Auslander Regularity of p-adic Distribution Algebras

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Abstract

Given a compact p-adic Lie group over an arbitrary base field we prove that its distribution algebra is Fréchet-Stein with Auslander regular Banach algebras whose global dimensions are bounded above by the dimension of the group. As an application, we show that nonzero coadmissible modules coming from smooth or, more general, $U(\mathfrak{g})$-finite representations have a maximal grade number (codimension) equal to the dimension of the group.

1 Introduction

Given a locally L-analytic group G where $L \subseteq \mathbb{C}_p$ is a finite extension of $\mathbb{Q}_p$ P. Schneider and J. Teitelbaum recently developed a systematic framework to study locally analytic G-representations in topological p-adic vector spaces (cf. [ST1-6]). At the center of this theory lies a certain subcategory $\mathcal{C}_G$ of modules over the K-valued locally analytic distribution algebra $D(G, K)$, $K \subseteq \mathbb{C}_p$ being a complete and discretely valued extension of L. The category $\mathcal{C}_G$ (the *coadmissible* modules) is contravariantly equivalent to the category of admissible locally analytic G-representations via the functor "passage to the strong dual". The construction of $\mathcal{C}_G$ relies on the fact that, when G is compact, the algebra $D(G, K)$ is Fréchet-Stein. The latter means that $D(G, K)$ equals a projective limit of certain noetherian Banach algebras $D_r(G, K)$ with flat transition maps. Furthermore, Schneider and Teitelbaum prove ([ST5], Thm. 8.9) that, in case $L = \mathbb{Q}_p$ and G is compact, the ring $D(G, K)$ is "almost" Auslander regular: the rings $D_r(G, K)$ are Auslander regular with a global dimension bounded above by the dimension of the manifold G. This result allows to establish a well-behaved dimension theory on $\mathcal{C}_G$ where the usual grade number serves as a codimension function. On the other hand, it allows to deduce important properties of the locally analytic duality functor (in the sense of [ST6]) e.g. its involutivity.

The present work establishes the regularity property of $D(G, K)$ over arbitrary base fields. Its main result is the

Theorem. *Let G be a compact locally L-analytic group. Then $D(G,K)$ has the structure of a K-Fréchet-Stein algebra where the corresponding Banach algebras are Auslander regular rings whose global dimensions are bounded above by the dimension of G.*

We remark straightaway that the desired Banach algebras arise in the same fashion as in [ST5] i.e. they come as quotient Banach algebras via the map $D(G_0,K) \to D(G,K)$. Here, G_0 denotes the underlying locally $\mathbb{Q}_p$-analytic group and the map is dual to embedding locally L-analytic functions into locally $\mathbb{Q}_p$-analytic functions on G.

The brief outline of the paper is as follows. After reviewing the notions of uniform pro-p groups and Fréchet-Stein algebras in our setting (sections 2-3) we begin by investigating certain standard subgroups H of G serving as locally L-analytic analogues of uniform groups (section 4). In section 5 we generalize the filtration methods developed in [ST5] to the base field L. As a result we may deduce the regularity properties for the algebras $D_r(H,K)$ when r is "sufficiently small" and H is as above. We then use the existence of a well-behaved filtration of $D(H,K)$ by Fréchet-Stein subalgebras (arising functorially from the lower p-series of H) to deduce the main result (sections 6-7). We finish by describing the resulting dimension theory on $\mathcal{C}_G$. As an application we show that coadmissible modules coming from smooth or, more general, $U(\mathfrak{g})$-finite G-representations (as studied in [ST1]) are zero-dimensional.

Notations: Throughout, $\mathbb{Q}_p \subseteq L \subseteq K \subseteq \mathbb{C}_p$ is a chain of complete intermediate fields where $L/\mathbb{Q}_p$ is finite and K is discretely valued. If not otherwise stated G denotes a compact locally L-analytic group, $\mathfrak{g}_L$ its Lie algebra and exp an exponential map. Let $\kappa := 1$ resp. $\kappa := 2$ if p is odd resp. even.

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2 Uniform pro-p groups and p-valuations

We begin by clarifying the relation between uniform pro-p groups (as introduced in [DDMS]) and certain p-valued groups introduced in [ST5]. Let us first recall the definitions of these groups and their basic properties.

Let G be a topologically finitely generated pro-p group. Denoting by G^l

the subgroup of G generated by l-th powers the *lower p-series* $(P_i(G))_{i\geq 1}$ is inductively defined via

$$P_1(G) := G,\ P_{i+1}(G) := P_i(G)^p[P_i(G), G]$$

for all $i \geq 1$. The subgroups $P_i(G)$ are (topologically) characteristic in G and constitute a fundamental system of open neighbourhoods for $1 \in G$. The group G is called *powerful* if the commutator of G is contained in G^p resp. G^4 (in case p odd resp. even). In this case $P_{i+1}(G) = P_i(G)^p = G^{p^i}$. Finally, G is called *uniform* if it is powerful and the index $(P_i(G) : P_{i+1}(G)),\ i \geq 1$ does not depend on i. A uniform group G has a unique locally $\mathbb{Q}_p$-analytic structure: for any minimal (ordered) set of topological generators $h_1, ..., h_d$ the map

$$(x_1, ..., x_d) \mapsto h_1^{x_1} \cdots h_d^{x_d}$$

is a bijective global chart $\mathbb{Z}_p^d \to G$. In this situation, the subgroup $P_i(G)$ equals the image of $p^{i-1}\mathbb{Z}_p^d$ and is a uniform group itself.

