FORMS OF AN AFFINOID DISC AND RAMIFICATION

TOBIAS SCHMIDT

Abstract. Let k be a complete nonarchimedean field and let X be an affinoid closed disc over k. We classify the tamely ramified twisted forms of X. Generalizing classical work of P. Russell on inseparable forms of the affine line we construct explicit families of wildly ramified forms of X. We finally compute the class group and the Grothendieck group of forms of X in certain cases.

1. Introduction

Let k be a field complete with respect to a nontrivial nonarchimedean absolute value and let $X = X(r)$ be an affinoid closed disc over k of some real radius $r > 0$. A form of X is an isomorphism class of a k-affinoid space that becomes isomorphic to X over some complete field extension $k \subseteq \ell$. In this paper we prove some classification results for such forms. Let $\mathcal{A}$ be the automorphism functor of X.

Let $k \subseteq \ell$ be an extension and let $H^1(\ell/k, \mathcal{A})$ be the corresponding pointed set of forms of X. The behavior of the latter set depends crucially on the ramification properties of the extension $k \subseteq \ell$. To simplify things in this introduction, let us assume for a moment that this extension is finite and Galois. If $k \subseteq k^{\mathrm{tr}} \subseteq \ell$ denotes the maximal tamely ramified subextension we have a short exact sequence

$$1 \longrightarrow H^1(k^{\mathrm{tr}}/k, \mathcal{A}) \longrightarrow H^1(\ell/k, \mathcal{A}) \longrightarrow H^1(\ell/k^{\mathrm{tr}}, \mathcal{A}_{k^{\mathrm{tr}}}).$$

Here $\mathcal{A}_{k^{\mathrm{tr}}}$ denotes the automorphism functor of the base change $X \otimes k^{\mathrm{tr}}$. We propose to study the two outer terms in the sequence. So suppose that $k \subseteq \ell$ is tamely ramified. We establish a canonical bijection of pointed sets

$$|\ell^{\times}|/|k^{\times}| \xrightarrow{\cong} H^1(\ell/k, \mathcal{A}), \quad \text{class of } |\mathrm{a}| \mapsto \text{class of } X(r|a|)$$

where $X(r|a|)$ denotes the closed disc over k of radius $r|a|$. In particular, there are no unramified forms of X. These results are in accordance with the corresponding results for the open (poly-)disc obtained by A. Ducros [10]. As in loc.cit. our proof depends on the theory of graded reduction as introduced into analytic geometry by M. Temkin [30]. For example, any affinoid k-algebra A leads, in a functorial way, to a $\mathbb{R}_+^{\times}$-graded algebra $\tilde{A}_\bullet$ over the graded field $\tilde{k}_\bullet$ (a graded field is a graded ring in which any nonzero homogeneous element is invertible). It is induced by the spectral semi norm filtration on A. In particular, the homogeneous degree 1 part $\tilde{A}_1$ equals the usual reduction of A. These graded rings are not local and so the local arguments of [10] do not apply in our affinoid situation. In turn, we only deal with 1-dimensional spaces which makes the functor $\mathcal{A}$ more accessible to computation. Since $k \subseteq \ell$ is tamely ramified, the extension between graded fields $\tilde{k}_\bullet \subseteq \tilde{\ell}_\bullet$ is Galois with Galois group isomorphic to $\mathrm{Gal}(\ell/k)$. We then use Galois descent properties of the automorphism functors of X and its graded reduction to establish the result. For more information we refer to the main body of the text.

We turn to the case of wild ramification. Here, we do not obtain a complete picture. As a starting point, if $k \subseteq \ell$ is a wildly ramified extension, its graded reduction $\tilde{k}_\bullet \subseteq \tilde{\ell}_\bullet$ is purely inseparable. We show that the classical construction of purely inseparable forms of the additive group by P. Russell [25] has a version over graded fields. Each of these graded forms can then be lifted to a form of the additive group on the analytic space X. This provides an abundance of forms of X. Following [12] we tentatively call these forms *of Russell type*. Let us give more details in the simplest case where X is actually the unit disc. Let $p = \mathrm{char}\ \tilde{k}_1 > 0$ and $\tilde{k}_\bullet^{\mathrm{alg}}$ denote a graded algebraic closure of the graded field $\tilde{k}_\bullet$. Let $k \subseteq \ell$ be an extension such that $(\tilde{k}_\bullet)^{p^{-n}} \subseteq \tilde{\ell}_\bullet$ for some $n \geq 1$. Let $\tilde{k}_1[F]$ be the endomorphism ring of the additive group over $\tilde{k}_1$ where F is the Frobenius morphism. The image U_n in the quotient $\tilde{k}_1[F]/(F^n)$ of the multiplicative monoid of separable endomorphisms is a group. Let $G_n = U_n \times \tilde{k}_1^\times$ be the direct product. In [25] the author constructs an explicit action of G_n on the pointed set U_n. We deduce a canonical injection of pointed sets

$$U_n/G_n \hookrightarrow H^1(\ell/k, \mathcal{A})$$

given by mapping the residue class mod (F^n) of a separable endomorphism $\sum_{i=1,\ldots,m} a_i F^i$ to the isomorphism class of the closed subgroup of the two dimensional additive group X^2 cut out by the equation

$$T_2^{p^n} = a_0 T_1 + a_1 T_1^p + \cdots + a_m T_1^{p^m}.$$

Here, T_i are two parameters on X^2. For example, U_1/G_1 equals the quotient of a certain $\tilde{k}_1^\times$-action on an infinite direct sum of copies of the space $\tilde{k}_1/\tilde{k}_1^p$. For more details we refer to loc.cit. and the main body of our text.

The forms of the additive group on X of Russell type have geometrically reduced graded reduction, i.e. $\tilde{A}_\bullet \otimes \tilde{k}_\bullet^{\mathrm{alg}}$ is reduced where A denotes the affinoid algebra of the form. In loc.cit. it is shown that there are many inseparable forms of the affine line that fail to have a group structure. In this light it is likely that, dropping the group structure, there are many more wildly ramified forms of the space X with geometrically reduced graded reduction. In the final part of our article we are concerned with basic invariants of such forms such as the Picard group and the Grothendieck group. We work under the assumptions that k is discretely valued and that all affinoid spaces are strict. These restrictions are technicalities and should not be essential in the end. Let Y be a wildly ramified form of X with geometrically reduced graded reduction. Let A be its affinoid algebra. Serre's theorem from algebraic K-theory [4] implies $K_0(A) = \mathbb{Z} \oplus \mathrm{Pic}(A)$ (and this holds for *any* form of X regardless of ramification and reduction properties). We then use a version of the K_0-part of Quillen's theorem [17] to obtain a canonical isomorphism $\mathrm{Pic}(A) \simeq \mathrm{Pic}(\tilde{A}_\bullet)$. Here, $\tilde{A}_\bullet$ is viewed as an abstract ring, i.e. we forget the gradation here and in the following. Since $\tilde{A}_\bullet$ is geometrically reduced, it equals a form of the affine line relative to the ring extension $\tilde{k}_\bullet \subseteq \tilde{\ell}_\bullet$. Let p^n be the degree of the latter (finite free) extension. A choice of uniformizer $\varpi \in \ell$ induces identifications

$$\tilde{k}_\bullet = \tilde{k}_1[t^{\pm p^n}] \quad \subseteq \quad \tilde{\ell}_\bullet = \tilde{k}_1[t^{\pm 1}]$$

with rings of Laurent polynomials. We then have the classical standard higher derivation associated with this p-radical extension [11]. To allow for $n > 1$ we build on K. Baba's generalization to higher exponents [3] of P. Samuel's classical p-radical descent theory [26] and give a fairly explicit description of $\mathrm{Pic}(\tilde{A}_\bullet)$ in terms of logarithmic derivatives. We deduce that the abelian group $\mathrm{Pic}(A)$ always has exponent p^n. We also deduce a criterion when $\mathrm{Pic}(A)$ is a cyclic group and thus a finite cyclic p-group. In this case, a generator is given by the logarithmic derivative of a parameter on the affine line. We discuss this criterion for forms of Russell type.
It is not unlikely that one may obtain more precise results for the Picard group by developing a graded version of p-radical descent theory and then incorporate the gradings into all our arguments. We leave this as an open question for future work.

We assemble some notions and results of graded commutative algebra in an appendix.

Acknowledgement. I thank Michael Temkin and Brian Conrad for explaining to me some points in analytic geometry.

2. Tamely ramified forms

2.1. G**-groups.** A reference for the following is [28, I.§5]. Let G be a finite group. A G*-group* is a set A with a G-action together with a group structure which is invariant under G (i.e. $g(aa') = g(a)g(a')$). Let A be a G-group. We denote by A^G or $H^0(G, A)$ the subgroup of A consisting of elements a with $g(a) = a$ for all $g \in G$. A *(1-)cocycle of* G *in* A is a map $g \mapsto a_g$ from G to A such that

$$a_{gh} = a_g \, g(a_h)$$

for all $g, h \in G$. Two cocycles a_g and a'_g are *cohomologous* if there is $b \in A$ such that $ba_g = a'_g g(b)$ for all $g \in G$. This is an equivalence relation on the set of cocycles and the set of equivalence classes is denoted by $H^1(G, A)$. The set $H^1(G, A)$ is a *pointed set*, i.e. a set with a distinguished element, namely the class of the trivial cocycle $a_g = 1$ for all $g \in G$. A *morphism of pointed sets* is a map preserving distinguished elements. There is an obvious notion of kernel and exact sequence for pointed sets and their morphisms.

The sets $H^0(G,A)$ and $H^1(G,A)$ are functorial in A and coincide with the usual cohomology groups of dimension 0 and 1 in case A is abelian.

Given a normal and G-invariant subgroup B of A the quotient $C = A/B$ is a G-group and there is an exact sequence of pointed sets

$$1 \longrightarrow B^G \longrightarrow A^G \longrightarrow C^G \longrightarrow H^1(G,B) \longrightarrow H^1(G,A) \longrightarrow H^1(G,C). \tag{2.1}$$

Given a normal subgroup $H \subseteq G$ there is an exact sequence of pointed sets

$$1 \longrightarrow H^1(G/H, A^H) \longrightarrow H^1(G,A) \longrightarrow H^1(H,A) \tag{2.2}$$

and the map $H^1(G/H, A^H) \to H^1(G,A)$ is injective.

2.2. **Twisted forms and Galois cohomology.** A reference for the following is [14, II.§9]. Consider an extension $R \subseteq S$ of commutative rings and a finite subgroup G of automorphisms of the R-algebra S. We assume that the extension is *Galois* with Galois group G. This means that S is a finitely generated projective R-module whose endomorphism ring $\mathrm{End}_R(S)$ admits a basis as (left) S-module consisting of $\sigma \in G$. The table of multiplication is therefore $(s\sigma)(t\tau) = s\sigma(t)\sigma\tau$ for all $s,t \in S$ and $\sigma,\tau \in G$.
Let B be an R-algebra and let $B_S := S \otimes_R B$. The ring B_S has the obvious G-action induced by $g(s\otimes b) = g(s)\otimes b$. Let $\mathrm{Aut}\, B_S$ be the group of automorphisms of the S-algebra B_S. Then $\mathrm{Aut}\, B_S$ becomes a G-group via $g.\alpha = g\alpha g^{-1}$, i.e.

$$(g.\alpha)(s\otimes b) = g(\alpha(g^{-1}(s)\otimes b))$$

where $\alpha \in \mathrm{Aut}\, B_S$. Let $\mathrm{Aut}\, B$ be the automorphism functor of B.

Let now (A) be an isomorphism class of R-algebras and A a representative of this class. We call (A) a *twisted form of B with respect to $R \subseteq S$* if there is an isomorphism of S-algebras

$$\beta : S\otimes_R A \xrightarrow{\cong} S \otimes_R B.$$

Let $H^1(S/R, \mathrm{Aut}\, B)$ be the set of such forms (by abuse of language we will also call *forms* the representatives A of a form (A)). It is a pointed set, the distinguished element being (B). Given a form (A,β) the map $g \mapsto \theta_g$ where

$$\theta_g = \beta g \beta^{-1} g^{-1}$$

is a cocycle of G in $\mathrm{Aut}\, B_S$. If (A',β) is a different representative of (A) with cocycle θ'_g, one has with $\alpha := \beta'\beta^{-1}$ that

$$\alpha\theta_g = \beta' g\beta^{-1}g^{-1} = \beta' g \beta'^{-1} g^{-1} g \beta'\beta^{-1}g^{-1} = \theta'_g g.\alpha$$

for $g \in G$, i.e. θ_g and θ'_g are cohomologous. We obtain a bijection of pointed sets

$$H^1(S/R, \mathrm{Aut}\, B) \xrightarrow{\cong} H^1(G, \mathrm{Aut}\, B_S) \tag{2.3}$$

according to loc.cit., II. Thm. 9.1. The inverse map is explicitly given as follows. Let θ_g be a cocycle of G in Aut B_S. Given $g \in G$ we let $\bar{g} := \theta_g g$. Hence any $\bar{g}$ acts S-semilinearly on B_S. Let

$$A := \{x \in B_S : \bar{g}(x) = x \text{ for all } g \in G\}.$$

Then A is a form of B the isomorphism $\beta : A_S \xrightarrow{\cong} B_S$ being induced by the inclusion $A \subset B_S$. The class (A) is the preimage of the class of θ_g.

2.3. **Graded forms.** Let $\Gamma = \mathbb{R}_+^\times$ (or any other commutative multiplicative group). The reader is referred to the appendix for all occuring notions from graded ring theory that we will use in the following. All occuring graded rings will be Γ-graded rings and so we will frequently omit the group Γ from the notation. Let k be a graded field and denote by $\rho : k^\times \to \Gamma$ its grading.

Let $k \subseteq \ell$ be an extension (finite or not) of graded fields and let B be a graded k-algebra. Then $B_\ell = \ell \otimes_k B$ is a graded ℓ-algebra with respect to the tensor product grading

$$(B_\ell)_\gamma = \sum_{\delta\tau=\gamma} \ell_\delta \otimes B_\tau$$

for $\gamma \in \Gamma$. Let $\mathrm{Aut}^{\mathrm{gr}} B$ be the automorphism functor of B. A graded k-algebra A is a *graded* twisted form of B with respect to $k \subseteq \ell$ if there is an isomorphism (of degree 1) of graded ℓ-algebras $A_\ell \xrightarrow{\cong} B_\ell$. As above we then have the pointed set $H^1(\ell/k, \mathrm{Aut}^{\mathrm{gr}} B)$ of graded forms of B relative to the extension ℓ/k. Let $\mathrm{Aut}^{\mathrm{gr}} B_\ell$ be the group of automorphisms of the graded ℓ-algebra B_ℓ. If $k \subseteq \ell$ is a finite Galois extension with group G, then $\mathrm{Aut}^{\mathrm{gr}} B_\ell$ becomes a G-group in the same manner as above. In this case, we have the cohomology set $H^1(G, \mathrm{Aut}^{\mathrm{gr}} B_\ell)$ and the isomorphism of pointed sets

$$H^1(\ell/k, \mathrm{Aut}^{\mathrm{gr}} B) \xrightarrow{\cong} H^1(G, \mathrm{Aut}^{\mathrm{gr}} B_\ell) \tag{2.4}$$

constructed as in the ungraded case.

2.4. **Graded reductions.** We let $\Gamma = \mathbb{R}_+^\times$ have its usual total ordering. Let k be a field endowed with a nontrivial nonarchimedean absolute value $|.| : k \to \mathbb{R}_+$. Let A be a k-algebra with a submultiplicative nonarchimedean seminorm $|.| : A \to \mathbb{R}_+$ extending the absolute value on k (and therefore denoted by the same symbol). We let $\tilde{A}_\bullet$ be the graded ring equal to

$$\tilde{A}_\bullet := \oplus_{\gamma \in \Gamma} A_{\leq\gamma}/A_{<\gamma}$$

where $A_{\leq\gamma}$ and $A_{<\gamma}$ consists of the elements $a \in A$ with $|a| \leq \gamma$ and $|a| < \gamma$ respectively. Following [10] we call $\tilde{A}_\bullet$ the *graded reduction* of A in the sense of M. Temkin [29],[30]. The homogeneous part $\tilde{A}_1$ is a subring, the *residue ring* of A (in the sense of [6]).

If $r \in \Gamma$ and $a \in A$ is an element with $|a| \leq r$ we denote by $\tilde{a}_r$ the corresponding element in $A_{\leq r}/A_{<r}$. If $|a| = r$ we simply write $\tilde{a}$ instead of $\tilde{a}_r$ and call $\tilde{a}$ the *principal symbol* of a. If $|a| = 0$ we put $\tilde{a} = 0$.

In the case $A = k$ the graded reduction $\tilde{k}_\bullet$ is a graded field: indeed, any homogeneous nonzero element is a principal symbol $\tilde{a}$ and the principal symbol of a^{-1} provides the inverse. It is called the *graded residue field* of k. The homogeneous part $\tilde{k}_1$ is a field, the *residue field* of k (in the sense of [6]).

Suppose $k \subseteq \ell$ is a finite extension between nonarchimedean fields where the absolute value on ℓ restricts to the one on k. Let

$$e := (|\ell^\times| : |k^\times|) \quad \text{and} \quad f := [\tilde{\ell}_1 : \tilde{k}_1].$$

Then $\tilde{k}_\bullet \subseteq \tilde{\ell}_\bullet$ is a finite extension of graded fields with

$$ef = [\tilde{\ell}_\bullet : \tilde{k}_\bullet] \leq [\ell : k]$$

as follows from [10, Prop. 2.10].

Remark: According to [6, Prop. 3.6.2/4] the inequality $[\tilde{\ell}_\bullet : \tilde{k}_\bullet] \leq [\ell : k]$ being an equality is equivalent to the extension ℓ being k-*cartesian* (in the sense of loc.cit., Def. 2.4.1/1). In loc.cit., the ground field k is defined to be *stable* if this property holds for any finite extension of k. Any nonarchimedean field complete with respect to a discrete valuation is stable (loc.cit., Prop. 3.6.2/1). Any nonarchimedean field which is spherically complete (or equivalently, maximally complete [13]) is stable [6, Prop. 3.6.2/12]. Finally, any nonarchimedean complete field which is algebraically closed is stable (loc.cit., Prop. 3.6.2/12).

