mathcal{C}$ over G/B *is the category* $\mathcal{C}(G/B)$ *defined by:*

(Obj) an object of $\mathcal{C}(G/B)$ *is a couple* $(\mathcal{F}, \phi_B)$ *where* $\mathcal{F}$ *is an object of* $\mathcal{C}(G)$ *and*

$$\phi_B : p_1^* \mathcal{F} \longrightarrow \alpha_{G,B}^* \mathcal{F}$$

is a morphism in $\mathcal{C}(G \times B)$ *satisfying the following cocycle condition: considering the maps*

$$G \times B \times B \longrightarrow G$$

$$p_1 = p_1 \circ (\mathrm{id}_G \times \alpha_{B,B}) = p_1 \circ p_{12}$$

$$q := \alpha_{G,B} \circ (\mathrm{id}_G \times \alpha_{B,B}) = \alpha_{G,B} \circ (\alpha_{G,B} \times \mathrm{id}_B)$$

$$r := p_1 \circ (\alpha_{G,B} \times \mathrm{id}_B) = \alpha_{G,B} \circ p_{12},$$

the diagram in $\mathcal{C}(G \times B \times B)$

$$\begin{array}{ccc} p_1^* \mathcal{F} & \xrightarrow{(\mathrm{id}_G \times \alpha_{B,B})^* \phi_B} & q^*(\mathcal{F}, \phi_B) \\ & {\scriptstyle p_{12}^* \phi_B} \searrow \quad \nearrow {\scriptstyle (\alpha_{G,B} \times \mathrm{id}_B)^* \phi_B} & \\ & r^* \mathcal{F} & \end{array}$$

is commutative ;

(Hom) *a morphism* $(\mathcal{F}^1, \phi_B^1) \to (\mathcal{F}^2, \phi_B^2)$ *in* $\mathcal{C}(G/B)$ *is a morphism* $\varphi : \mathcal{F}^1 \to \mathcal{F}^2$ *in* $\mathcal{C}(G)$ *such that the diagram in* $\mathcal{C}(G \times B)$

$$\begin{array}{ccc} p_1^*\mathcal{F}^1 & \xrightarrow{p_1^*\varphi} & p_1^*\mathcal{F}^2 \\ \downarrow{\scriptstyle \phi_B^1} & & \downarrow{\scriptstyle \phi_B^2} \\ \alpha_{G,B}^*\mathcal{F}^1 & \xrightarrow{\alpha_{G,B}^*\varphi} & \alpha_{G,B}^*\mathcal{F}^2 \end{array}$$

is commutative.

1.2.2. If $B \subset G$ is an inclusion of monoids, then an object of $\mathcal{C}(G/B)$ is a couple $(\mathcal{F}, \phi_B)$ as in 1.2.1 which is required to satisfy the additional condition that the morphism in $\mathcal{C}(G)$

$$\varepsilon^*(\phi_B) := (\mathrm{id}_G \times e_B)^*\phi_B : \mathcal{F} \longrightarrow \mathcal{F}$$

is equal to the identity. Homomorphisms in $\mathcal{C}(G/B)$ remain the same as in the case of semigroups.

1.2.3. If $B \subset G$ is an inclusion of groups, then given an object $(\mathcal{F}, \phi_B)$ of $\mathcal{C}(G/B)$ as in 1.2.2, the morphism ϕ_B in $\mathcal{C}(G \times B)$ is automatically an isomorphism, whose inverse is equal to $i^*(\phi_B) := (\alpha_{G,B} \times i_B)^*(\phi_B)$. The category $\mathcal{C}(G/B)$ coincides therefore with the category attached to the underlying inclusion of monoids.

1.3 Equivariant categories on the virtual quotient

1.3.1. By taking the direct product $\mathrm{id}_G \times \bullet$ of all the maps appearing in the definition 1.1.1 of the semigroupoid G/B, we get a semigroupoid $G \times G/B$, whose source and target maps are

$$(G \times G) \times B \underset{p_1}{\overset{\alpha_{G\times G,B}}{\rightrightarrows}} G \times G.$$

Then given $\mathcal{C}$ we define the category $\mathcal{C}(G \times G/B)$ exactly as we defined the category $\mathcal{C}(G/B)$, but now using the semigroupoid $G \times G/B$ instead of G/B. Applying once more $\mathrm{id}_G \times \bullet$, we also get the semigroupoid $G \times G \times G/B$ with source and target maps

$$(G \times G \times G) \times B \underset{p_1}{\overset{\alpha_{G\times G\times G,B}}{\rightrightarrows}} G \times G \times G,$$

and then the category $\mathcal{C}(G \times G \times G/B)$.