On the other hand, there is the notion of a *p-valuation* ω on an abstract group G introduced in [Laz]. This is a real valued function

$$\omega : G \setminus \{1\} \longrightarrow (1/(p-1), \infty)$$

satisfying

1. $\omega(gh^{-1}) \geq \min\ (\omega(g), \omega(h))$,
2. $\omega(g^{-1}h^{-1}gh) \geq \omega(g) + \omega(h)$,
3. $\omega(g^p) = \omega(g) + 1$

for all $g, h \in G$. As usual one puts $\omega(1) = \infty$. A p-valuation gives rise to a natural filtration of G by subgroups defining a topology on G. A p-valued group (G, ω), complete with respect to this topology, is called *p-saturated* if any $g \in G$ such that $\omega(g) > p/(p-1)$ is a p-th power.

Now let G be a compact locally $\mathbb{Q}_p$-analytic group endowed with a p-valuation ω. It follows from [Laz], III.3.1.3/9 and III.3.2.1 that the topology on G is defined by ω whence G is complete. Furthermore, G admits an *ordered basis*. This is an ordered set of topological generators $h_1, ..., h_d$ of G such that the map $(x_1, ..., x_d) \mapsto h_1^{x_1} \cdots h_d^{x_d}$ is a bijective global chart $\mathbb{Z}_p^d \to G$ and satisfies

$$\omega(h_1^{x_1} \cdots h_d^{x_d}) = \min_{i=1,\ldots,d} (\omega(h_i) + v_p(x_i)). \tag{1}$$

Here, v_p is the p-adic valuation on $\mathbb{Z}_p$. In [ST5], Sect. 4 the authors introduce the class of compact locally $\mathbb{Q}_p$-analytic groups G carrying a p-valuation ω with ordered basis $h_1, ..., h_d$ that satisfy the following additional axiom

(HYP) (G,ω) is $p-$saturated and the ordered basis $h_1,...,h_d$ of G satisfies $\omega(h_i)+\omega(h_j)>p/(p-1)$ for any $1\leq i\neq j\leq d$.

If $d=1$ the second condition is redundant.

We show that this class and the class of uniform groups are closely related (comp. also [loc.cit.], remark after Lem. 4.3).

Proposition 2.1 *Let G be a uniform pro-p group of dimension d. Then G has a p-valuation ω satisfying* (HYP). *It is given as follows: for $g\in P_i(G)\setminus P_{i+1}(G)$ put $\omega(g):=i$ resp. $\omega(g):=i+1$ in case $p\neq 2$ resp. $p=2$. In particular, ω is integrally valued. Any ordered system of topological generators $h_1,...,h_d$ for G is an ordered basis for ω in the sense of* (HYP) *with the property*

$$\omega(h_1)=...=\omega(h_d)=\kappa.$$

Conversely, if $p\neq 2$ then any compact locally $\mathbb{Q}_p$-analytic group with a p-valuation satisfying (HYP) *is a uniform pro-p group.*

Proof: We refer to [DDMS] for all basic properties of uniform groups that we use. Let G be a uniform pro-p group. Define ω as in the proposition. If $d>1$ then clearly $\omega(h_i)+\omega(h_j)>p/(p-1)$ for $i\neq j$. Moreover, $\omega(g)>p/(p-1)$ for $g\in G$ implies $g\in P_2(G)$ and the group $P_2(G)$ consists (as a set) precisely of the p-th powers of G. So for the first statement it remains to see that ω really is a p-valuation: the axiom 1. is clear. The map $x\mapsto x^p$ is a bijection $P_i(G)/P_{i+1}(G)\rightarrow P_{i+1}(G)/P_{i+2}(G)$ for all i whence 3. Now the lower p-series of any pro-p group satisfies

$$[P_i(G),P_j(G)]\leq P_{i+j+\kappa-1}(G)$$

for all $i,j\geq 1$ which gives 2. Finally, any ordered set of (topological) generators $h_1,...,h_d$ is an ordered basis and must lie in $P_1(G)\setminus P_2(G)$. In particular, $\omega(h_i)=\kappa$ for all i.

Conversely, assume $p\neq 2$ and let a compact locally $\mathbb{Q}_p$-analytic group G be given together with a p-valuation ω satisfying (HYP). One has $\omega([g,h])>p/(p-1)$ for all $g,h\in G$ according to (HYP) and the equation (1) above. Since G is p-saturated this implies $[G,G]\subseteq G^p$ i.e. G is powerful. It is also torsionfree as there is no p-torsion according to 3. Hence, G is uniform. $\square$

3 $D(G,K)$ as a Fréchet-Stein algebra

We review the construction of a Fréchet-Stein structure on the algebra $D(G,K)$ since this is central to our work. We refer to [ST5] for all details.