Denote by p the *characteristic exponent* of the field k_1, i.e. $p = \operatorname{char} k_1$ in case $\operatorname{char} k_1 > 0$ and $p = 1$ else. The extension $k \subseteq \ell$ is called *tamely ramified* if the residue field extension $\tilde{k}_1 \subseteq \tilde{\ell}_1$ is separable and e is prime to p. This is equivalent to the extension of graded fields $\tilde{k}_\bullet \subseteq \tilde{\ell}_\bullet$ being separable [10, Prop. 2.10]. We call $k \subseteq \ell$ *wildly ramified* if the residue field extension $\tilde{k}_1 \subseteq \tilde{\ell}_1$ is purely inseparable and e is a p-power.

Lemma 2.5. *The extension $k \subseteq \ell$ is wildly ramified if and only if $\tilde{k}_\bullet \subseteq \tilde{\ell}_\bullet$ is purely inseparable.*

Proof. If $\tilde{k}_1 \subseteq \tilde{\ell}_1$ is purely inseparable, so is the extension $\tilde{k}_\bullet \subseteq \tilde{\ell}_1 \cdot \tilde{k}_\bullet$. Take $\tilde{x} \in \tilde{\ell}_\bullet$ for some $x \in \ell^\times$. If $e = p^n$, say, then $|x^{p^n}| \in |k^\times|$. This means there is $z \in \tilde{\ell}_1$ and a homogeneous nonzero $w \in \tilde{k}_\bullet$ such that $\tilde{x}^{p^n} = zw$ and therefore $\tilde{x}$ is purely inseparable over $\tilde{\ell}_1 \cdot \tilde{k}_\bullet$. Conversely, suppose that $\tilde{k}_\bullet \subseteq \tilde{\ell}_\bullet$ is purely inseparable. The homogeneous minimal polynomial f over $\tilde{k}_\bullet$ of any nonzero homogeneous element x of $\tilde{\ell}_\bullet$ is purely inseparable. If $x \in \tilde{\ell}_1$, then f has coefficients in $\tilde{k}_1$. Hence $\tilde{\ell}_1$ is purely inseparable over $\tilde{k}_1$. Since $[\tilde{\ell}_\bullet : \tilde{k}_\bullet]$ is a p-power, so is e. $\square$

Remark: Let ℓ/k be an arbitrary k-cartesian finite extension. If $f = 1$ (e.g. if ℓ/k is wildly ramified with $\tilde{k}_1$ being perfect), then $[\ell : k] = e$ and so ℓ/k is *totally ramified*.

2.5. **Tame ramification and Galois cohomology.** Let k be a field complete with respect to a nontrivial nonarchimedean absolute value $|.|$. Let p be the characteristic exponent of the residue field $\tilde{k}_1$ of k.

We will work relatively to a finite tamely ramified field extension $k \subseteq \ell$ which is Galois with group G. Let $n = [\ell : k]$. The finite extensions between the fields $\tilde{k}_1 \subseteq \tilde{\ell}_1$ and the graded fields $\tilde{k}_\bullet \subseteq \tilde{\ell}_\bullet$ are then normal [10, Prop. 2.11] and therefore Galois and have degrees f and n respectively. Since G acts by isometries on ℓ we have two natural homomorphisms

$$G = \mathrm{Gal}(\ell/k) \xrightarrow{\cong} \mathrm{Gal}(\tilde{\ell}_\bullet/\tilde{k}_\bullet) \longrightarrow G(\tilde{\ell}_1/\tilde{k}_1) \tag{2.6}$$

for the corresponding Galois groups. Both maps are surjective and the first map is even an isomorphism (loc.cit., Prop. 2.11). Let $I = I(\ell/k)$ be the *inertia subgroup*, i.e. the kernel of the composite homomorphism. Then $\mathrm{Gal}(\ell/k)/I \xrightarrow{\cong} \mathrm{Gal}(\tilde{\ell}_1/\tilde{k}_1)$ and $n = ef$ implies $\#I = e$. Hence I has no p-torsion which implies the familiar isomorphism

$$I \xrightarrow{\cong} \mathrm{Hom}(|\ell^\times|/|k^\times|, \tilde{\ell}_1^\times),\ g \mapsto \psi_g \tag{2.7}$$

where $\psi_g(\gamma \bmod |k^\times|)$ equals the reduction of $\frac{g(x)}{x}$ for some element $x \in \ell^\times$ of absolute value γ (loc.cit., Prop. 2.14). In particular, I is abelian.

In the following we will collect some results on cohomology groups associated with the ring of integers ℓ° in ℓ and its residue field $\tilde{\ell}_1$. These results are certainly well-known but we give complete proofs for lack of suitable reference.

Lemma 2.8. *One has $H^n(G, \tilde{\ell}_1) = 0$ for all $n \geq 1$.*

Proof. We have $H^n(I, \tilde{\ell}_1) = 0$ for all $n \geq 1$ since the order $\#I$ is invertible in $\tilde{\ell}_1$ and therefore the group ring $\tilde{\ell}_1[I]$ is semisimple. Now I acts trivially on $\tilde{\ell}_1$ and the normal basis theorem [27, Prop. X.§1.1] implies $H^n(G/I, \tilde{\ell}_1) = 0$. The assertion follows therefore from the exact inflation-restriction sequence

$$0 \longrightarrow H^n(G/I, (\tilde{\ell}_1)^I) \longrightarrow H^n(G, \tilde{\ell}_1) \longrightarrow H^n(I, \tilde{\ell}_1)$$

(loc.cit., Prop. VII.§6.5). □

Let $\ell^\circ \subset \ell$ be the subring of elements x with $|x| \leq 1$. Let G act trivially on the value group $|\ell^\times|$. We have the G-equivariant short exact sequence

$$1 \longrightarrow (\ell^\circ)^\times \longrightarrow \ell^\times \xrightarrow{|.|} |\ell^\times| \longrightarrow 1.$$

Proposition 2.9. *The above sequence induces an isomorphism*

$$|\ell^\times|/|k^\times| \xrightarrow{\cong} H^1(G, (\ell^\circ)^\times)$$

given by $\gamma = |x| \mapsto \psi(\gamma)$ where $\psi(\gamma)_g = \frac{gx}{x}$ for all $g \in G$.

Proof. The long exact cohomology sequence gives

$$k^\times = H^0(G,\ell^\times) \xrightarrow{|.|} H^0(G,|\ell^\times|) \xrightarrow{\delta} H^1(G,(\ell^\circ)^\times) \longrightarrow H^1(G,\ell^\times) = 1$$

where the right hand identity is *Hilbert theorem 90* (e.g. [27, Prop. X.§1.2]). The boundary map δ gives the required map. $\square$

The group G/I acts on I via conjugation since I is abelian and it acts on $\tilde{\ell}_1^\times$ via the natural action. We therefore have an induced action of G/I on the set of group homomorphisms $\mathrm{Hom}(I,\tilde{\ell}_1^\times)$ via $(g.f)(h) = gf(g^{-1}h)$ for all $h \in I$.

Lemma 2.10. *Let $\ell^{\circ\circ}$ be the kernel of the reduction map $\ell^\circ \to \tilde{\ell}_1$. There is an isomorphism*

$$|\ell^\times|/|k^\times| \xrightarrow{\cong} \mathrm{Hom}(I,\tilde{\ell}_1^\times)^{G/I} = \mathrm{Hom}(I,\tilde{\ell}_1^\times)$$

mapping $\gamma = |x|$ to $g \mapsto (\psi(\gamma)_g \bmod \ell^{\circ\circ})$.

Proof. Quite generally, if M, A are abelian G-groups where G acts trivially on M and the sets of group homomorphisms $M^* := \mathrm{Hom}(M,A)$ and $M^{**} := \mathrm{Hom}(M^*,A)$ have their induced G-actions, then the natural bidual homomorphism $M \to M^{**}$ is equivariant. This is elementary. Now let $M := |\ell^\times|/|k^\times|$ and $A := \tilde{\ell}_1^\times$ and recall the isomorphism $I \xrightarrow{\cong} (|\ell^\times|/|k^\times|)^*$ from (2.7). Since $\#|\ell^\times|/|k^\times| = e = \#I$ the bidual map $|\ell^\times|/|k^\times| \xrightarrow{\cong} (|\ell^\times|/|k^\times|)^{**}$ is bijective. By the above remark it is equivariant (and so the action on both sides is trivial). But $I \xrightarrow{\cong} (|\ell^\times|/|k^\times|)^*$ is also equivariant. Indeed, let $g_0 \in G, g \in I$ and $\gamma = |x|$. Then

$$\frac{(g_0.g)(x)}{x} = \frac{g_0 g g_0^{-1}(x)}{x} = g_0\left(\frac{g g_0^{-1}(x)}{g_0^{-1}(x)}\right)$$

and since $\gamma = |g_0^{-1}(x)|$ passing to the reduction mod $\ell^{\circ\circ}$ implies $\psi_{g_0.g}(\gamma) = g_0(\psi_g(\gamma)) = (g_0.\psi_g)(\gamma)$. The bidual map gives therefore an equivariant isomorphism

$$|\ell^\times|/|k^\times| \xrightarrow{\cong} \mathrm{Hom}(I,\tilde{\ell}_1^\times)$$

(and so the action on both sides is trivial). The definition of the map (2.7) shows that this isomorphism has the required form. $\square$

Proposition 2.11. *The reduction map $\ell^\circ \to \tilde{\ell}_1$ induces an isomorphism*

$$|\ell^\times|/|k^\times| = H^1(G,(\ell^\circ)^\times) \xrightarrow{\cong} H^1(G,\tilde{\ell}_1^\times).$$

Proof. The inflation-restriction exact sequence

$$1 \longrightarrow H^1(G/I,\tilde{\ell}_1^\times) \longrightarrow H^1(G,\tilde{\ell}_1^\times) \longrightarrow H^1(I,\tilde{\ell}_1^\times)^{G/I}$$

together with the *Hilbert theorem 90* and the preceding lemma yields an injection

$$H^1(G,\tilde{\ell}_1^\times) \hookrightarrow \mathrm{Hom}(I,\tilde{\ell}_1^\times) = |\ell^\times|/|k^\times|.$$

Precomposing this injection with the map $|\ell^\times|/|k^\times| = H^1(G,(\ell^\circ)^\times) \longrightarrow H^1(G,\tilde{\ell}_1^\times)$ gives the identity on $|\ell^\times|/|k^\times|$. This yields the assertion. $\square$

2.6. **Spectral norms and affinoid algebras.** We work in the framework of Berkovich analytic spaces [5]. However, we deal only with affinoid algebras and our methods are entirely algebraic. We let k be a nonarchimedean field which is complete with respect to a nontrivial absolute value $|.|$. Let $\Gamma = \mathbb{R}_+^\times$.

In the following, graded reductions of affinoid algebras A are always computed relatively to their spectral seminorm

$$|f| = \sup_{x \in \mathscr{M}(A)} |f|_x$$

for $f \in A$ and where $\mathscr{M}(A)$ denotes the corresponding affinoid space. If the ring A is reduced, this seminorm is a norm and induces the Banach topology on A [6, Prop. 6.2.1/4 and Thm. 6.2.4/1].

Suppose $k \subseteq \ell$ is an extension between nonarchimedean fields where the absolute value on ℓ restricts to the one on k.

Lemma 2.12. *Let A be a reduced k-affinoid algebra with its spectral norm. If the ring $\tilde{\ell}_\bullet \otimes_{\tilde{k}_\bullet} \tilde{A}_\bullet$ is reduced, then so is the ring $A_\ell := \ell \hat{\otimes}_k A$. In this case, the graded reduction of the ℓ-affinoid algebra A_ℓ equals $\tilde{\ell}_\bullet \otimes_{\tilde{k}_\bullet} \tilde{A}_\bullet$.*

Proof. Consider the tensor product norm on A_ℓ. The corresponding graded reduction equals $\tilde{\ell}_\bullet \otimes_{\tilde{k}_\bullet} \tilde{A}_\bullet$ because the extension of graded fields $\tilde{k}_\bullet \subseteq \tilde{\ell}_\bullet$ is free. Since this graded reduction is reduced, the tensor product norm must be power-multiplicative (i.e. $||f^n|| = ||f||^n$) and therefore A_ℓ must be reduced. Since the tensor product norm is furthermore a complete norm on A_ℓ, it must be equal to the spectral norm on the ℓ-affinoid algebra A_ℓ [6, Prop. 6.2.3/3]. □

Let $r \in \Gamma$ and denote by $B = k\{r^{-1}T\}$ the k-affinoid algebra of the closed disc of radius r. We write $B = k\{T\}$ in case $r = 1$. The spectral seminorm on B is a multiplicative norm given by

$$|\sum_n a_n T^n| = \sup_n |a_n| r^n.$$

The associated graded reduction is given by $\tilde{B}_\bullet = \tilde{k}_\bullet[r^{-1}T]$ according to [30, Prop. 3.1.i].

Lemma 2.13. *Let A be a reduced k-affinoid algebra such that $\tilde{\ell}_\bullet \otimes_{\tilde{k}_\bullet} \tilde{A}_\bullet$ is reduced for any complete extension field ℓ of k. Let $s \in \mathbb{R}_+$. If $f : k\{s^{-1}T\} \to A$ is a homomorphism of k-algebras whose graded reduction is an isomorphism, then f is an isomorphism.*

Proof. Since the graded reduction of f is injective, f is isometric and therefore injective. Let $B' := k\{s^{-1}T\}$. Consider any complete extension field $k \subseteq \ell$. If the induced map $f_\ell := \ell \hat{\otimes}_k f$ between $B'_\ell := \ell \hat{\otimes}_k B'$ and $A_\ell := \ell \hat{\otimes}_k A$ is surjective, then so is f. Indeed, f is strict and has therefore closed image. We thus have a strict exact sequence $B' \to A \to \operatorname{coker} f \to 0$ of k-Banach spaces. By results of L. Gruson [24, Lem. A5] we obtain an embedding $\operatorname{coker} f \hookrightarrow \ell \hat{\otimes}_k \operatorname{coker} f = \operatorname{coker} f_\ell$, which proves the claim.

There is a complete extension field $k \subseteq \ell'$ such that $B'_{\ell'}$ and $A_{\ell'}$ are both strictly ℓ'-affinoid and such that $|B'_{\ell'}| = |\ell'|$, cf. [5, 2.1]. We let ℓ be the completion of an algebraic closure of ℓ'. The graded reductions of the ℓ-affinoid algebras B'_ℓ and A_ℓ are given by $\tilde{\ell}_\bullet \otimes_{\tilde{k}_\bullet} \tilde{B'}_\bullet$ and $\tilde{\ell}_\bullet \otimes_{\tilde{k}_\bullet} \tilde{A}_\bullet$ respectively. For B' this follows from [30, Prop. 3.1(i)]. For A this follows from our hypothesis and Lem. 2.12. We see that the graded reduction of f_ℓ is the base change to $\tilde{\ell}_\bullet$ of $\tilde{f}_\bullet$ and therefore still an isomorphism. In particular, the ordinary reduction $(\tilde{f_\ell})_1$ is an isomorphism. Since $(\tilde{A_\ell})_\bullet$ and $(\tilde{B'_\ell})_\bullet$ are reduced, A_ℓ and B'_ℓ are reduced (Lem. 2.12). According to [6, Prop. 3.4.1/3] the field ℓ is algebraically closed. Hence, ℓ is a stable field and $|\ell^\times|$ is divisible (loc.cit., Prop. 3.6.2/12). We may now apply loc.cit., Cor. 6.4.2/2 to conclude from the bijectivity of $(\tilde{f_\ell})_1$ that f_ℓ is bijective. □

In the following we are interested in affinoid algebras A that allow an isomorphism

$$\beta : \ell \hat{\otimes}_k A \xrightarrow{\cong} \ell \hat{\otimes}_k B$$

of ℓ-algebras (and hence of ℓ-affinoid algebras). We denote the pointed set of isomorphism classes of such k-algebras A by $H^1(\ell/k, \operatorname{Aut} B)$. If ℓ/k is a finite Galois extension and if A is an *arbitrary* k-algebra with an isomorphism β as above, then A is k-affinoid [5, Prop. 2.1.14(ii)]. Hence our present notation is consistent with subsection 2.2.

Let us now assume that $k \subseteq \ell$ is a finite Galois extension. If $G := \operatorname{Gal}(\ell/k)$ denotes the Galois group, the isomorphism (2.3) implies $H^1(\ell/k, \operatorname{Aut} B) \simeq H^1(G, \operatorname{Aut} B_\ell)$. We therefore will focus on the Galois cohomology $H^1(G, \operatorname{Aut} B_\ell)$.

Let (A, β) be a form of B. Recall that a commutative ring of dimension 1 which is noetherian and integrally closed is called a *Dedekind domain*.

Lemma 2.14. *The ring A is a Dedekind domain.*

Proof. Any affinoid algebra is noetherian. Since β exhibits B_ℓ as a finite free A-module, A is an integral domain of dimension 1. The semilinear G-action on B_ℓ extends naturally to its fraction field $\operatorname{Frac}(B_\ell)$ with $\operatorname{Frac}(B_\ell)^G = \operatorname{Frac}(A)$. Suppose $f \in \operatorname{Frac}(A)$ is integral over A. Since B_ℓ is integrally closed, we have $f \in B_\ell \cap \operatorname{Frac}(B_\ell)^G = (B_\ell)^G = A$. □

The isomorphism β restricts to an inclusion $A \hookrightarrow B_\ell$ which is strict with respect to spectral seminorms. In particular, $\tilde{A}_\bullet \hookrightarrow (\tilde{B}_\ell)_\bullet$ and, hence, $\tilde{A}_\bullet$ is an integral domain. Since $(\tilde{B}_\ell)_\bullet$ is a graded $\tilde{\ell}_\bullet$-algebra, the inclusion extends to a homomorphism

$$\tilde{\ell}_\bullet \otimes_{\tilde{k}_\bullet} \tilde{A}_\bullet \to (\tilde{B}_\ell)_\bullet \tag{2.15}$$

of graded $\tilde{\ell}_\bullet$-algebras. It will play a central role in the following. In general, it is neither injective nor surjective (cf. example after proof of [30, Prop. 3.1]).