1.3.2. A morphism $f : G \times G \to G$ is *B-equivariant* if the diagram

$$\begin{array}{ccc} (G \times G) \times B & \xrightarrow{f \times \mathrm{id}_B} & G \times B \\ \downarrow{\scriptstyle \alpha_{G\times G,B}} & & \downarrow{\scriptstyle \alpha_{G,B}} \\ G \times G & \xrightarrow{f} & G \end{array}$$

commutes. Then there is a well-defined *pull-back functor*

$$f^* : \mathcal{C}(G/B) \longrightarrow \mathcal{C}(G \times G/B),$$

given by the rules $(\mathcal{F}, \phi_B) \mapsto (f^*\mathcal{F}, (f \times \mathrm{id}_B)^*\phi_B)$ and $\varphi \mapsto f^*\varphi$. One defines similarly the B-equivariant morphisms $f : G \times G \times G \to G \times G$ and the associated pull-back functors $f^* : \mathcal{C}(G \times G/B) \to \mathcal{C}(G \times G \times G/B)$.

1.3.3. With this preparation, we will now be able to define the G-equivariant version of the category $\mathcal{C}(G/B)$. It relies on the semigroupoid $G\backslash G$ consisting of the source and target maps

$$G \times G \underset{p_2}{\overset{\alpha_{G,G}}{\rightrightarrows}} G$$

together with the composition

$$\begin{array}{rcl} (G\times G)_{\alpha_{G,G}}\times_{G\;p_2}(G\times G) & \longrightarrow & G\times G \\ \big((g_1,g_0),(g_2,g_1g_0)\big) & \longmapsto & (g_2g_1,g_0). \end{array}$$

Note that the source and target maps $\alpha_{G,G}$ and p_2 are B-equivariant.

1.3.4. Definition. *The* (G-)equivariant (fiber of the) category $\mathcal{C}$ over G/B *is the category* $\mathcal{C}^G(G/B)$ *defined by:*

(Obj) an object of $\mathcal{C}^G(G/B)$ *is a triple* $(\mathcal{F},\phi_B,{}_G\phi)$ *where* $(\mathcal{F},\phi_B)$ *is an object of* $\mathcal{C}(G/B)$ *and*

$${}_G\phi : p_2^*(\mathcal{F},\phi_B)\longrightarrow \alpha_{G,G}^*(\mathcal{F},\phi_B)$$

is an isomorphism *in* $\mathcal{C}(G\times G/B)$ *satisfying the following cocycle condition: considering the* B*-equivariant maps*

$$G\times G\times G\longrightarrow G$$

$$p_3$$

$$q:=\alpha_{G,G}\circ(\alpha_{G,G}\times \mathrm{id}_G)=\alpha_{G,G}\circ(\mathrm{id}_G\times\alpha_{G,G})$$

$$r:=p_2\circ(\mathrm{id}_G\times\alpha_{G,G})=\alpha_{G,G}\circ p_{23},$$

and the B*-equivariant maps* $\alpha_{G,G}\times\mathrm{id}_G$*,* p_{23}*,* $\mathrm{id}_G\times\alpha_{G,G}$ *from* $G\times G\times G$ *to* $G\times G$*, the diagram in* $\mathcal{C}(G\times G\times G/B)$

$$\begin{array}{ccccc} p_3^*(\mathcal{F},\phi_B) & & \xrightarrow{(\alpha_{G,G}\times\mathrm{id}_G)^*{}_G\phi} & & q^*(\mathcal{F},\phi_B) \\ & \searrow^{p_{23}^*{}_G\phi} & & \nearrow_{(\mathrm{id}_G\times\alpha_{G,G})^*{}_G\phi} & \\ & & r^*(\mathcal{F},\phi_B) & & \end{array}$$

is commutative ;