Recall that a (two-sided) K-Fréchet algebra is called *Fréchet-Stein* if there is a sequence $q_1 \leq q_2 \leq ...$ of algebra seminorms on A defining its Fréchet topology and such that for all $n \in \mathbb{N}$ the completion A_n of A with respect to q_n is a noetherian K-Banach algebra and a flat A_{n+1}-module via the natural map $A_{n+1} \to A_n$. This applies to the algebra $D(G,K)$ of K-valued locally analytic distributions on G. Recall that $D(G,K)$ equals the strong dual of the locally convex vector space $C^{an}(G,K)$ of K-valued locally L-analytic functions on G equipped with the convolution product. Let G_0 be the underlying locally $\mathbb{Q}_p$-analytic group. Choose a normal open subgroup $H_0 \subseteq G_0$ which is a uniform pro-p group. According to Prop. 2.1 the lower p-series of H_0 induces an integrally valued p-valuation ω on H_0 satisfying (HYP) with $h_1, ..., h_d$ chosen to be any (ordered) minimal set of generators of H_0. Thus, the construction of [ST5], Sect. 4 for distribution algebras of such p-valued groups applies: the bijective global chart $\mathbb{Z}_p^d \to H_0$ for the manifold H_0 given by

$$(x_1, ..., x_d) \mapsto h_1^{x_1} ... h_d^{x_d}$$

induces a topological isomorphism $C^{an}(H_0, K) \simeq C^{an}(\mathbb{Z}_p^d, K)$ on K-valued locally analytic functions. In this isomorphism the right-hand side is a space of classical Mahler series and the dual isomorphism $D(H_0, K) \simeq D(\mathbb{Z}_p^d, K)$ therefore realizes $D(H_0, K)$ as a space of noncommutative power series. More precisely, putting $b_i := h_i - 1 \in \mathbb{Z}[G]$, $\mathbf{b}^\alpha := b_1^{\alpha_1} ... b_d^{\alpha_d}$ for $\alpha \in \mathbb{N}_0^d$ the Fréchet space $D(H_0, K)$ equals all convergent series

$$\lambda = \sum_{\alpha \in \mathbb{N}_0^d} d_\alpha \mathbf{b}^\alpha$$

with $d_\alpha \in K$ such that the set $\{|d_\alpha| r^{\kappa|\alpha|}\}_\alpha$ is bounded for all $0 < r < 1$. Here $|.|$ denotes the normalized absolute value on $\mathbb{C}_p$. The value $\lambda(f) \in K$ of such a series on a function $f \in C^{an}(H_0, K)$ with Mahler expansion

$$f(\mathbf{x}) = \sum_{\alpha \in \mathbb{N}_0^d} c_\alpha \binom{\mathbf{x}}{\alpha}, \ c_\alpha \in K$$

is given by $\sum_{\alpha \in \mathbb{N}_0^d} c_\alpha d_\alpha$. The family of norms $||.||_r, \ 0 < r < 1$ defined via

$$||\lambda||_r := \sup_\alpha |d_\alpha| r^{\kappa|\alpha|}$$

defines the Fréchet topology on $D(H_0, K)$. Restricting to the subfamily $p^{-1} < r < 1, \ r \in p^{\mathbb{Q}}$ these norms are multiplicative and the norm completions $D_r(H_0, K)$ are K-Banach algebras realizing a Fréchet-Stein structure on $D(H_0, K)$.

Choose representatives $g_1 = 1, ..., g_r$ for the cosets in G/H and define on $D(G_0, K) = \oplus_i D(H_0, K) g_i$ the norms $|| \sum_i \lambda_i g_i ||_r := \max_i ||\lambda_i||_r$. The completions $D_r(G_0, K)$ give suitable Banach algebras for $D(G_0, K)$.

Finally, $D(G,K)$ is equipped with the quotient norms coming from the map $D(G_0,K) \to D(G,K)$. The latter arises as the transpose to the embedding $C^{an}(G,K) \subseteq C^{an}(G_0,K)$. Again, norm completions give suitable Banach algebras.

We conclude with another important feature of $D(H_0,K)$ in case of a locally $\mathbb{Q}_p$-analytic group H_0 which is uniform. Each algebra $D_r(H_0,K)$ carries the (separated and exhaustive) norm filtration defined by the additive subgroups

$$F_r^s D_r(H_0,K) := \{\lambda \in D_r(H_0,K),\ ||\lambda||_r \leq p^{-s}\},$$

$$F_r^{s+} D_r(H_0,K) := \{\lambda \in D_r(H_0,K),\ ||\lambda||_r < p^{-s}\}$$

for $s \in \mathbb{R}$ with graded ring

$$gr_r^{\cdot}\, D_r(H_0,K) := \oplus_{s\in\mathbb{R}}\ gr_r^s D_r(H_0,K)$$

where $gr_r^s D_r(H_0,K) := F_r^s D_r(H_0,K)/F_r^{s+} D_r(H_0,K)$. For a nonzero $\lambda \in D_r(H_0,K)$ denote by $\deg(\lambda) \in \mathbb{R}$ the *degree* of λ in the filtration. The *principal symbol* $\sigma(\lambda) \neq 0$ of λ is then given by

$$\lambda + F_r^{s+} D_r(H_0,K) \in gr_r^s D_r(H_0,K) \subseteq gr_r^{\cdot}\, D_r(H_0,K)$$

where $s = \deg(\lambda)$. Note that $gr^{\cdot} K \simeq k[\epsilon_0, \epsilon_0^{-1}]$ where k denotes the residue field and ϵ_0 is the principal symbol of a prime element for K.