The maximal tamely ramified subextension

$$k \subseteq k^{\mathrm{tr}} \subseteq \ell$$

induces, according to (2.2), a short exact sequence

$$1 \longrightarrow H^1(\mathrm{Gal}(k^{\mathrm{tr}}/k), \mathrm{Aut}\, B_{k^{\mathrm{tr}}}) \longrightarrow H^1(G, \mathrm{Aut}\, B_\ell) \longrightarrow H^1(\mathrm{Gal}(\ell/k^{\mathrm{tr}}), \mathrm{Aut}\, B_\ell)$$

where the map $H^1(\mathrm{Gal}(k^{\mathrm{tr}}/k), \mathrm{Aut}\, B_{k^{\mathrm{tr}}}) \longrightarrow H^1(G, \mathrm{Aut}\, B_\ell)$ is injective. In the following we will examine the outer two terms in this sequence.

Let

$$\sqrt{|k^\times|} := \{\alpha \in \mathbb{R}_+ : \alpha^n \in |k^\times| \text{ for some } n \geq 1\}.$$

If $r \in \sqrt{|k^\times|}$, we may enlarge ℓ so that $r = |\epsilon| \in |\ell^\times|$ for some $\epsilon \in \ell^\times$ and compose β with the isomorphism $B_\ell = \ell\{r^{-1}T\} \xrightarrow{\cong} \ell\{T\}$ induced by $T \mapsto \epsilon T$. We will therefore restrict ourselves in the following to the two cases

$$r = 1 \quad \text{or} \quad r \notin \sqrt{|k^\times|}.$$

2.7. **Tamely ramified forms.** We keep all assumptions, but we suppose additionally that the finite Galois extension $k \subseteq \ell$ is *tamely ramified*. In this case we can determine all corresponding forms of B based on the following key proposition.

Proposition 2.16. *The graded homomorphism (2.15) is bijective.*

Proof. Let $F := \mathrm{Frac}_\Gamma(\tilde{A}_\bullet)$ be the graded fraction field of $\tilde{A}_\bullet$, cf. appendix. Since $k \subseteq \ell$ is tamely ramified, the finite extension $\tilde{k}_\bullet \subseteq \tilde{\ell}_\bullet$ is separable. The finite graded F-algebra $F \otimes_{\tilde{k}_\bullet} \tilde{\ell}_\bullet$ is therefore étale in the sense of [10, 1.14.4] and therefore reduced. Thus, the subring $\tilde{A}_\bullet \otimes_{\tilde{k}_\bullet} \tilde{\ell}_\bullet \hookrightarrow F \otimes_{\tilde{k}_\bullet} \tilde{\ell}_\bullet$ is reduced. Now Lem. 2.12 implies that (2.15) is isomorphic to the graded reduction of β. In particular, it is bijective. $\square$

The proposition implies a map of pointed sets

$$H^1(\ell/k, \mathrm{Aut}\, B) \longrightarrow H^1(\tilde{\ell}_\bullet/\tilde{k}_\bullet, \mathrm{Aut}^{\mathrm{gr}} \tilde{B}_\bullet), \quad (A) \mapsto (\tilde{A}_\bullet).$$

By (2.6) we may identify $G = G(\ell/k) \simeq G(\tilde{\ell}_\bullet/\tilde{k}_\bullet)$. The obvious group homomorphism $\mathrm{Aut}\ B_\ell \to \mathrm{Aut}^{\mathrm{gr}}(\tilde{B}_\ell)_\bullet$ is then G-equivariant and induces a map of pointed sets

$$H^1(G, \mathrm{Aut}\ B_\ell) \to H^1(G, \mathrm{Aut}^{\mathrm{gr}}(\tilde{B}_\ell)_\bullet).$$

The following lemma follows by unwinding the definitions of the maps involved.

Lemma 2.17. *The diagram*

$$\begin{tikzcd}
H^1(\ell/k, \mathrm{Aut}\, B) \arrow[r] \arrow[d, "\simeq"] & H^1(\tilde{\ell}_\bullet/\tilde{k}_\bullet, \mathrm{Aut}^{\mathrm{gr}} \tilde{B}_\bullet) \arrow[d, "\simeq"] \\
H^1(G, \mathrm{Aut}\ B_\ell) \arrow[r] & H^1(G, \mathrm{Aut}^{\mathrm{gr}}(\tilde{B}_\ell)_\bullet)
\end{tikzcd}$$

is commutative. Here, the vertical arrows come from (2.3) and (2.4).

An automorphism α of the graded $\tilde{\ell}_\bullet$-algebra $(\tilde{B}_\ell)_\bullet = \tilde{\ell}_\bullet[r^{-1}T]$ is, in particular, an automorphism of the underlying algebra and maps T to $aT+b$ with $a,b \in \tilde{\ell}_\bullet$. Since α preserves the grading, one has $a \in \tilde{\ell}_1^\times$ and a G-equivariant map $\mathrm{Aut}^{\mathrm{gr}}(\tilde{B}_\ell)_\bullet \mapsto \tilde{\ell}_1^\times, \alpha \mapsto a$.

Proposition 2.18. *The map induces an isomorphism* $H^1(G, \mathrm{Aut}^{\mathrm{gr}}(\tilde{B}_\ell)_\bullet) \xrightarrow{\cong} H^1(G, \tilde{\ell}_1^\times)$.

Proof. Suppose $r \notin \sqrt{|k^\times|}$. Then $b=0$ and already the map $\alpha \mapsto a$ is an isomorphism. Suppose on the contrary $r=1$. Then $\alpha \mapsto a$ induces an exact G-equivariant sequence

$$0 \longrightarrow \tilde{\ell}_1 \longrightarrow \mathrm{Aut}^{\mathrm{gr}}(\tilde{B}_\ell)_\bullet \longrightarrow \tilde{\ell}_1^\times \longrightarrow 1.$$

It is equivariantly split. Indeed, given $a \in \tilde{\ell}_1^\times$ the map induced by $T \mapsto aT$ lies in $\mathrm{Aut}^{\mathrm{gr}}(\tilde{B}_\ell)_\bullet$ and this gives the equivariant splitting $\tilde{\ell}_1^\times \to \mathrm{Aut}^{\mathrm{gr}}(\tilde{B}_\ell)_\bullet$. By (2.1) the sequence

$$H^1(G, \tilde{\ell}_1) \longrightarrow H^1(G, \mathrm{Aut}^{\mathrm{gr}}(\tilde{B}_\ell)_\bullet) \longrightarrow H^1(G, \tilde{\ell}_1^\times)$$

is exact. We have a section for the right hand arrow which is therefore surjective. It remains to see $H^1(G, \tilde{\ell}_1) = 0$. This follows from Lem. 2.8. $\square$

If $a \in \ell$ with $|a| = 1$, then $T \mapsto aT$ gives an element of Aut B_ℓ. This gives a map

$$|\ell^\times|/|k^\times| = H^1(G, (\ell^\circ)^\times) \longrightarrow H^1(G, \mathrm{Aut}\ B_\ell). \tag{2.19}$$

The first identity here is due to Prop. 2.9. Conversely, any $\alpha \in \mathrm{Aut}\ B_\ell$ preserves the spectral norm whence

$$\alpha(T) = a_0 + a_1 T + a_2 T^2 + \cdots$$

with $a_i \in \ell$ and

$$|a_0| \le r, \quad |a_1| = 1 \ \text{ and } \ |a_i| r^i < r \text{ for all } 2 \le i.$$

Mapping α to a_1 induces a map

$$H^1(G, \mathrm{Aut}\ B_\ell) \longrightarrow H^1(G, (\ell^\circ)^\times)$$

with a section given by (2.19). Let us make the forms of B corresponding to the injection

$$|\ell^\times|/|k^\times| = H^1(G, (\ell^\circ)^\times) \hookrightarrow H^1(G, \mathrm{Aut}\ B_\ell)$$

explicit. Let x be an element of $\ell^\times$. For $g \in G$ let θ_g be the automorphism of the ℓ-algebra B_ℓ whose value on T is given by $\frac{g(x)}{x}T$. The form of B corresponding to the class of $\gamma = |x|$ in $|\ell^\times|/|k^\times|$ equals, up to isomorphism, the subring $A \subseteq B_\ell$ of invariants under the semilinear G-action given by $\bar{g} = \theta_g g$, cf. subsection 2.2 and proof of Prop. 2.9. For an element $f = \sum_n a_n T^n \in B$ we compute

$$\bar{g}(f) = \sum_n g(a_n)(\theta_g(T))^n = \sum_n g(a_n) \frac{g(x^n)}{x^n} T^n$$

and thus

$$\bar{g}(f) = f \text{ for all } g \in G \iff g(a_n x^n) = a_n x^n \text{ for all } n, g \iff a_n x^n \in k \text{ for all } n.$$

The subring A is therefore given by the ring of all series

$$\sum_n b_n(x^{-1}T)^n \quad \text{where } b_n \in k \quad \text{and } |b_n|s^n \to 0 \quad \text{for } n \to \infty$$

where $s = \gamma^{-1}r$. Mapping $x^{-1}T$ to T induces an isomorphism

$$A \xrightarrow{\cong} k\{s^{-1}T\}.$$

This shows that the forms of B that correspond to the finite subset $H^1(G, (\ell^\circ)^\times)$ are given by the k-affinoid discs of radii $\gamma_i r$ where $\gamma_i \in |\ell^\times|$ runs through a system of representatives for the classes in $|\ell^\times|/|k^\times|$.

Proposition 2.20. *One has a commutative diagram of bijections of pointed sets*

$$\begin{tikzcd} H^1(G, \mathrm{Aut}\ B_\ell) \arrow[r, "\simeq"] \arrow[d, "\simeq"] & H^1(G, \mathrm{Aut}^{\mathrm{gr}}(\tilde{B}_\ell)_\bullet) \arrow[d, "\simeq"] \\ H^1(G, (\ell^\circ)^\times) \arrow[r, "\simeq"] & H^1(G, \tilde{\ell}_1^\times) \end{tikzcd}$$

where the lower horizontal arrow comes from Prop. 2.11.

Proof. The commutativity of the diagram follows by inspection. Let (A, β) be a form of B and let $c \in H^1(G, \mathrm{Aut}\ B_\ell)$ be its cohomology class. Let $\tilde{c}$ be its image in $H^1(G, \mathrm{Aut}^{\mathrm{gr}}(\tilde{B}_\ell)_\bullet)$. Since the lower horizontal arrow and the right vertical arrow in the diagram are isomorphisms, there is $x \in \ell^\times$ such that $\tilde{c} = [\tilde{\theta}]$ where

$$\tilde{\theta}_g(T) = (\frac{gx}{x} \bmod \ell^{\circ\circ})T \in (\tilde{B}_\ell)_\bullet$$

for all $g \in G$. If $\tilde{x} \in \tilde{\ell}_\bullet$ denotes the principal symbol of x, we have $(\frac{gx}{x} \bmod \ell^{\circ\circ}) = \frac{g\tilde{x}}{\tilde{x}}$. Let $\gamma = |x|$. Now $\tilde{A}_\bullet$ equals, up to a graded isomorphism, the invariant $\tilde{k}_\bullet$-subalgebra of $(\tilde{B}_\ell)_\bullet$ under the semilinear G-action defined by $\bar{g} := \tilde{\theta}_g g$. A computation as above shows that this $\tilde{k}_\bullet$-subalgebra is given by the algebra of all polynomials

$$\sum_n b_n(\tilde{x}^{-1}T)^n \quad \text{where } b_n \in \tilde{k}_\bullet.$$

It has the induced grading from $(\tilde{B}_\ell)_\bullet$, i.e. a polynomial as above is homogeneous of degree s if and only if $b_n\tilde{x}^{-n}$ is homogeneous of degree sr^{-n}, i.e. if and only if b_n is homogeneous of degree $s\gamma^n r^{-n}$. Mapping $\tilde{x}^{-1}T$ to T induces a graded isomorphism from this $\tilde{k}_\bullet$-subalgebra to $\tilde{k}_\bullet[s^{-1}T]$ where $s := \gamma^{-1}r$. We thus have an isomorphism of graded $\tilde{k}_\bullet$-algebras

$$\tilde{A}_\bullet \xrightarrow{\cong} \tilde{k}_\bullet[s^{-1}T]. \tag{2.21}$$

Let $\tilde{f}$ be the preimage of T under this isomorphism and $f \in A$ a representative. Mapping T to f induces a homomorphism $\psi : k\{s^{-1}T\} \to A$ of k-affinoid algebras whose graded reduction is an isomorphism. If ℓ' is any complete extension field of k, then $\tilde{\ell'}_\bullet \otimes_{\tilde{k}_\bullet} \tilde{A}_\bullet$ is reduced according to (2.21). Since the ring A is reduced, Lem. 2.13 implies that ψ is an isomorphism. This shows that c lies in the image of $H^1(G,(\ell^\circ)^\times) \hookrightarrow H^1(G, \mathrm{Aut}\, B_\ell)$. The left vertical arrow of our diagram is therefore an isomorphism and hence, so is any arrow in the diagram. □

We summarize this discussion in the following statement.

Theorem 2.22. *The forms of B with respect to the tamely ramified extension $k \subseteq \ell$ are given by the finitely many k-affinoid discs of radii $\gamma_i r$ where $\gamma_i \in |\ell^\times|$ runs through a system of representatives for the classes in $|\ell^\times|/|k^\times|$.*

3. Forms of Russell type and wild ramification

3.1. **The additive group.** Let k be a field of characteristic $p > 0$ and let $\mathbb{G}_a$ be the additive group over k. The endomorphism ring of $\mathbb{G}_a$ equals the noncommutative polynomial ring $k[F]$ with relations $Fa = a^p F$ for $a \in k$ where F corresponds to the Frobenius endomorphism [9, Prop. II.§3.4.4]. We let $k[F]^*$ be the multiplicative monoid consisting of separable endomorphisms, i.e. the set of all polynomials $\sum_i a_i F^i$ with $a_0 \neq 0$. For any n the left ideal $k[F]F^n$ of $k[F]$ is a two-sided ideal and we let U_n be the image of $k[F]^*$ under the projection $k[F] \to k[F]/k[F]F^n$. Then U_n is a multiplicative group.

If T denotes a parameter for $\mathbb{G}_a$, then F induces the ring homomorphism given by $T \mapsto T^p$ which we denote by F as well. Mapping

$$\tau = \sum_i a_i F^i \mapsto \tau(T)$$

yields a bijection of $k[F]$ with the set of p-polynomials $a_0 T + a_1 T^p + \cdots + a_m T^{p^m}$ in the (commutative) polynomial ring $k[T]$. The separable endomorphisms correspond to those p-polynomials with $a_0 \neq 0$.

Any endomorphism φ of k extends to an endomorphism of $k[F]$ via

$$\tau = \sum_i a_i F^i \mapsto \varphi(\tau) = \sum_i \tau(a_i) F^i.$$

If φ equals the n-th Frobenius $a \mapsto a^{p^n}$ on k we write $\tau^{(n)} := \varphi(\tau)$. In this case $k[F]^{(n)}$ denotes the image of $k[F]$ under $\tau \mapsto \tau^{(n)}$. This image is $k[F]$ if and only if k is perfect.

Let A be a Hopf algebra over k with comultiplication c and let $T = 0$ be the unit element of $\mathbb{G}_a$. The set of group homomorphisms $\mathrm{Hom}(\mathrm{Spec}(A), \mathbb{G}_a)$ is in bijection via $\psi \mapsto \psi(T)$ with the set of elements $x \in A$ that satisfy $c(x) = 1 \otimes x + x \otimes 1$.

3.2. **Graded Frobenius.** Let $\Gamma = \mathbb{R}_+^\times$. All graded rings are Γ-graded rings.

Let A be a graded ring such that A_1 has characteristic $p > 0$. Let $\rho = \rho_A$ be its grading function. Let $n \in \mathbb{Z}$. We define a new grading $\rho^{\langle n \rangle}$ on A by

$$\rho^{\langle n \rangle}(x) := \rho(x)^{1/p^n}$$

for any nonzero homogeneous $x \in A$. We denote by $A^{\langle n\rangle}$ the ring A equipped with the new grading $\rho^{\langle n\rangle}$, i.e. $A^{\langle n\rangle} = \oplus_\gamma A_\gamma^{\langle n\rangle}$ where

$$A_\gamma^{\langle n\rangle} = A_{\gamma^{p^n}}$$

for all $\gamma \in \Gamma$. Obviously $A_1^{\langle n\rangle} = A_1$.

Let k be a graded field such that k_1 has characteristic $p > 0$. Then $k^{\langle n\rangle}$ is a graded field. If A is a graded k-algebra, then $A^{\langle n\rangle}$ is a graded $k^{\langle n\rangle}$-algebra. The Frobenius map

$$\phi^n : k \to k^{\langle n\rangle}, x \mapsto x^{p^n}$$

is a graded homomorphism (of degree 1) and we may form the graded tensor product

$$\theta^{\langle n\rangle} A = k^{\langle n\rangle} \otimes_{\phi^n,k} A.$$

It is a graded $k^{\langle n\rangle}$-algebra via multiplication on the left hand factor and comes with the Frobenius homomorphism

$$F_A^{\langle n\rangle} : \theta^{\langle n\rangle} A \longrightarrow A^{\langle n\rangle},\ x \otimes a \mapsto xa^{p^n}.$$

The map $F_A^{\langle n\rangle}$ is a graded homomorphism (of degree 1) of graded $k^{\langle n\rangle}$-algebras.

We briefly discuss the relation between the functor $\theta^{\langle n\rangle}$ and base change. Let $\bar{k}$ be a graded algebraic closure of k, cf. appendix. Suppose that $k \subseteq \ell \subseteq \bar{k}$ is a graded extension field which is purely inseparable over k. If $\ell^{p^n} \subseteq k$ for some n, then associativity of the graded tensor product gives

$$\theta^{\langle n\rangle} A \simeq k^{\langle n\rangle} \otimes_{\bar{\phi}^n,\ell} A_\ell \tag{3.1}$$

with the graded map $\bar{\phi}^n : \ell \to k^{\langle n\rangle}, x \mapsto x^{p^n}$. Let

$$k^{p^{-n}} := \{x \in \bar{k} : x^{p^n} \in k\},$$

a graded subfield of $\bar{k}$ and a purely inseparable extension of k. If $k^{p^{-n}} \subseteq \ell$, we have in turn

$$\ell \otimes_{\psi^n,k^{\langle n\rangle}} \theta^{\langle n\rangle} A = \ell \otimes_{\psi^n,k^{\langle n\rangle}} (k^{\langle n\rangle} \otimes_{\bar{\phi}^n,\ell} A_\ell) \simeq A_\ell \tag{3.2}$$

where $\psi^n : k^{\langle n\rangle} \to \ell, x \mapsto x^{1/p^n}$.