(Hom) a morphism $(\mathcal{F}^1,\phi_B^1,{}_G\phi^1)\to(\mathcal{F}^2,\phi_B^2,{}_G\phi^2)$ *in* $\mathcal{C}^G(G/B)$ *is a morphism* $\varphi:(\mathcal{F}^1,\phi_B^1)\to(\mathcal{F}^2,\phi_B^2)$ *in* $\mathcal{C}(G/B)$ *such that the diagram in* $\mathcal{C}(G\times G/B)$

$$\begin{array}{ccc} p_2^*(\mathcal{F}^1,\phi_B^1) & \xrightarrow{p_2^*\varphi} & p_2^*(\mathcal{F}^2,\phi_B^2) \\ {\scriptstyle {}_G\phi^1}\downarrow & & \downarrow{\scriptstyle {}_G\phi^2} \\ \alpha_{G,G}^*(\mathcal{F}^1,\phi_B^1) & \xrightarrow{\alpha_{G,G}^*\varphi} & \alpha_{G,G}^*(\mathcal{F}^2,\phi_B^2) \end{array}$$

is commutative (which by definition means that the diagram in $\mathcal{C}(G\times G)$

$$\begin{array}{ccc} p_2^*\mathcal{F}^1 & \xrightarrow{p_2^*\varphi} & p_2^*\mathcal{F}^2 \\ {\scriptstyle {}_G\phi^1}\downarrow & & \downarrow{\scriptstyle {}_G\phi^2} \\ \alpha_{G,G}^*\mathcal{F}^1 & \xrightarrow{\alpha_{G,G}^*\varphi} & \alpha_{G,G}^*\mathcal{F}^2 \end{array}$$

is commutative).

1.3.5. If $B\subset G$ is an inclusion of monoids, then an object of $\mathcal{C}^G(G/B)$ is a triple $(\mathcal{F},\phi_B,{}_G\phi)$ as in 1.3.4, where now the object $(\mathcal{F},\phi_B)$ of $\mathcal{C}(G/B)$ is as in 1.2.2, which is required to satisfy the additional condition that the morphism in $\mathcal{C}(G)$

$$(e_G\times\mathrm{id}_G)^*{}_G\phi:\mathcal{F}\longrightarrow\mathcal{F}$$

is equal to the identity. Homomorphisms in $\mathcal{C}^G(G/B)$ remain the same as in the case of semigroups.

1.3.6. As in the non-equivariant setting, cf. 1.2.3, if $B\subset G$ is an inclusion of groups, then the category $\mathcal{C}^G(G/B)$ coincides with the category attached to the underlying inclusion of monoids.

1.4 Induction of representations

From now on, the fixed base scheme is a field k and $\mathcal{C}$ is the fibered category of vector bundles.

1.4.1. Definition. *The category* $\mathrm{Rep}(B)$ *of* right representations of the k-semigroup scheme B on finite dimensional k-vector spaces *is defined as follows:*

(Obj) an object of $\mathrm{Rep}(B)$ *is a couple* $(M, \alpha_{M,B})$ *where* M *is a finite dimensional* k*-vector space and*

$$\alpha_{M,B} : M \times B \longrightarrow M$$

is a morphism of k*-schemes such that*

$$\forall (m, b_1, b_2) \in M \times B \times B, \quad \alpha_{M,B}(\alpha_{M,B}(m, b_1), b_2) = \alpha_{M,B}(m, \alpha_{B,B}(b_1, b_2)).$$

(Hom) a morphism $(M^1, \alpha_{M^1,B}) \to (M^2, \alpha_{M^2,B})$ *in* $\mathrm{Rep}(B)$ *is a* k*-linear map* $f : M^1 \to M^2$ *such that*

$$\forall (m, b) \in M^1 \times B, \quad f(\alpha_{M_1,B}(m, b)) = \alpha_{M_2,B}(f(m), b).$$

1.4.2. We define an *induction functor*

$$\mathcal{I}nd_B^G : \mathrm{Rep}(B) \longrightarrow \mathcal{C}^G(G/B)$$

as follows. Let $(M, \alpha_{M,B})$ be an object of $\mathrm{Rep}(B)$. Set $\mathcal{F} := G \times M \in \mathcal{C}(G)$. There are canonical identifications $p_1^*\mathcal{F} = G \times M \times B$ and $\alpha_{G,B}^*\mathcal{F} = G \times B \times M$ in $\mathcal{C}(G \times B)$. Set

$$\begin{array}{rcl} \phi_B : G \times M \times B & \longrightarrow & G \times B \times M \\ (g, m, b) & \mapsto & (g, b, \alpha_{M,B}(m, b)). \end{array}$$