Theorem 3.1 *Mapping $\sigma(b_i) \mapsto X_i$ yields a $gr^{\cdot} K$-algebra isomorphism of $gr_r^{\cdot}\, D_r(H_0,K)$ onto the polynomial ring $(gr^{\cdot} K)[X_1,...,X_d]$.*

Proof: [ST5], Thm. 4.5. □

We will prove an analogue over L of this result in section 5.

4 Results on certain standard groups

Over the ground field $\mathbb{Q}_p$ the Fréchet-Stein theory of the distribution algebra depends heavily on the presence of uniform subgroups: every compact locally $\mathbb{Q}_p$-analytic group contains an open normal uniform subgroup ([DDMS], Cor. 8.34). It is the principal aim of this section to generalize this latter fact in a suitable way to the ground field L.

We begin by reviewing the notion of a standard group. Let $\mathfrak{m}$ be the maximal ideal in the valuation ring $\mathfrak{o}$ of L. A locally L-analytic group is called *standard* of *level* h, $h \in \mathbb{N}$ if it admits a global chart onto $(\mathfrak{m}^h)^d \subseteq L^d$ such that the group operation is given by a single power series without constant term and with coefficients in $\mathfrak{o}^d$. A standard group of level 1 will simply be called standard. Any compact locally L-analytic group contains an open subgroup which is standard ([B-L], III. 7.3 Thm. 4). In case $L = \mathbb{Q}_p$ a locally $\mathbb{Q}_p$-analytic group is called *standard** if it admits a global chart onto $p^\kappa \mathbb{Z}_p^d$ such that the group operation is given by a single power series without constant term and with coefficients in $\mathbb{Z}_p^d$. A standard* group is uniform according to [DDMS], Thm. 8.31. We deduce a lemma from the proof of this result.

Lemma 4.1 *Suppose G is a locally $\mathbb{Q}_p$-analytic group which is standard* with respect to the global chart $\psi : G \to p^\kappa \mathbb{Z}_p^d$. Denote by $e_i \in \mathbb{Z}_p^d$ the i-th unit vector. The elements $h_i := \psi^{-1}(p^\kappa e_i)$ constitute a minimal set $h_1, ..., h_d$ of topological generators of G.*

Proof: By the proof of [loc.cit.] ψ induces a group isomorphism

$$G/P_2(G) \to p^\kappa \mathbb{Z}_p^d / p^{\kappa+1} \mathbb{Z}_p^d$$

where $P_2(G)$ equals the Frattini subgroup of G. Hence, $h_1, ..., h_d$ is a minimal set of topological generators. $\square$

Let $n = [L : \mathbb{Q}_p]$, e' be the ramification index of $L/\mathbb{Q}_p$ and u a uniformizer for $\mathfrak{o}$. Let $l \in \mathbb{N},\ l \geq 2$ and let G be locally L-analytic of dimension d.

Lemma 4.2 *Suppose G is standard of level le' with respect to a global chart ψ. Then it is standard of level 1 with respect to $u^{1-le'} \cdot \psi$. Its scalar restriction G_0 is standard* with respect to $G_0 \to (p^\kappa \mathfrak{o})^d \to p^\kappa \mathbb{Z}_p^{nd}$ where the first map equals $p^{\kappa - l} \cdot \psi$ and the second is induced by an arbitrary choice of $\mathbb{Z}_p$-basis for $\mathfrak{o}$.*

Proof: Let $\mathfrak{y}_1, ..., \mathfrak{y}_d$ be the L-basis of $\mathfrak{g}_L$ induced by ψ and regard ψ as a map $G \to \Gamma$ where $\Gamma = \oplus_j \mathfrak{m}^{le'} \mathfrak{y}_j \subseteq \mathfrak{g}_L$. By standardness we have for $g, h \in G$ and $\psi(g) = \sum_j \lambda_j \mathfrak{y}_j,\ \psi(h) = \sum_j \mu_j \mathfrak{y}_j$, with $\lambda_j, \mu_j \in \mathfrak{m}^{le'}$ that

$$\psi(gh) = \sum_j F_j(\lambda_1, ..., \lambda_d, \mu_1, ..., \mu_d) \mathfrak{y}_j$$

where $F_j(X_1, ..., X_d, Y_1, ..., Y_d) \in \mathfrak{o}[[(X_s), (Y_s)]]$ without constant term.