3.3. **Graded forms of Russell type.** We keep the assumptions. Let A be a graded k-algebra. We call A a *graded Hopf algebra* if A is a Hopf algebra relative to the ring k in the usual sense [31] and if the relevant maps -comultiplication, inverse, augmentation- are graded homomorphisms (of degree 1). In this case $\theta^{\langle n\rangle} A$ and $A^{\langle n\rangle}$ are graded Hopf algebras for all $n \in \mathbb{Z}$. One verifies that $F_A^{\langle n\rangle}$ respects these structures.

Example: Let $r \in \Gamma$ and consider $B = k[r^{-1}T]$. The Hopf algebra structure on B coming from $\mathbb{G}_a$ with origin $T = 0$ is given by

$$c(T) = 1 \otimes T + T \otimes 1,\ \ i(T) = -T \ \text{ and } \ \epsilon(T) = 0$$

and is therefore graded. Conversely, any structure of graded Hopf algebra on B arises this way. Indeed, the discussion before [25, Lemma 1.2] carries over to the graded setting and hence, if t is a homogeneous generator of the kernel of the augmentation, then necessarily $c(t) = 1 \otimes t + t \otimes 1$ and $i(t) = -t$. The canonical isomorphism

$$\theta^{\langle n \rangle}(k[r^{-1}T]) \xrightarrow{\cong} k^{\langle n \rangle}[r^{-1}T], \quad x \otimes a \mapsto xa$$

is graded and transforms the Frobenius $F_B^{\langle n \rangle}$ into the graded homomorphism

$$k^{\langle n \rangle}[r^{-1}T] \longrightarrow (k[r^{-1}T])^{\langle n \rangle} = k^{\langle n \rangle}[r^{-1/p^n}T], \quad T \mapsto T^{p^n}. \tag{3.3}$$

Let $r \in \Gamma$. A p-polynomial $f(T) = a_0T + a_1T^p + \cdots + a_mT^{p^m} \in k[r^{-1}T]$ is homogeneous of degree $\gamma \in \Gamma$ if and only if $a_i \in k$ is homogeneous of degree γr^{-p^i}.

Let $n \geq 1$ and let ℓ be the graded field

$$\ell := k^{p^{-n}} = \{a \in \bar{k} : a^{p^n} \in k\}.$$

It is a purely inseparable (possibly infinite) extension of k. Let $r \in \Gamma$ and let A be a graded form of $B := k[r^{-1}T]$ with respect to ℓ/k that is also a graded Hopf algebra. We start with the following general construction. Let $s \in \Gamma$ and choose

$$f(T_1) = a_0T_1 + a_1T^p + \cdots + a_mT^{p^m} \in k[r^{-p^n}T_1],$$

a homogeneous p-polynomial of degree s^{p^n} with $a_0 \neq 0$. The graded quotient

$$A := k[r^{-p^n}T_1, s^{-1}T_2]/(T_2^{p^n} - f(T_1))$$

inherits a Hopf structure from $k[r^{-p^n}T_1, s^{-1}T_2]$ coming from the 2-dimensional additive group of with unit element $T_1 = T_2 = 0$. Since $a_0 \neq 0$ the tensor product $\ell \otimes_k A$ is reduced.

For any polynomial $h \in k[T]$ we let h^{ϕ^n} be the polynomial that comes from h by applying ϕ^n to its coefficients. Now let x respectively y be the residue class of T_1 respectively T_2 in the quotient A. We then have

$$1 \otimes a_0x = 1 \otimes y^{p^n} - 1 \otimes a_mx^{p^m} - \ldots - 1 \otimes a_1x^p = (1 \otimes y^{p^{n-1}} - (a_m^{p^{n-1}} \otimes x^{p^{m-1}} + \ldots + a_1^{p^{n-1}} \otimes x))^p =: t_1^p$$

in $\theta^{\langle n \rangle}A$ with a homogeneous element t_1 of degree $s^{p^{n-1}}$. Let $h := a_0^{-1}f$ and $b_i := a_0^{-1}a_i$. Then

$$1 \otimes y^{p^n} = 1 \otimes f(x) = f^{\phi^n}(1 \otimes x) = h^{\phi^n}(1 \otimes a_0x) = h^{\phi^n}(t_1^p) = h^{\phi^{n-1}}(t_1)^p$$

in $\theta^{\langle n \rangle}A$. Since $\theta^{\langle n \rangle}A$ is reduced, this implies

$$1 \otimes y^{p^{n-1}} = h^{\phi^{n-1}}(t_1) \tag{3.4}$$

in $\theta^{\langle n \rangle}A$ viewing the coefficients of $h^{\phi^{n-1}}$ as elements of $k^{\langle n \rangle}$. Writing this out yields

$$t_1 = 1 \otimes y^{p^{n-1}} - b_m^{p^{n-1}}t_1^{p^m} - \ldots - b_1^{p^{n-1}}t_1^p = (1 \otimes y^{p^{n-2}} - (b_m^{p^{n-2}}t_1^{p^{m-1}} + \ldots + b_1^{p^{n-2}}t_1))^p =: t_2^p$$

in $\theta^{\langle n\rangle}A$ with a homogeneous element t_2 of degree $s^{p^{n-2}}$. Now

$$1\otimes y^{p^{n-1}} = h^{\phi^{n-1}}(t_1) = h^{\phi^{n-1}}(t_2^p) = h^{\phi^{n-2}}(t_2)^p$$

implies $1\otimes y^{p^{n-2}} = h^{\phi^{n-2}}(t_2)$. Comparing with (3.4) we continue in this way and eventually find a homogeneous element $t := t_n \in \theta^{\langle n\rangle}A$ of degree s such that $1\otimes y = h(t)$ and $1\otimes a_0 x = t^{p^n}$. Since $\theta^{\langle n\rangle}A$ is generated over $k^{\langle n\rangle}$ by $1\otimes y$ and $1\otimes a_0x$ the inclusion $k^{\langle n\rangle}[t]\subseteq \theta^{\langle n\rangle}A$ is surjective. Now t is transcendental over $k^{\langle n\rangle}$ and there is an isomorphism of graded ℓ-algebras

$$A_\ell \xrightarrow{\cong} \ell[s^{-1}T],\ \ t\mapsto T \tag{3.5}$$

according to (3.2). In particular, $A\otimes_k A$ is reduced. Since $c(x) = 1\otimes x + x\otimes 1$ we may deduce from $1\otimes a_0 x = t^{p^n}$ that $\bar c(t) = 1\otimes t + t\otimes 1$ where $\bar c$ denotes the comultiplication in $\theta^{\langle n\rangle}A$ induced from A. It follows that the above isomorphism is a Hopf isomorphism when the target has its graded Hopf structure coming from the additive group with unit element $T=0$.

For the choice $s\in\rho(\ell^\times)r$ in Γ this yields graded Hopf algebras A over k which are forms of B with respect to ℓ/k.

On the other hand, let A be a graded form of $B := k[r^{-1}T]$ with respect to ℓ/k that is also a graded Hopf algebra. Let c be the comultiplication of A and let $\bar c$ be the induced comultiplication on the base change $\theta^{\langle n\rangle}A$. According to (3.1) we have an isomorphism of $k^{\langle n\rangle}$-algebras

$$\theta^{\langle n\rangle}A \xrightarrow{\cong} k^{\langle n\rangle}[r^{-1}T] \tag{3.6}$$

which we use as an identification. Via transport of structure we view the right hand side as a graded Hopf algebra over $k^{\langle n\rangle}$. If t denotes a homogeneous generator of the kernel of the augmentation, then $t = T-a$ with $a\in (k^{\langle n\rangle})_r$. By replacing T with $T-a$ we may therefore assume that $\bar c(T) = 1\otimes T + T\otimes 1$. Using (3.6) as an identification, we may write

$$T = \sum_i a_i\otimes y_i \tag{3.7}$$

with $a_i\in k^{\langle n\rangle}$ and $y_i\in A$ where each $a_i\otimes y_i$ is homogeneous of degree r. We may assume that the a_i are a linearly independent family of homogeneous elements in the graded k-vector space $k^{\langle n\rangle}$. Then each $y_i\in A$ is homogeneous with degree, say, r_i. Moreover,

$$\sum_i a_i\otimes c(y_i) = \bar c(\sum_i a_i\otimes y_i) = \bar c(T) = 1\otimes T + T\otimes 1 = \sum_i a_i\otimes y_i\otimes 1 + a_i\otimes 1\otimes y_i$$

implies

$$c(y_i) = 1\otimes y_i + y_i\otimes 1$$

for all i. By the discussion in subsection 3.1, $1\otimes y_i$ equals under the identification (3.6) a p-polynomial $f_i\in k^{\langle n\rangle}[r^{-1}T]$ which is homogeneous of degree r_i.

Consider now the graded subring $k[y_1,...,y_n] \subseteq A$ generated by the y_i. By (3.7) the inclusion map becomes surjective after the faithfully flat base change $\phi^n : k \to k^{\langle n\rangle}$ and therefore $k[y_1,...,y_n] = A$. Hence the graded fraction field $F := \mathrm{Frac}_\Gamma(A)$ of A is generated over k by $y_1,...,y_n$. Since $\ell \otimes_k F = \mathrm{Frac}_\Gamma(\ell[r^{-1}T])$ it follows that $\mathrm{trdeg}_k(F) = 1$ (Lem. A.6) and there is an element, say y, among the homogeneous elements y_i which forms a separating transcendence basis of F/k (Lem. A.5). Let s be the degree of $y \in A$. By our above discussion we have

$$1 \otimes y = f(T) \in \theta^{\langle n\rangle} A = k^{\langle n\rangle}[r^{-1}T]$$

with a p-polynomial homogeneous of degree s

$$f(T) = a_0 T + a_1 T^p + \cdots a_m T^{p^m} \in k^{\langle n\rangle}[r^{-1}T]$$

with $a_m \neq 0$. Applying the graded homomorphism $F_A^{\langle n\rangle}$ to $T \in k^{\langle n\rangle}[r^{-1}T] = \theta^{\langle n\rangle}A$ yields a homogeneous element $x := F_A^{\langle n\rangle}(T)$ of degree r in $A^{\langle n\rangle}$ that satisfies

$$y^{p^n} = F_A^{\langle n\rangle}(1 \otimes y) = F_A^{\langle n\rangle}(f(T)) = f(F_A^{\langle n\rangle}(T)) = f(x)$$

in $A^{\langle n\rangle}$ and $c(x) = 1 \otimes x + x \otimes 1$.

By choice of y the extension of graded fields $F/k(y)$ is separable and so is the extension $F/k(x,y)$. But for all i we have in $A^{\langle n\rangle}$ that $y_i^{p^n} = f_i(x)$. We read this equation in A which implies that y_i is purely inseparable over $k(x)$. Since $F = k(y_1,...,y_n)$ this means that $F/k(x,y)$ is purely inseparable. Consequently, $F = k(x,y)$. We also see that $A = k[y_1,...,y_n]$ is a graded integral extension of $k[x,y]$.

On the other hand, $\theta^{\langle n\rangle}$ preserves graded direct sums and so the degree $[F : k(y)]$ equals the degree of the free graded $\theta^{\langle n\rangle}k(y)$-module $\theta^{\langle n\rangle}F$. Since this module equals $\mathrm{Frac}_\Gamma(k^{\langle n\rangle}[r^{-1}T])$ by (3.6), this degree equals the monomial degree in T of the element $1 \otimes y = f(T)$ of $\theta^{\langle n\rangle}A = k^{\langle n\rangle}[r^{-1}T]$ which is p^m. Since $F = k(x,y)$, the homogeneous polynomial

$$h(T) := -y^{p^n} + f(T) = -y^{p^n} + a_0T + a_1T^p + \cdots + a_mT^{p^m} \in k(y)[r^{-p^n}T]$$

of degree s^{p^n}, which annihilates x and has monomial degree p^m, must be irreducible. The separability of x over $k(y)$ implies then that $a_0 \neq 0$ ([10, 1.14.1]). Since $a_i \in k$ and y is transcendental over k the graded Gauss lemma (cf. proof of Lem. A.3) shows that $h(T)$ is also irreducible in $k[y][r^{-p^n}T]$. We thus have an isomorphism of graded Hopf k-algebras

$$k[r^{-p^n}T_1, s^{-1}T_2]/(T_2^{p^n} - f(T_1)) \xrightarrow{\cong} k[x,y] \tag{3.8}$$

induced by $T_1 \mapsto x, T_2 \mapsto y$ where the Hopf structure on the source comes from the 2-dimensional additive group with unit element $T_1 = T_2 = 0$. At this point we need a lemma.

Lemma 3.9. *The graded domain $k[x,y]$ is graded integrally closed.*

Proof. We first show that the graded local ring $(k[x,y])_Q$ for any $Q \in \mathrm{Spec}_\Gamma(k[x,y])$ is integrally closed. Since this graded local ring is graded noetherian it suffices, according to Lem. A.2, to check that its maximal homogeneous ideal is generated by a non-nilpotent homogeneous element. To do this let $R := k[y]$ and consider the graded ring extension

$$R \to R[r^{-p^n}T]/(h) =: S.$$

Of course, $S \simeq k[x,y]$. The formal derivative of the polynomial h is the unit $h' = a_0 \neq 0$ in $R[r^{-p^n}T]$. The usual argument proving that so-called standard-étale extensions are étale and therefore unramified (e.g. [14, Thm. 4.2]) carries over to the graded setting: for the prime homogeneous ideal $P = Q \cap R$ of R with graded residue field $\kappa(P) = R_P/PR_P$ the graded $\kappa(P)$-algebra $\kappa(P) \otimes_R S_Q$ equals a finite product of separable finite graded field extensions of $\kappa(P)$ and, at the same time, is a graded local ring because of $PS_Q \subseteq QS_Q$. It follows that S_Q/PS_Q is a finite and separable graded field extension of $\kappa(P)$ and, in particular, P generates the maximal homogeneous ideal of S_Q. Since R, being isomorphic to $k[r^{-1}T]$ via $y \mapsto T$, is a graded principal ideal domain, its local graded rings are graded discrete valuation rings and so P is generated by a homogeneous non-nilpotent element (Lem. A.2). Thus, S_Q is integrally closed for any Q and therefore $S \simeq k[x,y]$ is integrally closed, cf. last sentence in A.2. This proves the lemma. $\square$

Since A is integral over $k[x,y]$ the lemma together with $F = k(x,y)$ implies that $A = k[x,y]$. The isomorphism (3.8) allows us to apply to A our previous discussion which yields an isomorphism $A_\ell \xrightarrow{\cong} \ell[s^{-1}T]$, cf. (3.5). In view of (3.6) we conclude $s \in \rho(\ell^\times)r$ in Γ. We may summarize the whole discussion in the following theorem.

Theorem 3.10. *Let $\ell = k^{p^{-n}}$ and $B := k[r^{-1}T]$. Let $s \in \rho(\ell^\times)r$ in Γ. Let*

$$f(T_1) = a_0T_1 + a_1T^p + \cdots + a_mT^{p^m} \in k[r^{-p^n}T_1]$$

be a p-polynomial homogeneous of degree s^{p^n} with $a_0 \neq 0$. The graded Hopf k-algebra

$$A := k[r^{-p^n}T_1, s^{-1}T_2]/(T_2^{p^n} - f(T_1))$$

is a graded form of B with respect to ℓ/k. Conversely, any graded Hopf k-algebra, which is a form of B with respect to ℓ/k, is of this type.

Remark: In [25] P. Russell determines all forms of the additive group over a field of positive characteristic. Our above theorem is a version for graded fields of loc.cit., Thm. 2.1 and its proof. Following [12, Part I.§2] we tentatively call the graded forms appearing in the above theorem *of Russell type*. It is easy to see that a graded version of [25, Lem. 1.1.i] holds: any A as above admits a trivialization $A_{\ell'} \xrightarrow{\cong} B_{\ell'}$ which is defined over a *finite* subextension ℓ'/k inside ℓ.

To determine how many graded forms of Russell type exist, one needs to know when such a form is trivial and when two such forms are isomorphic. In the ungraded case there is a very precise answer to these two questions (loc.cit., Cor. 2.3.1 and Thm. 2.5). In the presence of gradings the arguments of loc.cit. that lead to this answer seem to allow no direct generalization. For example, the Frobenius map $T \mapsto T^p$ is *not* a graded degree 1 endomorphism of B if $r \neq 1$. Hence, at this point we will say something only in case $r = 1$ and leave the general case for future work.

In case $r = 1$ the grading ρ_A of a form A of Russell type has image in $\rho(k^\times)$. Hence the graded algebra A is *induced* from A_1 in the sense of graded Frobenius reciprocity [22, Thm. 2.5.5]. In particular, the whole situation is governed by the homogeneous parts of degree 1. To be more precise, any p-polynomial $f(T) \in k[T]$ homogeneous of degree 1 lies in $k_1[T]$ and uniquely corresponds to an endomorphism of $\mathbb{G}_{a,k_1}$, the additive group over k_1 (subsection 3.1). Moreover, $\ell_1 = (k_1)^{p^{-n}}$. Recall that $k_1[F]^{(n)}$ equals the image of $k_1[F]$ under the map $\tau \mapsto \tau^{(n)}$. Let $B = k[T]$ as graded k-algebra (i.e. $r = 1$).

Corollary 3.11. *Let $f(T_1) = a_0T_1 + a_1T^p + \cdots + a_mT^{p^m} \in k[T_1]$ be a p-polynomial homogeneous of degree 1 with $a_0 \neq 0$ and let τ be its endomorphism of $\mathbb{G}_{a,k_1}$. The graded Hopf k-algebra*

$$A(\tau) := k[T_1, T_2]/(T_2^{p^n} - f(T_1))$$

is a graded form of B with respect to the extension $\ell = k^{p^{-n}}$ of k. The form $A(\tau)$ is trivial if and only if $\tau c \in k_1[F]^{(n)}$ for some $c \in k_1^\times$. In particular, k_1 is perfect if and only if all $A(\tau)$ are trivial.