Then $(\mathcal{F}, \phi_B)$ is an object of $\mathcal{C}(G/B)$. Next, there are canonical identifications $p_2^*\mathcal{F} = G \times G \times M$ and $\alpha_{G,G}^*\mathcal{F} = G \times G \times M$ in $\mathcal{C}(G \times G)$. Set

$$_G\phi := \mathrm{id}_{G \times G \times M} .$$

Then $_G\phi$ is an isomorphism $p_2^*(\mathcal{F}, \phi_B) \to \alpha_{G,G}^*(\mathcal{F}, \phi_B)$ in $\mathcal{C}(G \times G/B)$, and $((\mathcal{F}, \phi_B), {}_G\phi)$ is an object of $\mathcal{C}^G(G/B)$.

Let $f : (M^1, \alpha_{M^1,B}) \to (M^2, \alpha_{M^2,B})$ be a morphism in $\mathrm{Rep}(B)$. Then

$$\mathrm{id}_G \times f : \mathcal{F}^1 = G \times M^1 \longrightarrow \mathcal{F}^2 = G \times M^2$$

defines a morphism $\varphi : ((\mathcal{F}^1, \phi_B^1), {}_G\phi^1) \to ((\mathcal{F}^2, \phi_B^2), {}_G\phi^2)$ in $\mathcal{C}^G(G/B)$.

These assignments are functorial.

1.4.3. Lemma. *The functor* $\mathcal{I}nd_B^G$ *is faithful. Suppose moreover that the* k*-semigroup scheme* G *has the following property:*

There exists a k-point of G which belongs to all the $G(\overline{k})$-left cosets in $G(\overline{k})$, and the underlying k-scheme of G is locally of finite type.

Then the functor $\mathcal{I}nd_B^G$ *is fully faithful.*

Proof. Faithfulness is obvious. Now let $\varphi : \mathcal{I}nd_B^G(M^1) = G \times M^1 \to \mathcal{I}nd_B^G(M^2) = G \times M^2$. The compatibilty of φ with $_G\phi^i$, $i = 1, 2$, reads as

$$\mathrm{id}_G \times \varphi = \alpha_{G,G}^*\varphi : G \times G \times M^1 \longrightarrow G \times G \times M^2.$$

For $g \in G(\overline{k})$, denote by $\phi_g : M_{\overline{k}}^1 \to M_{\overline{k}}^2$ the fiber of φ over g. Taking the fiber at (g', g) in the above equality implies that $\varphi_g = \varphi_{g'g}$ for all $g, g' \in G(\overline{k})$, i.e. φ_g depends only on the left coset $G(\overline{k})g$, hence is independent of g if all the left cosets share a common point. Assuming that such a point exists and is defined over k, let $f : M^1 \to M^2$ be the corresponding k-linear endomorphism. Then $\varphi - \mathrm{id}_G \times f$ is a linear morphism between two vector bundles on G, which vanishes on each geometric fiber. Then it follows from Nakayama's Lemma that $\varphi - \mathrm{id}_G \times f = 0$ on G, at least if the latter is locally of finite type over k. □

1.4.4. Definition. *When the functor $\mathcal{I}nd_B^G$ is fully faithful, we call its essential image the* category of induced vector bundles on G/B, *and denote it by $\mathcal{C}^G_{\mathcal{I}nd}(G/B)$:*

$$\mathcal{I}nd_B^G : \mathrm{Rep}(B) \xrightarrow{\sim} \mathcal{C}^G_{\mathcal{I}nd}(G/B) \subset \mathcal{C}^G(G/B).$$

1.4.5. If $B \subset G$ is an inclusion of monoids, then an object of $\mathrm{Rep}(B)$ is a couple $(M, \alpha_{M,B})$ as in 1.4.1 which is required to satisfy the additional condition that the k-morphism

$$\alpha_{M,B} \circ (\mathrm{id}_M \times e_B) : M \longrightarrow M$$

is equal to the identity. Homomorphisms in $\mathrm{Rep}(B)$ remain the same as in the case of semigroups.