Let us first prove the statement concerning G_0: take as an L-basis of $\mathfrak{g}_L$ the elements $\mathfrak{x}_j' := p^{l-\kappa} \mathfrak{y}_j$. Then $\Gamma = \oplus_j \mathfrak{m}^{le'} \mathfrak{y}_j = \oplus_j \mathfrak{m}^{\kappa e'} \mathfrak{x}_j'$ and for $g, h \in G$ and $\psi(g) = \sum_i \lambda_j \mathfrak{x}_j',\ \psi(h) = \sum_j \mu_j \mathfrak{x}_j'$, with $\lambda_j, \mu_j \in \mathfrak{m}^{\kappa e'}$ we have

$$\psi(gh) = \sum_j F_j(\lambda_1 p^{l-\kappa}, ..., \lambda_d p^{l-\kappa}, \mu_1 p^{l-\kappa}, ..., \mu_d p^{l-\kappa}) \mathfrak{y}_j.$$

Since $F_j \in \mathfrak{o}[[(X_s),(Y_s)]]$ has no constant term and $l \geq 2$ (i.e. $p^{l-\kappa} \in \mathbb{Z}_p$) we get

$$\psi(gh) = \sum_j F''_j(\lambda_1, ..., \lambda_d, \mu_1, ..., \mu_d)\mathfrak{x}'_j \tag{2}$$

where $F''_j \in \mathfrak{o}[[(X_s),(Y_s)]]$ without constant term. By definition G is standard of level $\kappa e'$ with respect to

$$G \xrightarrow{\psi} \Gamma = \oplus_j \mathfrak{m}^{\kappa e'}\mathfrak{x}'_j \longrightarrow (\mathfrak{m}^{\kappa e'})^d. \tag{3}$$

Now choosing a $\mathbb{Z}_p$-basis $v_1, ..., v_n$ of $\mathfrak{o}$ yields $\mathfrak{m}^{\kappa e'} = \oplus_i p^\kappa \mathbb{Z}_p v_i$. From (3) together with the $\mathbb{Q}_p$-basis $\{v_i\mathfrak{x}'_j\}$ of $\mathfrak{g}_L$ we obtain the global chart

$$G_0 \xrightarrow{\psi} \Gamma = \oplus_j \oplus_i p^\kappa \mathbb{Z}_p v_i \mathfrak{x}'_j \longrightarrow (p^\kappa \mathbb{Z}_p)^{nd}$$

for G_0 which is as claimed. By (2) it follows for $g, h \in G_0$ and $\psi(g) = \sum_{ij} \lambda_{ij} v_i \mathfrak{x}'_j$, $\psi(h) = \sum_{ij} \mu_{ij} v_i \mathfrak{x}'_j$, with $\lambda_{ij}, \mu_{ij} \in p^\kappa \mathbb{Z}_p$ that

$$\begin{aligned} \psi(gh) = & \sum_j F''_j((\textstyle\sum_r \lambda_{rs} v_r)_s, (\sum_r \mu_{rs} v_r)_s)\mathfrak{x}'_j \\ = & \sum_j \sum_i G_{ij}((\lambda_{rs}), (\mu_{rs})) v_i \mathfrak{x}'_j. \end{aligned}$$

The nd functions G_{ij} are given by power series with coefficients in $\mathbb{Z}_p$ and no constant term. Hence, G_0 is standard* with a global chart as claimed.

It remains to see that G is standard of level 1 with respect to the claimed chart. But this follows as above by taking $\mathfrak{x}_j := u^{le'-1}\mathfrak{y}_j$ as L-basis of $\mathfrak{g}_L$. □

Proposition 4.3 *Any compact locally L-analytic group G contains an open normal standard subgroup H such that H_0 is standard*. The corresponding global chart is given by*

$$H \xrightarrow{\exp^{-1}} \oplus_j \mathfrak{m}\mathfrak{x}_j \longrightarrow \mathfrak{m}^d$$

where exp *is the exponential map and $\mathfrak{x}_1, ..., \mathfrak{x}_d$ a suitable L-basis of $\mathfrak{g}_L$.*

Proof: Choose a basis $\mathfrak{y}'_1, ..., \mathfrak{y}'_d$ of $\mathfrak{g}_L$, endow $\mathfrak{g}_L$ with the maximum norm and the space of L-linear maps on $\mathfrak{g}_L$ with the operator norm. Scaling the basis we may assume that the Hausdorff series converges on

$$\Lambda' := \oplus_j \mathfrak{m}\mathfrak{y}'_j$$

(viewed as a subset of L^d via $\mathfrak{y}'_1, ..., \mathfrak{y}'_d$) turning it into a locally L-analytic group. We obtain an isomorphism of locally L-analytic groups $\exp : \Lambda' \to G'$ onto an open subgroup $G' \subseteq G$.

Scaling the basis we may also assume that $|\mathrm{ad}\,\mathfrak{x}| < p^{-\frac{1}{p-1}}$ for all $\mathfrak{x} \in \Lambda'$ and that $\mathrm{Ad}(g) = \sum_{k\in\mathbb{N}_0} 1/k!\ (\mathrm{ad}\,\mathfrak{x})^k$ for all $g = \exp(\mathfrak{x}) \in G'$. Hence, $|\mathrm{Ad}(g)| = 1$ and Λ' is $\mathrm{Ad}(g)$-stable for all $g \in G'$. Furthermore, let $\mathcal{R}'$ be a (finite) system of representatives for the cosets G/G' and put