Proof. Passing to the homogeneous part of degree 1 induces a map from the set of isomorphism classes of the $A(\tau)$ to the set of forms of $\mathbb{G}_{a,k_1}$ with respect to the extension of characteristic p-fields $k_1 \subseteq \ell_1$. The map is surjective by [25, Thm 2.5] and injective by graded Frobenius reciprocity, cf. Lem. A.1. Then [25, Cor. 2.3.1] gives the first statement. If k_1 is perfect, then $k_1[F] = k_1[F]^{(n)}$. If k_1 is not perfect, there is a p-polynomial whose coefficient a_1 at T^p is not a p-power in k_1. If τ is the corresponding endomorphism, then $\tau c \notin k_1[F]^{(1)}$ for any $c \in k_1^\times$ and hence $A(\tau)$ is nontrivial. $\square$

Recall that $H^1(\ell/k, \mathrm{Aut}^{\mathrm{gr}} B)$ equals the pointed set of graded forms of B with respect to ℓ/k. Recall that the group U_n equals the image of $k[F]^*$ under the projection $k[F] \to k[F]/k[F]F^n$. Let

$$G_n := U_n \times k_1^\times$$

be the direct product of the multiplicative groups U_n and $k_1^\times$. There is an action of G_n on the pointed set U_n given by

$$(\bar{\sigma}, c).\bar{\tau} := (\sigma^{(n)}\tau c^{-1}) \bmod (F^n).$$

Here, $\bar{\sigma} = \sigma \bmod (F^n)$ is the class of $\sigma \in k[F]^*$ in $U_n \subseteq k[F]/k[F]F^n$ and similarly for τ. Moreover, $\sigma^{(n)}$ equals the image of σ under the extension of ϕ^n to $k[F]$. Let U_n/G_n be the pointed set of orbits corresponding to this action.

Example: In loc.cit., p. 534, there is the following description of the set U_n/G_n in the simplest case $n = 1$. Let W_0 be a complementary $\mathbb{F}_p$-subspace to $k_1^p \subseteq k_1$ (in case $p \neq 1$) and for each $i \geq 1$ let W_i be a copy of W_0. Then $U_1 = k_1^\times$ acts linearly on the space

$W = \oplus_{i\geq 1} W_i$ by $c.(\sum_i a_i) := \sum_i c^{p(1-p^i)} a_i$. Mapping $\sum_i a_i$ to $1 + \sum_i a_i F^i \in k[F]^*$ induces an isomorphism of pointed sets

$$W/k^\times \xrightarrow{\cong} U_1/G_1.$$

Back in the case of general $n \geq 1$, the proof of the preceding corollary and loc.cit., Thm. 2.5 imply the following corollary.

Corollary 3.12. *Let $\ell = k^{p^{-n}}$ and $r = 1$. The map*

$$k_1[F]^* \to H^1(\ell/k, \mathrm{Aut}^{\mathrm{gr}} B), \quad \tau \mapsto \text{isomorphism class of } A(\tau)$$

induces an inclusion between pointed sets

$$U_n/G_n \hookrightarrow H^1(\ell/k, \mathrm{Aut}^{\mathrm{gr}} B).$$

3.4. **A class of wildly ramified forms.** We let $\Gamma = \mathbb{R}_+^\times$ and let k be a nonarchimedean field which is complete with respect to a nontrivial absolute value $|.|$ and whose residue field $\tilde{k}_1$ has characteristic $p > 0$.

Let $n \geq 1$. Let $k \subseteq \ell$ be a complete field extension whose absolute value restricts to the one on k and which has the property $(\tilde{k}_\bullet)^{p^{-n}} \subseteq \tilde{\ell}_\bullet$.

Let $r \in \Gamma$ and $B := k\{r^{-1}T\}$. Let $s \in \Gamma$ and $\tilde{f}(T_1) = a_0 T_1 + a_1 T^p + ... + a_m T^{p^m} \in \tilde{k}_\bullet[r^{-p^n} T_1]$ be a homogeneous p-polynomial with $a_0 \neq 0$ of degree s^{p^n}. Let $\tilde{A}_\bullet$ be the form of $\tilde{B}_\bullet = \tilde{k}_\bullet[r^{-1}T]$ corresponding to these data via the theorem above. Let $f(T_1)$ be a homogeneous lift of $\tilde{f}$ in $k\{r^{-p^n} T_1\}$ and consider the affinoid k-algebra

$$A := k\{r^{-p^n} T_1, s^{-1} T_2\}/(T_2^{p^n} - f(T_1)).$$

The quotient norm on A gives a filtration whose graded ring equals $\tilde{A}_\bullet$ [6, Cor. 1.1.9/6]. Since $\tilde{A}_\bullet$ is reduced, the quotient norm is power-multiplicative and, thus, equals the spectral seminorm (loc.cit., Prop. 6.2.3/3). Hence, this seminorm is a norm and A is reduced. Also, $\tilde{A}_\bullet$ is the graded reduction of the affinoid algebra A. Since $\tilde{A}_\bullet$ is a form of $\tilde{B}_\bullet$ the graded reduction of $A_\ell = \ell \hat{\otimes}_k A$ equals $\tilde{\ell}_\bullet \otimes_{\tilde{k}_\bullet} \tilde{A}_\bullet \simeq \tilde{\ell}_\bullet[r^{-1}T]$ according to Lem. 2.12. Let $\tilde{f}$ be the preimage of T under the latter isomorphism and f a homogeneous lift in A_ℓ. Mapping T to f induces a homomorphism $\psi : \ell\{r^{-1}T\} \to A_\ell$ of ℓ-affinoid algebras whose graded reduction is an isomorphism. If ℓ' is any complete extension field of k containing ℓ then $\tilde{\ell'}_\bullet \otimes_{\tilde{k}_\bullet} \tilde{A}_\bullet$ is reduced since $\tilde{A}_\bullet$ is a form of $\tilde{B}_\bullet$. Lem. 2.13 yields now that ψ is an isomorphism and so A is a form of B with respect to $k \subseteq \ell$. We summarize:

Proposition 3.13. *Any form of $\tilde{k}_\bullet[r^{-1}T]$ of Russell type with respect to $\tilde{k}_\bullet \subseteq \tilde{\ell}_\bullet$ lifts to a wildly ramified form of $k\{r^{-1}T\}$ with respect to $k \subseteq \ell$ and this map is injective on isomorphism classes of forms.*

Let $r = 1$. Consider the pointed set of orbits U_n/G_n from the preceding subsection. Composing the inclusion of the preceding proposition with the one from Cor. 3.12 yields the following corollary.

Corollary 3.14. *In case $r = 1$, there is a canonical inclusion of pointed sets*

$$U_n/G_n \hookrightarrow H^1(\ell/k, \mathrm{Aut}\, B).$$

4. Picard groups and p-radical descent

All rings are commutative and unital. All ring homomorphisms are unital.

4.1. **Krull domains.** We recall some divisor theory of Krull domains [7, VII.1] thereby fixing some notation. Let A be an integral domain with quotient field K. The ring A is called a *Krull domain* if there exists a family of discrete valuations $(v_i)_{i\in I}$ on K such that A equals the intersection of the valuation rings of the v_i and such that for $x \neq 0$ almost all $v_i(x)$ vanish. A noetherian Krull domain of dimension 1 is the same as a Dedekind domain (loc.cit., Cor. VII.1.3).

Let A be a Krull domain and $P(A)$ the set of height 1 prime ideals of A. The free abelian group $D(A)$ on $P(A)$ is called the *divisor group* of A. Given $P \in P(A)$ the localization A_P of A is a discrete valuation ring. Let v_P be the associated valuation. The *divisor map*

$$\mathrm{div}_A : K^\times \longrightarrow D(A)\ ,\ x \mapsto \sum_{P\in P(A)} v_P(x)P$$

is then a well-defined group homomorphism giving rise to the *divisor class group*

$$\mathrm{Cl}(A) := D(A)/\mathrm{Im}\ \mathrm{div}_A.$$

If $A' \subseteq A$ is a subring which is a Krull domain itself and if $P' \in P(A'),\ P \in P(A)$ are height 1 prime ideals with $P \cap A' = P'$ let $e(P/P') \in \mathbb{N}$ denote the ramification index of P over P'. Then $j(P') := \sum e(P/P')P$ induces a well-defined group homomorphism

$$j : D(A') \to D(A)$$

where the sum runs through all $P \in P(A)$ with $P \cap A' = P'$. If $A' \subseteq A$ is an *integral* ring extension, then j factors through a group homomorphism

$$\bar{\jmath} : \mathrm{Cl}(A') \to \mathrm{Cl}(A)$$

[7, Prop. VII.§1.14].

4.2. **p-radical descent.** Let A be for a moment an arbitrary commutative ring and $m \geq 0$ a natural number. Recall that a *higher derivation* ∂ of *rank* m on A is an ordered tuple

$$\partial = (\partial_0, ..., \partial_m)$$

of additive endomorphisms ∂_k of A satisfying the convolution formula

$$\partial_k(ab) = \sum_{j=0,\dots,k} \partial_j(a)\partial_{k-j}(b) \tag{4.0}$$

for all $a, b \in A$ and such that $\partial_0 = \mathrm{id}_A$ [11, IV.9]. Let

$$A[T]_m := A[T]/(T^{m+1})$$

be the m-truncated polynomial ring over A with its augmentation map

$$\epsilon : A[T]_m \longrightarrow A,\ \ f \mapsto f \bmod (T).$$

The higher derivation ∂ induces an injective ring homomorphism

$$A \to A[T]_m,\ \ a \mapsto \sum_{j=0,\ldots,m} \partial_j(a)T^j$$

which we again denote by ∂. Obviously, $\epsilon \circ \partial = \mathrm{id}_A$. Conversely, any ring homomorphism $\partial : A \to A[T]_m$ with the property $\epsilon \circ \partial = \mathrm{id}_A$ induces in an obvious way a higher derivation of rank m on A. An element $a \in A$ such that $\partial(a) = a$ is called a *constant*. The constants form a subring of A. We say that ∂ is *nontrivial* if there exists an element in A which is not a constant. In this case, we define the *order* $\mu(\partial)$ of ∂ as

$$\mu(\partial) := \min\{1 \leq j \leq m \mid \partial_j \neq 0\}.$$

We assume in the following that the ring A has characteristic $p > 0$. In this case we introduce for a nontrivial ∂ the *exponent* $n(\partial)$ as

$$n(\partial) := \min\{n \mid m < \mu(\partial) \cdot p^n\}.$$

Note that $n(\partial) \geq 1$.

Lemma 4.1. *Let ∂ be nontrivial. Any power $a^{p^{n(\partial)}}$ with $a \in A$ is a constant.*

Proof. Suppose $\partial_j(a) \neq 0$ for some $j \geq 1$. Then $\mu(\partial) \leq j$ and hence $m < j \cdot p^{n(\partial)}$. It follows that $(\partial_j(a)T^j)^{p^{n(\partial)}} = 0$ in $A[T]_m$. Since ∂ is a homomorphism and $A[T]_m$ has characteristic p we obtain

$$\partial(a^{p^{n(\partial)}}) = \partial(a)^{p^{n(\partial)}} = \sum_{j=0,\ldots,m} (\partial_j(a)T^j)^{p^{n(\partial)}} = a^{p^{n(\partial)}}.$$

$\square$

If $C \subseteq A$ is a subring with $\partial_j(C) \subseteq C$ for all $j = 0, .., m$ the ring C is called *invariant*. We obtain in this case a higher derivation on C of the same rank.

In [3] K. Baba uses the notion of higher derivation to build up a p-radical descent theory of higher exponent. We recall that the case of exponent one using ordinary derivations is classical and due to P. Samuel [26]. We shall only need a special case of Baba's theory as follows. Consider a Krull domain A of characteristic $p > 0$ with quotient field K and a nontrivial higher derivation ∂ of rank m on K such that the subring $A \subseteq K$ is invariant. Let $K' \subseteq K$ be the field of constants and let $A' := A \cap K'$. Then K' is the quotient field of A'. If $\{v_i\}_{i \in I}$ is a family of discrete valuations on K exhibiting A as a Krull domain, their restrictions to K' prove A' to be a Krull domain.
By the previous lemma $A' \subseteq A$ is an integral ring extension. We therefore have the canonical homomorphisms $j : D(A') \to D(A)$ and $\bar{\jmath} : \mathrm{Cl}(A') \to \mathrm{Cl}(A)$. On the other

hand, we have inside the group of units $(K[T]_m)^\times$ the subgroup of so-called *logarithmic derivatives*

$$\mathcal{L} := \{\frac{\partial(z)}{z} \,|\, z \in K^\times\}$$

as well as its subgroups

$$(4.1)\qquad \begin{aligned} \mathcal{L}_A &:= \ \mathcal{L} \cap (A[T]_m)^\times, \\ \mathcal{L}'_A &:= \ \{\tfrac{\partial(u)}{u} \,|\, u \in A^\times\} \end{aligned}$$

with $\mathcal{L}'_A \subseteq \mathcal{L}_A$.

Lemma 4.2. *The abelian group $\mathcal{L}$ has exponent $p^{n(\partial)}$ and one has*

$$\mathcal{L} = \{\frac{\partial(z)}{z} \,|\, z \in A - \{0\}\}.$$

Proof. If $z \in K^\times$ then $\partial(z)^{p^{n(\partial)}} = \partial(z^{p^{n(\partial)}}) = z^{p^{n(\partial)}}$ by the preceding lemma. This proves the first claim. Now consider $\frac{\partial(z)}{z}$ with $z \in K^\times$. Write $z = f/g$ with nonzero $f, g \in A$ and let $m := p^{n(\partial)} - 1$. As we have just seen $(\partial(g)/g)^m = (\partial(g)/g)^{-1}$ in $(K[T]_m)^\times$ and therefore

$$\partial(fg^m)/(fg^m) = (\partial(f)/f) \cdot (\partial(g)/g)^m = \partial(f/g)/(f/g).$$

Since $fg^m \in A - \{0\}$ this yields the assertion. $\square$

The fundamental construction in [3], (1.5) produces an injective homomorphism

$$\Phi_A : \ker \bar{\jmath} \longrightarrow \mathcal{L}_A/\mathcal{L}'_A, \quad D \bmod \operatorname{div}_{A'}(K'^\times) \mapsto \frac{\partial(x_D)}{x_D} \bmod \mathcal{L}'_A$$

where $x_D \in K^\times$ is chosen such that

$$\operatorname{div}_A(x_D) = j(D).$$

We consider the following condition on the tuple (A, ∂):

$$(\heartsuit) \quad [K : K'] = p^{n(\partial)} \text{ and } \partial_{\mu(\partial)}(a) \in A^\times \text{ for some } a \in A.$$

Proposition 4.3. *If $(\heartsuit)$ holds, then Φ_A is an isomorphism.*

Proof. This is a special case of loc.cit., Thm. 1.6. Note that we have $r = 1$ in the notation of loc.cit. $\square$

Thus, if $(\heartsuit)$ holds and the class group of A vanishes, then the class group of the ring of constants A' has exponent $p^{n(\partial)}$ and admits a fairly explicit description in terms of logarithmic derivatives in the ring $A[T]_m$. This will be our main application.

We shall need some information how the property $(\heartsuit)$ behaves when the ring of constants varies. To this end we consider an injective homomorphism $A' \to B$ into an integrally closed domain B such that $A_B := B \otimes_{A'} A$ is an integral domain. Let $L' = \mathrm{Frac}(B)$ be the fraction field of B. Since A_B is integral over B, we then have

$$L := \mathrm{Frac}(A_B) = L' \otimes_{K'} K$$

for the fraction field of A_B [20, Lem. §9.1]. Since B is integrally closed, one has $L' \cap A_B = B$.

Lemma 4.4. *Suppose that A is flat as a module over A' and that the A'-submodule $\mathrm{Im}\,(\partial_0 - \partial) \subseteq A[T]_m$ associated with the homomorphism $\partial : A \to A[T]_m$ is flat. The field L admits a higher derivation*

$$\partial_L := \mathrm{id}_{L'} \otimes_{K'} \partial$$

of the same rank and order as ∂ such that the subring A_B is invariant. The field of constants with respect to ∂_L equals L'.

Proof. The free A-module $A[T]_m$ is flat as an A'-module whence $\mathrm{Tor}_1^{A'}(B, A[T]_m) = 0$. So tensoring the short exact sequence

$$0 \longrightarrow (T^{m+1}) \longrightarrow A[T] \longrightarrow A[T]_m \longrightarrow 0$$

over A' with B shows that $B \otimes_{A'} A[T]_m \simeq (A_B)[T]_m$. We therefore obtain a higher derivation $A_B \to (A_B)[T]_m$ by $\partial_B := \mathrm{id}_B \otimes_{A'} \partial$ of rank m. According to formula (4.0) all ∂_k are A'-linear whence all $(\partial_B)_k = \mathrm{id}_B \otimes_{A'} \partial_k$ are B-linear.
Let $N := \mathrm{Im}\,(\partial_0 - \partial)$. By assumption we have $\mathrm{Tor}_1^{A'}(B, N) = 0$. So tensoring the exact sequence of A'-modules

$$0 \longrightarrow A' \xrightarrow{\subseteq} A \xrightarrow{\partial_0 - \partial} N \longrightarrow 0$$

with B we obtain that B equals the ring of constants for ∂_B on A_B. Let $S := B - \{0\}$. Then $L = S^{-1}A_B = K \otimes_{K'} L'$. The ring homomorphim ∂_B is B-linear whence $\partial_L := S^{-1}\partial_B$ is an L'-linear ring homomorphism $L \to L[T]_m$ and a higher derivation of L of rank m. We have $(\partial_L)_k = S^{-1}(\partial_B)_k = L' \otimes_L \partial_k$ for all $k = 0, ..., m$ whence $\mu(\partial_L) = \mu(\partial)$. Moreover, $x/s \in L$ with $x \in A_B, s \in S$ is constant with respect to ∂_L if and only if x is constant with respect to ∂_B, i.e. if and only if $x \in B$. This shows that L' equals the field of constants with respect to ∂_L.