In particular, comparing with 1.3.5, the same assignments as in the case of semigroups define an induction functor

$$\mathcal{I}nd_B^G : \mathrm{Rep}(B) \longrightarrow \mathcal{C}^G(G/B).$$

Now set $e := e_B = e_G \in B(k) \subset G(k)$, the identity element. We define a functor *fiber at e*

$$\mathrm{Fib}_e : \mathcal{C}^G(G/B) \longrightarrow \mathrm{Rep}(B)$$

as follows. Let $(\mathcal{F}, \phi_B, {}_G\phi)$ be an object of $\mathcal{C}^G(G/B)$. Set $M := \mathcal{F}|_e$, a finite dimensional k-vector space. There are canonical identifications $(p_1^*\mathcal{F})|_{e\times B} = M \times B$, $(\alpha^*_{G,B}\mathcal{F})|_{e\times B} = (\alpha^*_{G,G}\mathcal{F})|_{B\times e} = \mathcal{F}|_B$ and $(p_2^*\mathcal{F})|_{B\times e} = B \times M$. Set

$$\alpha_{M,B} : M \times B \xrightarrow{\phi_B|_{e\times B}} \mathcal{F}|_B \xleftarrow[\sim]{{}_G\phi|_{B\times e}} B \times M \xrightarrow{p_2} M.$$

Then $(M, \alpha_{M,B})$ is an object of $\mathrm{Rep}(B)$.

Let $\varphi : (\mathcal{F}^1, \phi^1_B, {}_G\phi^1) \to (\mathcal{F}^2, \phi^2_B, {}_G\phi^2)$ be a morphism in $\mathcal{C}^G(G/B)$. Then

$$f = \varphi_e : \mathcal{F}^1|_e = M^1 \longrightarrow \mathcal{F}^2|_e = M^2$$

defines a morphism $(M^1, \alpha_{M^1,B}) \to (M^2, \alpha_{M^2,B})$ in $\mathrm{Rep}(B)$.

These assignments are functorial.

1.4.6. Proposition. *For an inclusion of k-monoid schemes $B \subset G$ with unit e, the functors $\mathcal{I}nd_B^G$ and Fib_e are equivalences of categories, which are quasi-inverse one to the other.*

Proof. Left to the reader. □

1.4.7. Analogous to the property 1.3.6 for equivariant vector bundles, we have that if $B \subset G$ is an inclusion of groups, then given an object $(M, \alpha_{M,B})$ of $\mathrm{Rep}(B)$, the right B-action on M defined by $\alpha_{M,B}$ factors automatically through the k-group scheme opposite to the one of k-linear automorphisms of M, the inverse of $\alpha_{M,B}(\bullet, b)$ being equal to $\alpha_{M,B}(\bullet, i_B(b))$ for all $b \in B$. The category $\mathrm{Rep}(B)$ coincides therefore with the category attached to the underlying monoid of B.

In particular, we have the functors $\mathcal{I}nd_B^G$ and Fib_e attached to the underlying inclusion of monoids $B \subset G$, for which Proposition 1.4.6 holds.

1.5 Grothendieck rings of equivariant vector bundles

1.5.1. For a k-semigroup scheme B, the category $\mathrm{Rep}(B)$ is abelian k-linear symmetric monoidal with unit. Hence, for an inclusion of k-semigroup schemes $B \subset G$ such that the functor $\mathcal{I}nd_B^G$ is fully faithful, the essential image $\mathcal{C}^G_{\mathcal{I}nd}(G/B)$ has the same structure. In particular, it is an abelian category whose Grothendieck group $K^G_{\mathcal{I}nd}(G/B)$ is a commutative ring, which is isomorphic to the ring $R(B)$ of right representations of the k-semigroup scheme B on finite dimensional k-vector spaces:

$$\mathcal{I}nd_B^G : R(B) \xrightarrow{\sim} K^G_{\mathcal{I}nd}(G/B).$$

1.5.2. If $B \subset G$ is an inclusion of monoids, then it follows from 1.4.6 that the category $\mathcal{C}^G(G/B)$ is abelian k-linear symmetric monoidal with unit. In particular, it is an abelian category whose Grothendieck group $K^G(G/B)$ is a commutative ring, which is isomorphic to the ring $R(B)$ of right representations of the k-monoid scheme B on finite dimensional k-vector spaces:

$$\mathcal{I}nd_B^G : R(B) \xrightarrow{\sim} K^G(G/B).$$

1.5.3. If $B \subset G$ is an inclusion of groups, then 1.5.2 applies to the underlying inclusion of monoids.

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