$$\Lambda := \cap_{g\in\mathcal{R}'}\mathrm{Ad}(g)\Lambda'.$$

Then Λ is $\mathrm{Ad}(g)$-stable for all $g \in G$. Now $p^t\Lambda$ is a subgroup of Λ' when $t \in \mathbb{N}$ is big enough. To see this choose $t \in \mathbb{N}$ big enough such that $g\exp(\mathfrak{x})g^{-1} = \exp(\mathrm{Ad}(g)\mathfrak{x})$ holds for all $\mathfrak{x} \in p^t\Lambda'$ and for all $g \in \mathcal{R}'$. Then $p^t\Lambda$ is stable under $*$, the group operation of Λ'. Indeed, if $\mathfrak{x}, \mathfrak{y} \in p^t\Lambda$ and $g \in \mathcal{R}'$ then writing $\mathfrak{x} = \mathrm{Ad}(g)\mathfrak{x}'$, $\mathfrak{y} = \mathrm{Ad}(g)\mathfrak{y}'$ with $\mathfrak{x}', \mathfrak{y}' \in p^t\Lambda'$ one may compute

$$\begin{aligned}\exp(\mathfrak{x} * \mathfrak{y}) = \exp(\mathrm{Ad}(g)\mathfrak{x}')\exp(\mathrm{Ad}(g)\mathfrak{y}') = &\ g\exp(\mathfrak{x}')\exp(\mathfrak{y}')g^{-1} \\ = &\ \exp(\mathrm{Ad}(g)(\mathfrak{x}' * \mathfrak{y}')).\end{aligned}$$

Observe here, that we have $\mathfrak{x}' * \mathfrak{y}' \in p^t\Lambda'$ since $p^t\Lambda'$ is a subgroup of Λ'. The calculation implies that $\mathfrak{x} * \mathfrak{y} \in \mathrm{Ad}(g)p^t\Lambda'$ and since this holds for any $g \in \mathcal{R}'$ we have $\mathfrak{x} * \mathfrak{y} \in p^t\Lambda$. Since $p^t\Lambda = -p^t\Lambda$ and $0 \in p^t\Lambda$ we obtain that $p^t\Lambda$ really is a subgroup of Λ' whence

$$M := \exp(p^t\Lambda)$$

is an open subgroup of G. Choose $\mathfrak{y}_1, ..., \mathfrak{y}_d \in \mathfrak{g}_L$ such that

$$p^t\Lambda = \oplus_j \mathfrak{m}\mathfrak{y}_j$$

and consider the global chart

$$\varphi : \exp(p^t\Lambda) \to p^t\Lambda = \oplus_j \mathfrak{m}\mathfrak{y}_j \to \mathfrak{m}^d.$$

We may assume that M is standard with respect to φ (otherwise pass to $\exp(\lambda^{-1}\mathfrak{m}^d)$ for sufficiently big $\lambda \in L$ and use the chart $\lambda \cdot \varphi$, see (the proof of) [B-L], III.7.3 Thm. 4). In this situation consider

$$\Gamma := \oplus_j \mathfrak{m}^{le'}\mathfrak{y}_j \subseteq \oplus_j \mathfrak{m}\mathfrak{y}_j = p^t\Lambda$$

for suitable $l \in \mathbb{N}$, $l \geq 2$ which will be chosen conveniently in the following. In the rest of the proof we show that the open subgroup

$$H := \exp(\Gamma)$$

of G will satisfy our requirements.

Let us first check normality: H is normal in M by [B-L], III.7.4 Prop. 6. Let $\mathcal{R}$ be a (finite) system of representatives for the cosets in G/M. Enlarging l if necessary we may assume that $g\exp(\mathfrak{x})g^{-1} = \exp(\mathrm{Ad}(g)\mathfrak{x})$

holds all $\mathfrak{x} \in \Gamma,\ g \in \mathcal{R}$. Now Λ is $\mathrm{Ad}(g)$-stable for all $g \in G$ and hence, so is $\Gamma = u^{le'-1}p^t\Lambda$ (u a uniformizer for $\mathfrak{o}$) since each $\mathrm{Ad}(g)$ is L-linear. Thus, $H = \exp(\Gamma)$ is stable under conjugation with elements from $\mathcal{R}$ and therefore it is normal in G.

Now since M is standard with respect to φ it follows from [loc.cit.] that H is standard of level le' with respect to $\varphi|_H$. Applying Lem. 4.2 we see that H is also standard of level 1 with a global chart given by

$$H \stackrel{\exp^{-1}}{\longrightarrow} \Gamma = \oplus_j \mathfrak{m}\mathfrak{x}_j \longrightarrow \mathfrak{m}^d$$

where $\mathfrak{x}_j := u^{le'-1}\mathfrak{y}_j$. Thus, the L-basis $\mathfrak{x}_1, ..., \mathfrak{x}_d$ is as desired. Finally, Lem. 4.2 also implies that the restricted group H_0 is standard*. □

Given a locally L-analytic group G a choice of L-basis $\mathfrak{x}_1, ..., \mathfrak{x}_d$ of $\mathfrak{g}_L$ gives rise to the map

$$\theta_L : (\sum_j x_j\mathfrak{x}_j) \mapsto \exp(x_1\mathfrak{x}_1) \cdots \exp(x_d\mathfrak{x}_d) \in G$$

defined on an open subset of $\mathfrak{g}_L = \oplus_j L\mathfrak{x}_j$ containing 0. It is locally L-analytic and étale at 0 and is called a *system of coordinates of the second kind* associated to the decomposition $\mathfrak{g}_L = \oplus_j L\mathfrak{x}_j$ ([B-L], III.4.3 Prop. 3).