□

Corollary 4.5. *Keep the assumptions of the preceding lemma. Suppose that the ring A_B is again a Krull domain. If (A, ∂) satisfies $(\heartsuit)$ then (A_B, ∂_L) satisfies $(\heartsuit)$. In this case, B is a Krull domain.*

Proof. Since $\mu(\partial_L) = \mu(\partial)$ one has $[L' : L] = [K' : K] = p^{n(\partial)} = p^{n(\partial_L)}$. Take $a \in A$ such that $\partial_{\mu(\partial)}(a) \in A^\times$. It follows that $\partial_{\mu(\partial_L)}(1 \otimes a) = 1 \otimes \partial_{\mu(\partial)}(a) \in (A_B)^\times$. This shows $(\heartsuit)$ for (A_B, ∂_L). We have explained above that in this case, $B = L' \cap A_B$ is again a Krull domain. □

We apply these results to the following special case [11, IV.9]. Let k be a field of characteristic $p > 0$ and let $A := k[t^{\pm 1}]$ be the ring of Laurent polynomials in one variable t over k and let K be its fraction field. For any $m \geq 0$ we have the ring homomorphism

$$A \to A[T]_m, \; t \mapsto \text{class of } t + T$$

which is a higher derivation ∂ of rank m. Let $m' \geq 0$ and consider ∂ for $m := p^{m'} - 1$. Let $A' := k[t^{\pm p^{m'}}]$ and write K' for its quotient field. We have

$$\partial(t^{p^{m'}}) = \partial(t)^{p^{m'}} = (t + T)^{p^{m'}} = t^{p^{m'}}$$

and therefore ∂ is A'-linear. Since $K'[t] = K$ we obtain by localization at $A' - \{0\}$ a K'-linear higher derivation

$$\partial : K \to K[T]_m$$

of rank m that maps t to (the class of) $t + T$. Note that for $j = 0, ..., m$ each ∂_j is a K'-linear endomorphism on K whose effect on the K'-basis $1, t, ..., t^m$ of K is given by

$$\partial_j t^i = \binom{i}{j} t^{i-j} \tag{4.6}$$

where $\binom{i}{j} = 0$ if $j > i$. This is obvious from the binomial expansion of $\partial(t^i) = (t + T)^i$.

Lemma 4.7. *The nontrivial higher derivation ∂ has the invariants $\mu(\partial) = 1$ and $n(\partial) = m'$ and the subring A is invariant. The field of constants equals K' with $K' \cap A = A'$. The ordered pair (A, ∂) satisfies $(\heartsuit)$.*

Proof. First of all, $[K : K'] \leq p^{m'}$. From $\partial(t) = t + T$ we deduce $\partial_1(t) = 1$ whence $\mu(\partial) = 1$. The identity $n(\partial) = m'$ is then obvious. Let K'' be the field of constants of ∂. We have

$$K^{p^{m'}} \subseteq K' \subseteq K'' \subseteq K$$

and so the field extension K/K'' is purely inseparable. Hence the minimal polynomial of t over K'' has the form $X^{p^n} - a$ with some $a \in K''$ and some n. Suppose $n < m'$ so that $p^n \leq p^{m'} - 1 = m$. Then $t^{p^n} = a \in K''$ is a constant. But (4.6) implies $\partial_{p^n}(t^{p^n}) = 1$, a contradiction. Hence $[K : K''] = p^{m'}$ and our initial remark implies $K' = K''$. Since A' is integrally closed, we have $K' \cap A = A'$. Finally, $[K : K'] = p^{m'} = p^{n(\partial)}$ and $\partial_1(t) = 1 \in A^\times$. Thus the ordered pair (A, ∂) satisfies $(\heartsuit)$. $\square$

For any $m' \geq 0$ we call the homomorphism $\partial : K \to K[T]_m$ induced by $t \mapsto t + T$ *the standard higher derivation over k of exponent m'*. It has rank $m = p^{m'} - 1$.

5. Affinoid algebras and Quillen's theorem

We let $\Gamma = \mathbb{R}_+^\times$ and let k be a nonarchimedean field which is complete with respect to a nontrivial absolute value $|.|$. We assume that $|.|$ is a *discrete* valuation, i.e. $|k^\times| = |\pi|^{\mathbb{Z}}$ for some uniformizing element $\pi \in k$ with $0 < |\pi| < 1$. We then have

$$\tilde{k}_\bullet \xrightarrow{\cong} \tilde{k}_1[t^{\pm 1}]$$

by mapping the principal symbol of π to the variable t. In particular, the ring $\tilde{k}_\bullet$ is noetherian. Let p be the characteristic exponent of the residue field $\tilde{k}_1$ of k.

Let A be a strictly affinoid algebra over k which is an integral domain. Let d be the dimension of A. By the Noether normalization lemma [6, Cor. 6.2.2/2] there is an injective homomorphism $k\{T_1, ..., T_d\} \hookrightarrow A$ which is finite and strict with respect to spectral norms. By loc.cit., Prop. 6.2.2/4, there is a number $c \geq 1$ such that $|A| \subseteq |k|^{1/c}$. Let $q = |\pi|^{1/c}$. We then have a decreasing complete and exhaustive $\mathbb{Z}$-filtration on A

$$\cdots \subseteq F_{s+1}A \subseteq F_sA \subseteq \cdots$$

by the additive subgroups

$$F_sA := \{a \in A,\ |a| \leq q^{-s}\} = A_{q^{-s}}$$

for $s \in \mathbb{Z}$. Letting $gr_sA := F_sA/F_{s+1}A$ we have

$$\tilde{A}_\bullet = \oplus_{s\in\mathbb{Z}}\ gr_sA$$

for the graded reduction of the affinoid algebra A. The ring homomorphism $\tilde{k}_\bullet\{T_1, ..., T_d\} \hookrightarrow \tilde{A}_\bullet$ is finite according to [30, Prop. 3.1(ii)]. In particular, the ring $\tilde{A}_\bullet$ is noetherian.

Besides $\tilde{A}_\bullet$ we need to introduce the *Rees ring* $R(A)_\bullet$ of the filtered ring A [19]. To this end let $A[X^{\pm 1}]$ be the Laurent polynomial ring over A in a variable X with its $\mathbb{Z}$-gradation by degree in X. Then $R(A)_\bullet$ equals the subring

$$R(A)_\bullet = \oplus_{s\in\mathbb{Z}}(F_sA)X^s \subseteq A[X^{\pm 1}]$$

with its induced gradation. Write $(R(A)_\bullet)_{(X)}$ for its localization at the multiplicatively closed subset $\{X, X^2, ...\}$. One has the identities

$$(R(A)_\bullet)_{(X)} \simeq A[X^{\pm 1}] \quad \text{and} \quad R(A)_\bullet/XR(A)_\bullet \simeq \tilde{A}_\bullet. \tag{5.1}$$

Since A is reduced, the filtration $F_\bullet A$ on A is complete. Since $\tilde{A}_\bullet$ is noetherian, so is $R(A)_\bullet$ according to [19, Prop. II.1.2.3].

Recall that a commutative noetherian ring R is called *regular* if all its localizations at prime ideals are regular local rings. By the Auslander-Buchsbaum formula [1, Thm. 3.7] this is equivalent to R having finite *global dimension*, i.e. there is a number $0 \leq d < \infty$ such that any finitely generated module has a projective resolution of length $\leq d$. In this case, the global dimension $\mathrm{gld}(R)$ of R is the smallest of such numbers d and coincides with the (Krull) dimension of R.

Proposition 5.2. *Suppose that $\tilde{A}_\bullet$ is a regular ring. Then A and $R(A)_\bullet$ are regular rings with* $\mathrm{gld}(A) \leq \mathrm{gld}(\tilde{A}_\bullet)$ *and* $\mathrm{gld}(R(A)_\bullet) \leq 1 + \mathrm{gld}(\tilde{A}_\bullet)$.

Proof. This is the commutative case of the main result of [18]. $\square$

The regularity of $R(A)_\bullet$ implies that any finitely generated graded $R(A)_\bullet$-module has a finite resolution by finitely generated graded projective $R(A)_\bullet$-modules with degree zero morphisms [21, 6.5].

Let $K_0(A)$ and $K_0(\tilde{A}_\bullet)$ be the Grothendieck group of the ring A and $\tilde{A}_\bullet$ respectively. We have the following generalization of the K_0-part of Quillen's theorem [23] which is due to Li Huishi, M. van den Bergh and F. van Oystaeyen [17, Cor. 2.6].

Proposition 5.3. *Suppose $\tilde{A}_\bullet$ is regular. There is a group isomorphism*

$$\nu : K_0(A) \xrightarrow{\cong} K_0(\tilde{A}_\bullet) \tag{5.3}$$

such that $\nu([A]) = [\tilde{A}_\bullet]$.

Let us explain where the isomorphism comes from. It is defined via the following commutative diagram

$$\begin{array}{ccc} K_{0g}(R(A)_\bullet) & \xrightarrow{\beta} & K_0(A) \\ \simeq \Big\downarrow i & & \simeq \Big\downarrow \nu \\ K_{0g}(\tilde{A}_\bullet) & \xrightarrow{\gamma} & K_0(\tilde{A}_\bullet) \end{array}$$

where the horizontal maps are surjective and the vertical maps are isomorphisms. Here, K_{0g} equals K_0 applied to the category of finitely generated projective graded modules over a $\mathbb{Z}$-graded ring with degree zero graded morphisms. Note here that a projective object in the graded category is the same as a graded module which is projective after forgetting gradations [21, Prop. 7.6.6]. The upper horizontal map comes from the homogeneous localization $R(A)_\bullet \to (R(A)_\bullet)_{(X)} = A[X^{\pm 1}]$ and the equivalence of categories $M \mapsto A[X^{\pm 1}] \otimes_A M$ between finitely generated A-modules and finitely generated $\mathbb{Z}$-graded $A[X^{\pm 1}]$-modules. Note that a quasi-inverse to this latter equivalence is given by passing from a $\mathbb{Z}$-graded $A[X^{\pm 1}]$-module to its homogeneous part of degree zero. The map γ comes from the functor "forgetting the gradation". The map i is induced from the canonical map $R(A)_\bullet \to R(A)_\bullet / XR(A)_\bullet = \tilde{A}_\bullet$ and the map ν is defined so as to make the diagram commutative. Let us make the map ν completely explicit. Let P be a finitely generated projective A-module and $P' := A[X^{\pm 1}] \otimes_A P$ the corresponding $\mathbb{Z}$-graded $A[X^{\pm 1}]$-module. Choose a finitely generated graded $R(A)_\bullet$-submodule P'' inside P' that generates P' over $A[X^{\pm 1}]$. Since $R(A)_\bullet$ is regular, there is a finite resolution by finitely generated projective graded $R(A)_\bullet$-modules and degree zero morphisms

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow P'' \longrightarrow 0.$$

Then

$$\nu([P]) = \sum_{i=0,\ldots,n} (-1)^i \, [P_i / XP_i].$$

using $R(A)_\bullet / XR(A)_\bullet = \tilde{A}_\bullet$.

Let $\mathrm{Pic}(R)$ be the *Picard group* of a commutative ring R, i.e. the abelian group (with tensor product) of isomorphism classes of finitely generated projective R-modules of rank 1.

Proposition 5.4. *Let A be of dimension 1 with $\tilde{A}_\bullet$ regular. The natural homomorphism*

$$\mathrm{Pic}(R(A)_\bullet) \longrightarrow \mathrm{Pic}((R(A)_\bullet)_{(X)})$$

is surjective.

Proof. To ease notation write $S := R(A)_\bullet$. Since S is regular, all its localizations at prime ideals are factorial rings by the Auslander-Buchsbaum-Nagata theorem [2]. Since S is an integral domain, X is not a zero divisor and therefore [4, Prop. 7.17] implies the claim. We briefly recall from loc.cit. how one produces preimages. Let $P \in \mathrm{Pic}(S_{(X)})$. Since S is an integral domain, P is isomorphic to an invertible ideal $I \subseteq S_{(X)}$. Let $I_0 = I \cap S$. The bidual $(I_0)^{**} = \mathrm{Hom}_S(\mathrm{Hom}_S(I_0, S), S)$ is a reflexive S-module and lies in $\mathrm{Pic}(S)$. Since I is a reflexive $S_{(X)}$-module, $(I_0)^{**}$ localizes to I. □

Remark: If in the situation of the proposition we have that $\tilde{A}_\bullet$ is an integral domain, then the surjection on Picard groups is even a bijection. Indeed, in this case $X \in R(A)_\bullet$ generates a prime ideal according to (5.1) and then [4, Prop. III.7.15] yields the assertion.

Back in the general case, let M, N be finitely generated projective A-modules of rank m, n respectively. The formula

$$\wedge^{m+n}(M \oplus N) = \sum_{k=0,\dots,m+n} \wedge^k(M) \otimes \wedge^{m+n-k}(N) = \wedge^m(M) \otimes \wedge^n(N)$$

shows that $[M] \mapsto [\wedge^m(M)]$ defines a group homomorphism $\det : K_0(R) \to \mathrm{Pic}(R)$. Let $\mathrm{rk}(M) \in \mathbb{N}$ denote the rank of a finitely generated projective A-module M (recall that A is an integral domain). We have the short exact sequence

$$0 \longrightarrow SK_0(A) \longrightarrow K_0(A) \xrightarrow{\mathrm{rk}\oplus\det} \mathbb{Z} \oplus \mathrm{Pic}(A) \longrightarrow 0. \tag{5.4}$$

The group $SK_0(A)$ consists of classes $[P]-[A^m]$ where P has some rank m and $\wedge^m P \cong A$. If A has dimension 1, then, since A is noetherian, $SK_0(A) = 0$ by Serre's theorem [4, Thm. IV.2.5]. We therefore have $K_0(A) \simeq \mathbb{Z} \oplus \mathrm{Pic}(A)$ in this case.

Proposition 5.5. *Let A be of dimension 1 such that the ring $\tilde{A}_\bullet$ is regular. There is a commutative diagram of abelian groups*

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "="] & K_0(A) \arrow[r, "\det"] \arrow[d, "\nu", "\simeq"'] & \mathrm{Pic}(A) \arrow[r] \arrow[d, "\bar{\nu}", "\simeq"'] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & K_0(\tilde{A}_\bullet) \arrow[r, "\det"] & \mathrm{Pic}(\tilde{A}_\bullet) \arrow[r] & 0
\end{tikzcd}$$

in which rows are exact and vertical maps are isomorphisms. The rank map induces a canonical splitting in both rows.

Proof. Since A is commutative, the map $1 \mapsto [A]$ defines an inclusion $\mathbb{Z} \hookrightarrow K_0(A)$ and similarly for $\tilde{A}_\bullet$. In particular, the upper row is a split exact sequence and in the lower row we have $\mathbb{Z} \subseteq \ker \det$. Since $\nu([A]) = [\tilde{A}_\bullet]$ the first square commutes. This implies a surjective homomorphism $\bar{\nu}$ making the second square commutative. It remains to show that $\bar{\nu}$ is injective. Mapping the isomorphism class of a rank 1 module P to the symbol $[P]$ in K_0 gives a set-theoretical section to both determinant maps. It suffices to see that ν preserves the image of these sections. So let P be a finitely generated projective A-module of rank 1 and let $[P] \in K_0(A)$. Since A is an integral domain, P is isomorphic to an invertible ideal $J \subseteq A$. Consider the invertible ideal

$$I = A[X^{\pm 1}] \otimes_A J = A[X^{\pm 1}]J \subseteq A[X^{\pm 1}]$$

and put $I_0 := I \cap R(A)_\bullet$. Let $P'' := (I_0)^{**}$ be the bidual of the $R(A)_\bullet$-module I_0. By the proof of the previous proposition, it is a reflexive $R(A)_\bullet$-module in $\mathrm{Pic}(R(A)_\bullet)$ that localizes to I. More precisely, P'' generates I over $A[X^{\pm 1}]$ by means of the duality isomorphism $I \xrightarrow{\cong} I^{**}$. Recall that $R(A)_\bullet$ is noetherian. Hence, $I_0 \subseteq R(A)_\bullet$ with its induced gradation is a finitely generated graded $R(A)_\bullet$-module and therefore $(I_0)^*$ has its natural $\mathbb{Z}$-gradation by degree of morphisms [22, Cor. 2.4.4]. Since also $(I_0)^*$ is finitely generated, a second application of loc.cit. implies that $P'' = (I_0)^{**}$ has a natural $\mathbb{Z}$-gradation. Similarly, using that $A[X^{\pm 1}]$ is noetherian, the module I^{**} has a natural $\mathbb{Z}$-gradation. It makes the duality isomorphism $I \xrightarrow{\cong} I^{**}$ into a graded morphism of degree zero (loc.cit., Prop. 2.4.9). It follows that the isomorphism $(P'')_{(X)} \simeq I$ is a graded isomorphism of degree zero. One therefore has $[P''] \in K_{0g}(R(A)_\bullet)$ with

$$\beta([P'']) = [J] = [P].$$

Consequently, $\nu([P]) = [P''/XP'']$. Since $P''/XP'' \in \mathrm{Pic}(\tilde{A}_\bullet)$, this completes the proof. □

6. Picard groups of forms

We keep the assumptions on the discretely valued field k. We will work relatively to a finite wildly ramified extension $k \subseteq \ell$. Let $p^n = [\tilde{\ell}_\bullet : \tilde{k}_\bullet]$ be the degree of the purely inseparable extension $\tilde{k}_\bullet \subseteq \tilde{\ell}_\bullet$. Then $p^n = [\ell : k]$ according to the first remark in subsection 2.4. We restrict to the case of positive characteristic char $\tilde{k}_1 = p > 0$. Let $\varpi \in \ell$ be a uniformizing element, i.e. $|\ell| = |\varpi|^{\mathbb{Z}}$ with $0 < |\varpi| < 1$. We then have $\tilde{\ell}_\bullet \xrightarrow{\cong} \tilde{\ell}_1[t^{\pm 1}]$ by mapping the principal symbol of ϖ to t. Under this identification $\tilde{k}_\bullet = \tilde{k}_1[t^{\pm p^n}]$. For simplicity, we assume in the following that ℓ/k is *totally ramified*, i.e. $f = 1$ and $\tilde{\ell}_1 = \tilde{k}_1$. The general case can be treated similarly but is more technical.

By our assumptions the ring extension $\tilde{k}_\bullet \subseteq \tilde{\ell}_\bullet$ is of the form considered at the end of section 4. In particular, we have the standard higher derivation associated to it.

Let $r \in \sqrt{|k^\times|}$ and $B := k\{r^{-1}T\}$. Let A be a form of B with respect to ℓ/k. We will study the Picard group of A under the assumption that A has *geometrically reduced* *graded reduction*, i.e.

$$\tilde{k}_\bullet^{\mathrm{alg}} \otimes_{\tilde{k}_\bullet} \tilde{A}_\bullet$$

is reduced where $\tilde{k}_\bullet^{\mathrm{alg}}$ denotes the graded algebraic closure of $\tilde{k}_\bullet$. By the usual argument it suffices to check that $\tilde{k}_\bullet^{p^{-1}} \otimes_{\tilde{k}_\bullet} \tilde{A}_\bullet$ is reduced.