Identifying the Lie algebras $\mathfrak{g}_L$ resp. $\mathfrak{g}_{\mathbb{Q}_p}$ of G resp. G_0 over $\mathbb{Q}_p$ (according to [B-VAR], 5.14.5) we consider the following condition on G:

Condition (L)*: There is an L-basis $\mathfrak{x}_1, ..., \mathfrak{x}_d$ of $\mathfrak{g}_L$ and a $\mathbb{Z}_p$-basis $v_1, ..., v_n$ of $\mathfrak{o}$ such that the system of coordinates of the second kind $\theta_{\mathbb{Q}_p}$ induced by the decomposition $\mathfrak{g}_{\mathbb{Q}_p} = \oplus_j \oplus_i \mathbb{Q}_p v_i\mathfrak{x}_j$ gives an isomorphism of locally $\mathbb{Q}_p$-analytic manifolds*

$$\theta_{\mathbb{Q}_p} : \oplus_j \oplus_i \mathbb{Z}_p v_i\mathfrak{x}_j \longrightarrow G_0.$$

The exponential satisfies $\exp(\lambda \cdot v_i\mathfrak{x}_j) = \exp(v_i\mathfrak{x}_j)^\lambda$ *for all* $\lambda \in \mathbb{Z}$.

Note that if G is pro-p and satisfies (L) with suitable bases v_i and $\mathfrak{x}_j$ then $\exp(\lambda \cdot v_i\mathfrak{x}_j) = \exp(v_i\mathfrak{x}_j)^\lambda$ for all $\lambda \in \mathbb{Z}$ extends to $\mathbb{Z}_p$-powers and so the elements

$$h_{ij} := \theta_{\mathbb{Q}_p}(v_i\mathfrak{x}_j) = \exp(v_i\mathfrak{x}_j)$$

are a minimal ordered system of topological generators for G_0.

For our purposes locally L-analytic groups that are uniform and satisfy the above condition (such as the additive group $\mathfrak{o}$) will play the role of uniform locally $\mathbb{Q}_p$-analytic groups. We show that there are enough of them.

Corollary 4.4 *Any compact locally L-analytic group G has a fundamental system of open normal subgroups H such that H_0 is uniform and satisfies* (L)*.*

Proof: It suffices to show that the group H constructed in the proof of the last proposition satisfies (L). This is because $H \subseteq G'$ (in the notation of this proof) and by construction, the open subgroup G' of G can be chosen as small as desired. We use the notation of this proof.

According to it (and in connection with Lem. 4.2) H_0 is standard* with respect to the bijective global chart

$$\psi : H_0 \xrightarrow{\exp^{-1}} \Gamma = \oplus_j \oplus_i p^\kappa \mathbb{Z}_p v_i \mathfrak{x}'_j \longrightarrow p^\kappa \mathbb{Z}_p^{nd}.$$

Here, $\mathfrak{x}'_1, ..., \mathfrak{x}'_d$ and $v_1, ..., v_n$ are bases of $\mathfrak{g}_L$ resp. $\mathfrak{o}$. Denoting by e_{ij} the ij-th unit vector in $\mathbb{Z}_p^{nd}$ put

$$h_{ij} := \psi^{-1}(p^\kappa e_{ij}) = \exp(p^\kappa v_i \mathfrak{x}'_j)$$

for $i = 1, ..., n,\ j = 1, ..., d$. Lem. 4.1 implies that these nd elements are a minimal system $h_{11}, h_{21}, ..., h_{nd}$ of topological generators for H_0. Putting $\mathfrak{z}_j := p^\kappa \mathfrak{x}'_j$ the map

$$\theta_{\mathbb{Q}_p} : \oplus_j \oplus_i \mathbb{Z}_p v_i \mathfrak{z}_j \longrightarrow H_0,\ \sum_{ij} \lambda_{ij} v_i \mathfrak{z}_j \mapsto \prod_j \prod_i h_{ij}^{\lambda_{ij}}$$

is a locally $\mathbb{Q}_p$-analytic isomorphism since H_0 is uniform (section 2). Because of $h_{ij} = \exp(v_i \mathfrak{z}_j)$ this map is by definition the system of coordinates of the second kind induced by the decomposition $\mathfrak{g}_{\mathbb{Q}_p} = \oplus_j \oplus_i \mathbb{Q}_p v_i \mathfrak{z}_j$. The last condition of (L) follows from this as well. $\square$

Corollary 4.5 *Suppose H is an open normal subgroup of G such that H_0 is uniform satisfying* (L). *Then any member in the lower p-series of H_0 is an open normal subgroup of G whose restriction is uniform satisfying* (L).