Suppose now this condition holds. Then Lem. 2.12 implies that $\tilde{A}_\bullet$ is a form of $\tilde{B}_\bullet$ for the extension $\tilde{k}_\bullet \subseteq \tilde{\ell}_\bullet$ which will allow us to use p-radical descent. We fix once and for all an isomorphism $\tilde{B}_\bullet \xrightarrow{\cong} \tilde{A}_\bullet \otimes_{\tilde{k}_\bullet} \tilde{\ell}_\bullet$ which we use as an identification in the following. We now forget all gradations and treat all graded rings as ordinary rings.

Put $m := p^n - 1$. Let K and K' be the (ordinary) fraction fields of $\tilde{\ell}_\bullet$ and $\tilde{k}_\bullet$. Since K has characteristic $p > 0$ we may consider the standard higher derivation

$$\partial : K \to K[S]_m$$

of exponent n on K given by $t \to t + S$ (end of section 4). According to Lem. 4.7 we have $\mu(\partial) = 1$ and the subring $\tilde{k}_\bullet$ is invariant. The field of constants equals K' with $K' \cap \tilde{\ell}_\bullet = \tilde{k}_\bullet$. The ordered pair $(\tilde{\ell}_\bullet, \partial)$ satisfies $(\heartsuit)$. Consider the injective homomorphism $\tilde{k}_\bullet \to \tilde{A}_\bullet$ into the integral domain $\tilde{A}_\bullet$. The ring $\tilde{B}_\bullet = \tilde{A}_\bullet \otimes_{\tilde{k}_\bullet} \tilde{\ell}_\bullet$ is an integral domain. Let $L' = \mathrm{Frac}(\tilde{A}_\bullet)$ be the (ordinary) fraction field of $\tilde{A}_\bullet$. Since $\tilde{B}_\bullet$ is integral over $\tilde{A}_\bullet$, we then have

$$L = \mathrm{Frac}(\tilde{B}_\bullet) = \tilde{A}_\bullet \otimes_{K'} K$$

for the (ordinary) fraction field of $\tilde{B}_\bullet$. Now $\tilde{k}_\bullet$ is a principal ideal domain and, hence, any torsion free module is flat. Hence, $\tilde{\ell}_\bullet$ is flat as a module over $\tilde{k}_\bullet$ and the $\tilde{k}_\bullet$-submodule $\mathrm{Im}\,(\partial_0 - \partial) \subseteq \tilde{\ell}_\bullet[S]_m$ associated with the homomorphism $\partial = \partial_{\tilde{\ell}_\bullet} : \tilde{\ell}_\bullet \to \tilde{\ell}_\bullet[S]_m$ is flat. According to Lem. 4.4 the field L admits a higher derivation

$$\partial_L := \mathrm{id}_{L'} \otimes_{K'} \partial$$

of the same rank and order as ∂ such that the subring $\tilde{B}_\bullet$ is invariant. The field of constants with respect to ∂_L equals L' and $L' \cap \tilde{B}_\bullet = \tilde{A}_\bullet$ is a Krull domain. Since $\tilde{B}_\bullet$ is a Krull domain, Cor. 4.5 implies that $(\tilde{B}_\bullet, \partial_L)$ satisfies $(\heartsuit)$. By Prop. 4.3 we therefore have the group isomorphism

$$\Phi_{\tilde{B}_\bullet} : \ker \bar{\jmath} \xrightarrow{\cong} \mathcal{L}_{\tilde{B}_\bullet}/\mathcal{L}'_{\tilde{B}_\bullet}, \quad D \bmod \mathrm{div}_{\tilde{A}_\bullet}(K'^\times) \mapsto \frac{\partial(x_D)}{x_D} \bmod \mathcal{L}'_{\tilde{B}_\bullet}$$

where $x_D \in K^\times$ is chosen such that

$$\mathrm{div}_{\tilde{B}_\bullet}(x_D) = j(D)$$

and

$$\bar{\jmath} : \mathrm{Cl}(\tilde{A}_\bullet) \to \mathrm{Cl}(\tilde{B}_\bullet)$$

is the canonical map. Since $\tilde{B}_\bullet = \tilde{\ell}_1[t^\pm, T]$ is a factorial domain [7, Prop. VII.§3.4.3 and Thm. VII.§3.5.2] we have $\mathrm{Cl}(\tilde{B}_\bullet) = 0$. Hence

$$\Phi_{\tilde{B}_\bullet} : \mathrm{Cl}(\tilde{A}_\bullet) \xrightarrow{\cong} \mathcal{L}_{\tilde{B}_\bullet}/\mathcal{L}'_{\tilde{B}_\bullet}$$

for the class group of the Krull domain $\tilde{A}_\bullet$.

Lemma 6.1. *The ring $\tilde{A}_\bullet$ is a noetherian regular Krull domain of global dimension* 2.

Proof. We have already explained above that $\tilde{A}_\bullet$ is a Krull domain. The ring extension $\tilde{A}_\bullet \hookrightarrow \tilde{B}_\bullet$ makes $\tilde{A}_\bullet$ a (bi-)module direct summand of $\tilde{B}_\bullet$ and $\tilde{B}_\bullet$ is a free $\tilde{A}_\bullet$-module via this extension. By [21, Thm. 7.2.8(i)] we have $\mathrm{gld}(\tilde{A}_\bullet) \leq \mathrm{gld}(\tilde{B}_\bullet) = 2$, hence $\tilde{A}_\bullet$ is regular. Its global dimension coincides then with its Krull dimension and [20, Ex.9.2] implies $\mathrm{gld}(\tilde{A}_\bullet) = 2$. Since $\tilde{B}_\bullet$ is noetherian, so is $\tilde{A}_\bullet$ [7, Cor. I.§3.5]. □

If $P \subseteq \tilde{A}_\bullet$ is a height 1 prime ideal of $\tilde{A}_\bullet$, the localization $(\tilde{A}_\bullet)_P$ is a discrete valuation ring. If $I \subseteq \tilde{A}_\bullet$ is an ideal, we let $v_P(I)$ be its valuation, i.e. the integer ν such that $I_P = P^\nu(\tilde{A}_\bullet)_P$ as ideals in $(\tilde{A}_\bullet)_P$. Since $\tilde{A}_\bullet$ is a regular Krull domain, [32, Cor. I.3.8.1] says that any divisor $D \in D(\tilde{A}_\bullet)$ is of the form $D = \sum_P v_P(I)P$ for a unique invertible ideal $I \subseteq \tilde{A}_\bullet$ and that $D \mapsto I$ induces an isomorphism $\mathrm{Cl}(\tilde{A}_\bullet) \xrightarrow{\cong} \mathrm{Pic}(\tilde{A}_\bullet)$. Invoking Prop. 5.5 we have $\bar{\nu} : \mathrm{Pic}(A) \xrightarrow{\cong} \mathrm{Pic}(\tilde{A}_\bullet)$. This proves the following proposition.

Proposition 6.2. *There are canonical isomorphisms*

$$\mathrm{Pic}(A) \xrightarrow{\cong} \mathrm{Pic}(\tilde{A}_\bullet) \xrightarrow{\cong} \mathrm{Cl}(\tilde{A}_\bullet) \xrightarrow{\cong} \mathcal{L}_{\tilde{B}_\bullet}/\mathcal{L}'_{\tilde{B}_\bullet}.$$

In particular, $\mathrm{Pic}(A)$ *has exponent p^n.*

We give a criterion in which the structure of these groups can be explicitly determined -at least up to a finite number of choices. To do this, let us write $\deg_T$ for the monomial degree in the variable T of an element in $\tilde{B}_\bullet = \tilde{\ell}_1[t^{\pm 1}, T]$ or of an element in $\tilde{B}_\bullet[S]_m$. Since $\partial_L(T) = T \bmod (S)$ we have that $\deg_T(\partial_L(T)) \geq 1$.

Proposition 6.3. *Suppose* $\deg_T(\partial_L(T)) = 1$*. Then $\mathcal{L}_{\tilde{B}_\bullet}/\mathcal{L}'_{\tilde{B}_\bullet}$ is a finite cyclic group of exponent p^n generated by the class of $\frac{\partial_L(T)}{T}$.*

Proof. We use the description of the group $\mathcal{L}$ from Lem. 4.2. This implies $\frac{\partial_L(T)}{T} \in \mathcal{L}_{\tilde{B}_\bullet}$ since ∂_L leaves $\tilde{B}_\bullet$ invariant. Moreover, let $f(T) \in \tilde{B}_\bullet - \{0\}$ be given such that $h := \frac{\partial_L(f)}{f} \in \mathcal{L}_{\tilde{B}_\bullet}$. Then $\partial_L(f) = f^\partial(\partial_L(T))$ where f^∂ refers to the polynomial in T that arises from f by applying ∂ to its coefficients. Our assumption implies

$$\deg_T(\partial_L(f)) = \deg_T(f^\partial) = \deg_T(f)$$

where the latter identity holds since ∂ is injective. Consequently,

$$\frac{\partial_L(f)}{f} \in (\tilde{B}_\bullet[S]_m)^\times \Longrightarrow \frac{\partial_L(f)}{f} \in (\tilde{\ell}_\bullet[S]_m)^\times$$

and so $\deg_T(h) = 0$. Write $f = \sum_{i,j} a_{ij} t^i T^j$ with $a_{ij} \in \tilde{\ell}_1$ with $i \in \mathbb{Z}$, $j \geq 0$. Write $\partial_L(T) = Tw$ with some $w \in \tilde{\ell}_\bullet[S]_m$. Thus, $w = \sum_{j=0,\dots,m} c_j S^j$ with $c_0 = 1$ and $c_j \in \tilde{\ell}_\bullet$ for $j > 0$. Now we compute

$$\sum_{i,j} a_{ij}(t+S)^i T^j w^j = \sum_{i,j} a_{ij} \partial(t)^i \partial_L(T)^j = \partial_L(\sum_{i,j} a_{ij} t^i T^j) = h \sum_{i,j} a_{ij} t^i T^j.$$

Since $f \neq 0$ there is j such that $a_{ij} \neq 0$. Since t, S, T are algebraically independent and since $\deg_T(h) = 0$ one may compare the coefficient of T^j on both sides which yields

$$w^j \, (\sum_i a_{ij}(t+S)^i)/(\sum_i a_{ij} t^i) = h.$$

Since $w = \partial_L(T)/T \in \mathcal{L}_{\tilde{B}_\bullet}$ one has $hw^{-j} \in \mathcal{L}_{\tilde{B}_\bullet}$. Using $K \cap \tilde{B}_\bullet = \tilde{\ell}_\bullet$ we obtain altogether

$$\sum_i a_{ij}(t+S)^i/(\sum_i a_{ij} t^i) = \partial(\sum_i a_{ij} t^i)/(\sum_i a_{ij} t^i) \in (K[S]_m)^\times \cap \tilde{B}_\bullet[S]_m \subseteq \tilde{\ell}_\bullet[S]_m.$$

In particular,

$$hw^{-j} = \partial(\sum_i a_{ij} t^i)/(\sum_i a_{ij} t^i) \in \mathcal{L}_{\tilde{\ell}_\bullet}$$

where $\mathcal{L}_{\tilde{\ell}_\bullet}$ refers to the integral logarithmic derivatives associated to the standard derivation ∂. Since the latter satisfies ($\heartsuit$) and since $\mathrm{Cl}(\tilde{\ell}_\bullet) = 0$ we have $\mathcal{L}_{\tilde{\ell}_\bullet} = \mathcal{L}'_{\tilde{\ell}_\bullet}$ (Prop. 4.3). Since ∂_L extends ∂ the inclusion $\tilde{\ell}_\bullet \subseteq \tilde{B}_\bullet$ induces $\mathcal{L}'_{\tilde{\ell}_\bullet} \subseteq \mathcal{L}'_{\tilde{B}_\bullet}$. Thus, $h \equiv w^j \mod \mathcal{L}'_{\tilde{B}_\bullet}$ in $\mathcal{L}_{\tilde{B}_\bullet}/\mathcal{L}'_{\tilde{B}_\bullet}$. Consequently, the class of w generates the group $\mathcal{L}_{\tilde{B}_\bullet}/\mathcal{L}'_{\tilde{B}_\bullet}$. □

We conclude by explaining the condition $\deg_T(\partial_L(T)) = 1$ in the context of forms of Russell type. For simplicity, we let A be a form of B of Russell type with parameters $n = 1$ and $s = r$ (in the notation of Thm. 3.10) and which admits a trivialization over $\tilde{\ell}_\bullet$. Up to isomorphism, such a form has graded reduction

$$\tilde{A}_\bullet = \tilde{k}_\bullet[r^{-p}T_1, r^{-1}T_2]/(T_2^p - f(T_1))$$

where $f(T_1) = a_0 T_1 + a_1 T_1^p + \dots + a_m T_1^{p^m} \in \tilde{k}_\bullet[r^{-p}T_1]$ is a homogeneous p-polynomial of degree r^p with $a_0 \neq 0$. Let x resp. y be the residue class of T_1 and T_2 in $\tilde{A}_\bullet$ respectively. According to subsection 3.3 we have

$$1 \otimes a_0 x = (1 \otimes y - (a_m \otimes x^{p^{m-1}} + \dots + a_1 \otimes x))^p =: t^p$$

in $\theta^{\langle 1 \rangle} A$ with a homogeneous element t of degree r. We have an isomorphism of graded $\tilde{\ell}_\bullet$-algebras

$$A_{\tilde{\ell}_\bullet} \xrightarrow{\cong} \tilde{\ell}_\bullet[r^{-1}T], \;\; t \mapsto T.$$

Under this isomorphism $1 \otimes y - (a_m^{1/p} \otimes x^{p^{m-1}} + \dots + a_1^{1/p} \otimes x) = T$ according to (3.2). Our condition comes down to the identity

$$\partial(1) \otimes y - (\partial(a_m^{1/p}) \otimes x^{p^{m-1}} + \dots + \partial(a_1^{1/p}) \otimes x)) = \partial_L(T) = fT + h$$

for elements $f, h \in \tilde{\ell}_\bullet[S]_m$. If this holds, we have

$$(1-f) \otimes y - ([\partial(a_m^{1/p}) - f a_m^{1/p}] \otimes x^{p^{m-1}} + ... + [\partial(a_1^{1/p}) - f a_1^{1/p}] \otimes x)) = h.$$

The elements $y, x^{p^{m-1}}, ..., x, 1$ are $\tilde{k}_\bullet$-linearly independent in $\tilde{k}_\bullet[x,y] = \tilde{A}_\bullet$ and therefore $\tilde{\ell}_\bullet[S]_m$-linearly independent in $\tilde{\ell}_\bullet[S]_m \otimes_{\tilde{k}_\bullet} \tilde{A}_\bullet = \tilde{B}_\bullet[S]_m$. It follows $h = 0, f = 1$ and $\partial(a_i^{1/p}) = a_i^{1/p}$ for all $i = 1, ..., m$. In other words, a_i has a p-th root in $\tilde{k}_\bullet$ for $i = 1, ..., m$. Conversely, this condition implies $\partial_L(T) = T$. The preceding two propositions show in this case that $\mathrm{Pic}(A) = 1$ and that A is a principal ideal domain (cf. Lem. 2.14).

Appendix A. Results from graded commutative algebra

All rings are assumed to be associative, commutative and unital. Let Γ be a commutative multiplicative group.

A.1. **Generalities.** Our reference for general graded rings is [22] but our exposition follows [8], [10] and [30]. A ring A is called Γ-*graded* or simply *graded* if it allows a direct sum decomposition as abelian groups

$$A = \oplus_{\gamma \in \Gamma} A_\gamma$$

such that $A_\gamma \cdot A_\delta \subseteq A_{\gamma\delta}$. The set $\coprod_{\gamma \in \Gamma} A_\gamma$ is the set of *homogeneous* elements. The decomposition of A yields a function

$$\rho_A : \coprod_\gamma (A_\gamma - \{0\}) \to \Gamma, \quad a \in A_\gamma - \{0\} \mapsto \gamma,$$

the *grading* of A. We have $1 \in A_1$ and A_1 is a subring of A; if $a \in A$ is homogeneous and invertible in A, then a^{-1} is homogeneous [22, Prop. 1.1.1]. We denote by $A^\times$ the group of invertible homogeneous elements in A. Then ρ yields a homomorphism $A^\times \to \Gamma$.
Any ring A may be viewed a graded ring by letting $A_1 = A$ and $A_\gamma = 0$ if $\gamma \neq 1$ (the *trivial gradation*). A *graded field* is a graded ring A with $A^\times = A - \{0\}$. Such a ring need not be a field in the usual sense as we will see below. On the other hand, every field k is a graded field via the trivial gradation.

If $S \subseteq A$ is a subset of homogeneous elements, then the smallest graded subring of A containing S is called the graded subring of A *generated* by S.

An ideal $I \subseteq A$ is called *graded* if it is generated by homogeneous elements or, equivalently, if $I = \oplus_{\gamma \in \Gamma}(I \cap A_\gamma)$. In this case, the quotient $A/I = \oplus_{\gamma \in \Gamma} A_\gamma/(I \cap A_\gamma)$ is graded.

A ring homomorphism $A \to B$ between graded rings A and B is *graded of degree* $\delta \in \Gamma$ if it maps A_γ into $B_{\delta\gamma}$ for all $\gamma \in \Gamma$. A homomorphism of degree 1 will simply be called *graded*. In this case, we call B a *graded algebra* over A.

Lemma A.1. *Let k be a graded field. Let A, B be two graded k-algebras with $Im\ \rho_A \subseteq Im\ \rho_k$. Then any k_1-algebra homomorphism $A_1 \to B_1$ extends uniquely to a homomorphism of graded k-algebras $A \to B$.*

Proof. Since $A_\gamma = k_\gamma \otimes_{k_1} A_1$ for any $\gamma \in \Gamma$, this is a simple instance of graded Frobenius reciprocity [22, Thm. 2.5.5]. □

A graded ring A is called *reduced* if there is no homogeneous nonzero nilpotent in A. A graded ring A is a *graded domain* if all nonzero homogeneous elements of A are not zero-divisors in A. We remark here that a graded ring which is reduced (resp. a graded domain) need not be reduced (resp. an integral domain) in the usual sense. However, this is true if Γ can be made into a totally ordered group (e.g. Γ is torsion free [16]), in which case one may argue with homogeneous parts of highest degree.

A *prime* homogeneous ideal is a homogeneous ideal $P \subsetneq A$ such that the graded ring A/P is a domain. Such an ideal need not be a prime ideal in the usual sense. We denote the set of such ideals by $\mathrm{Spec}_\Gamma(A)$ (the *graded prime spectrum* of A [22, 2.11]). Given such an ideal P we let A_P be the homogeneous localization, i.e. the localization at the multiplicative subset of the complement $A - P$ consisting of homogeneous elements. It is a *local* graded ring with a unique maximal homogeneous ideal. The rings A_P where P runs through $\mathrm{Spec}_\Gamma(A)$ are sometimes called *the* local rings of the graded ring A. If A is a graded domain and $P = (0)$, then A_P is a graded field, the *graded fraction field* of A. We denote it by $\mathrm{Frac}_\Gamma(A)$.

A graded domain is called a *graded discrete valuation ring* if it is a graded principal ideal domain (i.e. every homogeneous ideal admits a single homogeneous generator) with a unique nonzero prime homogeneous ideal. The proofs of [27, I.§2. Prop. 2 and 3] extend to the graded setting by working only with homogeneous elements. This implies the next lemma (cf. below for a discussion of the noetherian property and integral elements in the graded setting).

Lemma A.2. *For a graded ring A to be a graded discrete valuation ring it is necessary and sufficient to be a graded noetherian graded local ring with a maximal ideal generated by a non-nilpotent homogeneous element. A graded discrete valuation ring is integrally closed. The local rings of a graded principal ideal domain are graded discrete valuation rings.*

Let A be a graded ring. A *graded A-module* is an A-module M with a direct sum decomposition as abelian groups $M = \oplus_{\gamma \in \Gamma} M_\gamma$ such that $A_\delta M_\gamma \subseteq M_{\delta\gamma}$ for all $\delta, \gamma \in \Gamma$. For the straightforward notions of graded morphism of degree $\gamma \in \Gamma$, graded submodule, graded direct sum or graded tensor product we refer to [22].

If $\gamma \in \Gamma$ we let $A(\gamma)$ be the graded A-module whose underlying A-module equals A and whose gradation is given by $A(\gamma)_\delta = A_{\gamma\delta}$ for all $\delta \in \Gamma$. If M is a graded A-module and $(m_i)_i$ is a family of elements of M of respective degrees γ_i, there exists a unique graded A-linear map $\oplus_i A(\gamma_i^{-1}) \to M$ of degree 1 mapping $1 \in A(\gamma_i^{-1})$ to m_i for all i. The family $(m_i)_i$ is called *free* (resp. *generating* resp. *a basis*) if this map is injective (resp. surjective resp. bijective). The module M is called *of finite type* if there exists a generating family for M of finite cardinality.

Let k be a graded field. A graded k-module will be called a *graded k-vector space*. Any graded k-vector space M admits a basis and different bases have the same cardinality. This cardinality is called the *dimension* of M [30, Lem. 1.2].

A graded ring is *graded noetherian* if any homogeneous ideal of A is finitely generated. This is equivalent to the property that chains of homogeneous ideals in A satisfy the ascending chain condition. In this case, every graded A-submodule of a finitely generated graded A-module is finitely generated.

A.2. **Polynomial extensions.** Let A be a graded ring and $r_1, ..., r_n \in \Gamma^n$. We let

$$A[r_1^{-1}T_1, ..., r_n^{-1}T_n]$$

be the graded ring whose underlying ring equals the polynomial ring $A[T_1, ..., T_n]$ over A and where for all $s \in \Gamma$ the homogeneous elements of degree s are given by the polynomials $\sum_I a_I \mathbf{T}^I$ where $a_I \in A$ is homogeneous of degree $s\mathbf{r}^{-I}$. Here, $\mathbf{T}^I = T_1^{i_1} \cdots T_n^{i_n}$ and $\mathbf{r}^{-I} = (r_1^{-i_1}, ..., r_n^{-i_n}) \in \Gamma^n$ for $I = (i_1, ..., i_n) \in \mathbb{N}^n$. In particular, T_i is homogeneous of degree r_i. One has $A[r_1^{-1}T_1, ..., r_n^{-1}T_n]^\times = A^\times$.
If we want to refer to the usual degree in the variable T of the polynomial underlying an element f of $A[r^{-1}T]$ we will talk about the *monomial degree* of f. The proof of the Hilbert Basis theorem carries over, so $A[r_1^{-1}T_1, ..., r_n^{-1}T_n]$ is graded noetherian if A is graded noetherian [8, §4].

Let A be a graded domain. A nonzero homogeneous element $a \in A - A^\times$ is called *irreducible* if $a = bc$ with homogeneous elements $b, c \in A$ implies that $b \in A^\times$ or $c \in A^\times$. The graded domain A is called *factorial* if every nonzero homogeneous element from $A - A^\times$ is uniquely -up to rearrangement and up to an element from $A^\times$- a product of irreducible homogeneous elements in A. By the classical argument, a graded principal ideal domain is factorial.

Lemma A.3. *Let A be a graded domain which is factorial. Let $r \in \Gamma$. Then $A[r^{-1}T]$ is factorial. A monic homogeneous element in $A[r^{-1}T]$ is irreducible if and only if it is irreducible in* $\mathrm{Frac}_\Gamma(A)[r^{-1}T]$.

Proof. One has a graded version of the classical Gauss lemma [15, Thm. IV.2.1] working only with nonzero homogeneous elements. One may then copy the arguments from the ungraded case [15, Thm. IV.2.3]. □

If k is a graded field, then $k[r^{-1}T]$ is even a principal ideal domain by [8, Lemma 4.1].

Let $A \to B$ be an injective graded homomorphism between graded rings. If a homogeneous element $b \in B_r$ satisfies $f(b) = 0$ for some monic polynomial $f \in A[T]$ of positive monomial degree, then one may replace nonzero coefficients of f with suitable nonzero homogeneous parts (depending on γ) to find such an f which is homogeneous in $A[r^{-1}T]$.

In this case we call b *integral* over A. The set of elements of B whose homogeneous parts are integral over A forms a graded A-subalgebra A' of B, the *graded integral closure* of A in B. If $A = A'$ we call A *integrally closed* in B. If in this situation A is a graded domain and B its graded fraction field, we call A *integrally closed*.
The usual argument with Zorn's lemma using only homogeneous ideals implies that any homogeneous element $a \in A \setminus A^\times$ lies in a maximal homogeneous ideal. This implies $A = \cap_{P \in \mathrm{Spec}_\Gamma(A)} A_P$ for a graded domain A. In this case, A is integrally closed if and only if this holds for each A_P.

A.3. **Graded fields.** Let k be a graded field. Let p be the characteristic exponent of the field k_1.

We begin with an example. Let ρ be the grading of k and let $r \in \Gamma$ whose class in $\Gamma/\rho(k^\times)$ has infinite order. The homogeneous elements in $k[r^{-1}T]$ are then given by the monomials aT^j with $a \in k$ homogeneous and $j \in \mathbb{N}$: indeed, a second monomial bT^j of the same degree leads to $\rho(a)r^j = \rho(b)r^k$ which implies $j = i$ by the choice of r. The graded fraction field of $k[r^{-1}T]$ is therefore obtained by (homogeneous) localization at the single element T. We therefore have

$$k[r^{-1}T, rT^{-1}] := \mathrm{Frac}_\Gamma(k[r^{-1}T]) = k[r^{-1}T, rS]/(TS - 1) \tag{1.4}$$

in case r has infinite order in $\Gamma/\rho(k^\times)$. The underlying ring of $k[r^{-1}T, rT^{-1}]$ equals the Laurent polynomials over k in the variable T and so it is not a field in the usual sense. The homogeneous elements of $k[r^{-1}T, rT^{-1}]$ are given by aT^j with $a \in k$ homogeneous and $j \in \mathbb{Z}$.

A graded homomorphism $k \to \ell$ between graded fields is injective and called an *extension of graded fields.* If $x \in \ell^\times$ of degree $r \in \Gamma$, then the graded homomorphism

$$k[r^{-1}T] \to \ell, \quad T \mapsto x$$

has a homogeneous kernel I_x. If $I_x = 0$ resp. $I_x \neq 0$ the element x is called *transcendental* resp. *algebraic* over k. The latter case always occurs in case of a finite extension $k \subseteq \ell$. In the algebraic case we call the unique monic homogeneous generator f_x of I_x the *minimal homogeneous polynomial* of x over k.
The extension $k \subseteq \ell$ is called *normal* if f_x splits into a product of homogeneous polynomials in $\ell[r^{-1}T]$ of monomial degree 1 for every $x \in \ell^\times$ where $r = \rho_\ell(x)$. The classical argument involving Zorn's lemma applies to the graded setting and yields a *graded algebraic closure* $\bar{k}$ of k, unique up to k-isomorphism. If f_x has only simple roots in $\bar{k}$ then f_x and x are called *separable.* If f_x is of the form $f_x(T) = T^{p^n} - c$ with $c \in k^\times$, we call f_x purely inseparable and x purely inseparable of degree n over k. The union k^s of all separable elements of $\bar{k}$ (together with the zero element) is called the *graded separable closure of* k *in* $\bar{k}$. The union k^i of all purely inseparable elements (together with the zero element) is called the *graded purely inseparable closure of* k *in* $\bar{k}$. Both are graded fields and one has $k^s \cap k^i = k$.

A finite extension $k \subseteq \ell$ which is normal and separable is called *Galois*. In this case, the group of graded automorphisms of ℓ fixing k pointwise is denoted by $\mathrm{Gal}(\ell/k)$. There is a graded version of the main theorem of Galois theory [10, (1.16.1)]. In particular, $\#\mathrm{Gal}(\ell/k) = [\ell : k]$ and $\ell^{\mathrm{Gal}(\ell/k)} = k$.

Let $k \subseteq \ell$ be an extension of graded fields and $S \subseteq \ell$ a subset. We let $k(S)$ be the smallest graded subfield of ℓ that contains k and S. It equals the graded fraction field of the graded subring of ℓ generated by k and S. If $\ell = k(S)$ we say that ℓ is *generated by S over k*.

Let $k \subseteq \ell$ be an extension of graded fields. A subset $S \subseteq \ell^\times$ is said to be *algebraically independent over* k if the graded k-algebra homomorphism $k[r_1^{-1}T_1, ..., r_n^{-1}T_n] \to \ell$ defined by $T_i \mapsto s_i$ is injective for any finite subset $s_1, ..., s_n$ of S (with $r_i = \rho_\ell(s_i)$). A subset S which is a maximal algebraically independent set over k is called a *transcendence basis for ℓ/k*. As in the classical case, one shows that a transcendence basis always exists and that all such bases have the same cardinality, the *transcendence degree* $\mathrm{trdeg}_k(\ell)$ of the extension ℓ/k [8, §4]. Similarly, if the graded field ℓ is generated over k by a set $S \subseteq \ell$, one may choose a transcendence basis from S. We shall need a graded version on the existence of a *separating* transcendence basis in the following sense.

Lemma A.5. *Suppose $k \subseteq \ell$ is an extension of graded fields such that the graded ring $k^i \otimes_k \ell$ is reduced. If $S \subseteq \ell$ generates ℓ over k, there is a subset $S' \subseteq S$ with the properties:* (i) *S' is a transcendence basis of ℓ/k,* (ii) *the algebraic extension $k(S') \subseteq \ell$ is separable.*

Proof. Let r and $r_1, ..., r_n$ be elements of Γ. The graded ring $k[r_i^{-1}T]$ is a principal ideal domain and therefore factorial. Hence $A := k[r_1^{-1}T_1, ..., r_n^{-1}T_n]$ is factorial according to Lem. A.3. The latter lemma also implies that a monic homogeneous polynomial in $A[r^{-1}T]$ is irreducible if and only if it is irreducible in $\mathrm{Frac}_\Gamma(A)[r^{-1}T]$. We then have everything to extend the classical proof [15, Prop. VIII.4.1] on the existence of a separating transcendence basis for separable field extensions to the graded setting by working with homogeneous elements only. $\square$

Lemma A.6. *If K denotes the graded fraction field of $k[r^{-1}T]$, then* $\mathrm{trdeg}_k(K) = 1$.

Proof. According to [8, Lemma 4.8] we have the formula

$$\mathrm{trdeg}_k(K) = \mathrm{trdeg}_{k_1}(K_1) + \dim_{\mathbb{Q}}((\rho(K^\times)/\rho(k^\times)) \otimes_{\mathbb{Z}} \mathbb{Q}).$$

Suppose first that the image of r has finite order in $\Gamma/\rho(k^\times)$. There is a finite extension $k \subseteq \ell$ with $r = \rho(a)$ for some $a \in \ell^\times$. We then have an isomorphism $\ell[r^{-1}T] \xrightarrow{\cong} \ell[T]$ of graded ℓ-algebras by mapping $T \mapsto aT$. Since $\mathrm{trdeg}_k(K) = \mathrm{trdeg}_\ell(K(\ell))$ and since $K(\ell) = \mathrm{Frac}_\Gamma(\ell[r^{-1}T])$, we may therefore assume $r = 1$. But then $K_1 = k_1(T)$, the usual field of rational functions over k_1 and we are done. In the case, where the image of r has infinite order in $\Gamma/\rho(k^\times)$, we have $K = k[r^{-1}T, rT^{-1}]$ by (1.4). Hence $K_1 = k_1$. Since $(\rho(K^\times)/\rho(k^\times)) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the line generated by the image of r, the assertion follows in this case, too. $\square$

References

[1] M. Auslander and D. A. Buchsbaum. Homological dimension in local rings. *Trans. Amer. Math. Soc.*, 85:390–405, 1957.

[2] M. Auslander and D. A. Buchsbaum. Unique factorization in regular local rings. *Proc. Nat. Acad. Sci. U.S.A.*, 45:733–734, 1959.

[3] K. Baba. On p-radical descent of higher exponent. *Osaka J. Math.*, 18(3):725–748, 1981.

[4] H. Bass. *Algebraic K-theory*. W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[5] Vladimir G. Berkovich. *Spectral theory and analytic geometry over non-archimedean fields*, volume 33 of *Math. Surveys and Monographs*. American Mathematical Society, Providence, Rhode Island, 1990.

[6] S. Bosch, U. Güntzer, and R. Remmert. *Non-Archimedean analysis*. Springer-Verlag, Berlin, 1984.

[7] N. Bourbaki. *Commutative algebra. Chapters 1–7*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998.

[8] B. Conrad and M. Temkin. Descent for non-archimedean analytic spaces. *Preprint 2010. Available at: http : //math.huji.ac.il/ temkin/papers/Descent.pdf*.

[9] M. Demazure and P. Gabriel. *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs*. Masson & Cie, Éditeur, Paris, 1970. Avec un appendice ıt Corps de classes local par Michiel Hazewinkel.

[10] A. Ducros. Toute forme modérément ramifiée d'un polydisque ouvert est triviale. *Math. Z.*, 273(1-2):331–353, 2013.

[11] N. Jacobson. *Lectures in abstract algebra. III*. Springer-Verlag, New York, 1975. Theory of fields and Galois theory, Graduate Texts in Math., No. 32.

[12] T. Kambayashi, M. Miyanishi, and M. Takeuchi. *Unipotent algebraic groups*. Lecture Notes in Mathematics, Vol. 414. Springer-Verlag, Berlin, 1974.

[13] I. Kaplansky. Maximal fields with valuations. *Duke Math. J.*, 9:303–321, 1942.

[14] M.-A. Knus and M. Ojanguren. *Théorie de la descente et algèbres d'Azumaya*. Lecture Notes in Math., Vol. 389. Springer-Verlag, Berlin, 1974.

[15] S. Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2002.

[16] F. W. Levi. Ordered groups. *Proc. Indian Acad. Sci., Sect. A.*, 16:256–263, 1942.

[17] H. Li, M. Van den Bergh, and F. Van Oystaeyen. Note on the K_0 of rings with Zariskian filtration. *K-Theory*, 3(6):603–606, 1990.

[18] H. Li and F. van Oystaeyen. Global dimension and Auslander regularity of Rees rings. *Bull. Math. Soc. Belgique*, (serie A) XLIII:59–87, 1991.

[19] H. Li and F. van Oystaeyen. *Zariskian filtrations*, volume 2 of *K-Monographs in Mathematics*. Kluwer Academic Publishers, Dordrecht, 1996.

[20] H. Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1986.

[21] J. C. McConnell and J. C. Robson. *Noncommutative Noetherian rings*. Pure and Applied Mathematics (New York). John Wiley & Sons Ltd., Chichester, 1987.

[22] C. Năstăsescu and F. Van Oystaeyen. *Methods of graded rings*, volume 1836 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2004.

[23] D. Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.

[24] B. Rémy, A. Thuillier, and A. Werner. Bruhat-Tits theory from Berkovich's point of view. I. Realizations and compactifications of buildings. *Ann. Sci. Éc. Norm. Supér. (4)*, 43(3):461–554, 2010.

[25] P. Russell. Forms of the affine line and its additive group. *Pacific J. Math.*, 32:527–539, 1970.

[26] P. Samuel. Classes de diviseurs et dérivées logarithmiques. *Topology*, 3(suppl. 1):81–96, 1964.

[27] J.-P. Serre. *Local fields*, volume 67 of *Graduate Texts in Math.* Springer-Verlag, New York, 1979.
[28] J.-P. Serre. *Galois cohomology*. Springer Monographs in Math. Springer-Verlag, Berlin, english edition, 2002.
[29] M. Temkin. On local properties of non-Archimedean analytic spaces. *Math. Ann.*, 318(3):585–607, 2000.
[30] M. Temkin. On local properties of non-Archimedean analytic spaces. II. *Israel J. Math.*, 140:1–27, 2004.
[31] W. C. Waterhouse. *Introduction to affine group schemes*, volume 66 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1979.
[32] C. Weibel. An introduction to algebraic K-theory. *Preprint. Available at:* $http://www.math.rutgers.edu/\ weibel/Kbook.html$.

Institut für Mathematik, Humboldt-Universität zu Berlin, Rudower Chaussee 25, D-12489 Berlin, Germany
E-mail address: Tobias.Schmidt@math.hu-berlin.de