Proof: Let $H^m = \{x^{p^m}, x \in H\},\ m \in \mathbb{N}_0$ be a member in the lower p-series of H_0 with locally L-analytic structure as an open subgroup of G. It is thus open normal in G. Let $\mathfrak{x}_1, ..., \mathfrak{x}_d$ resp. $v_1, ..., v_n$ be bases of $\mathfrak{g}_L$ resp. $\mathfrak{o}$ such that the induced coordinate system $\theta_{\mathbb{Q}_p}$ realizes condition (L) for H_0. It immediately follows that H_0^m satisfies (L) with respect to the bases $p^m \mathfrak{x}_1, ..., p^m \mathfrak{x}_d$ and $v_1, ..., v_n$. $\square$

5 Filtrations

We wish to extend the filtration methods developed in [ST5] over $\mathbb{Q}_p$ to distribution algebras of certain locally L-analytic groups. In particular, we aim at an analogue over L of Thm. 3.1. The most natural way to do this is to realize $D(G,K)$ as a quotient of $D(G_0,K)$ and study the induced filtrations.

Let G be as usual a compact locally L-analytic group. Via the identification $\mathfrak{g}_L \simeq \mathfrak{g}_{\mathbb{Q}_p}$ over $\mathbb{Q}_p$ we view exp as an exponential map for G_0 as well. Recall that $\mathfrak{g}_L$ acts on $C^{an}(G_0,K)$ via differential operators in the usual way, i.e.

$$\mathfrak{x}f(g) := \frac{d}{dt} f(\exp(-t\mathfrak{x})g)\,|_{t=0}.$$

This gives an L-linear inclusion $L\otimes_{\mathbb{Q}_p}\mathfrak{g}_L \subseteq D(G_0,K)$ via associating to $1\otimes\mathfrak{x}$ the functional $f \mapsto (-\mathfrak{x})f(1)$. The kernel of the quotient map $D(G_0,K) \to D(G,K)$

$$I(G_0,K) := \{\lambda \in D(G_0,K) : \lambda|_{C^{an}(G,K)} \equiv 0\}$$

is a two-sided and closed ideal.

Lemma 5.1 *As a right ideal $I(G_0,K)$ is finitely generated by the elements*

$$F_{ij} := 1\otimes v_i\mathfrak{x}_j - v_i \otimes \mathfrak{x}_j \in D(G_0,K).$$

Here, $\mathfrak{x}_1,...,\mathfrak{x}_d$ is any L-basis of $\mathfrak{g}_L$ and $v_1,...,v_n$ is any $\mathbb{Q}_p$-basis of L.

Proof: For sake of clarity denote for the moment by δ_g the image of $g \in G$ in $D(G_0,K)$. Since $K[G] \subseteq D(G_0,K)$ is dense it suffices to show that $I := I(G_0,K)$ equals the closure of the K-vector space generated by the products

$$F_{ij}\delta_g \in D(G_0,K)$$

where $i = 1,...,n,\ j = 1,...,d,\ g \in G$. Let W denote the closure of the described vector space. According [Ko], Lem. 1.3.2 the subspace $C^{an}(G,K) \subseteq C^{an}(G_0,K)$ equals

$$\{f \in C^{an}(G_0,K) : (1\otimes v_i\mathfrak{x}_j - v_i\otimes\mathfrak{x}_j)f = 0 \text{ in } C^{an}(G_0,K) \text{ for all } i,j\}$$

and so $F_{ij}\delta_g \in I$ whence $W \subseteq I$. Choose a continuous functional ϕ on I vanishing on the subspace W. We show $\phi = 0$ whence $W = I$ by Hahn-Banach. For the strong dual I'_b we have the canonical isomorphism $I'_b \simeq C^{an}(G_0,K)/C^{an}(G,K)$ ([B-TVS], Cor. IV.2.2) whence $\phi = \bar{f}$ for some $\bar{f} \in C^{an}(G_0,K)/C^{an}(G,K)$. Then ϕ vanishing on W implies

$$F_{ij}\delta_g(f) = 0.$$

Recalling that multiplication in $D(G_0,K)$ is convolution one computes

$$\begin{aligned} 0 &= -F_{ij}\delta_g(f) \\ &= \delta_g(g' \longrightarrow -F_{ij}(g'' \longrightarrow f(g''g'))) \\ &= \delta_g((1\otimes v_i\mathfrak{x}_j)f - (v_i\otimes\mathfrak{x}_j)f) = (1\otimes v_i\mathfrak{x}_j - v_i\otimes\mathfrak{x}_j)f(g) \end{aligned}$$

Since i, j and $g \in G$ were arbitrary we have $f \in C^{an}(G, K)$ which means $\phi = 0$. $\square$

Remark: If $I_r(G_0, K)$ denotes the norm closure inside a Banach algebra $D_r(G_0, K)$ we have $I_r(G_0, K) = I(G_0, K)D_r(G_0, K)$ whence $I_r(G_0, K)$ is also finitely generated (as a right ideal of $D_r(G_0, K)$) by the F_{ij}.

From now on we additionally assume that G is uniform and satisfies (L). Let $\mathfrak{x}_1, ..., \mathfrak{x}_d$ resp. $v_1, ..., v_n$ be corresponding bases of $\mathfrak{g}_L$ resp. $\mathfrak{o}$. We may and will arrange

$$v_1 := 1 \in \mathbb{Z}_p$$

from now on, and thus $F_{1j} = 0$ for all j. Put $h_{ij} := \exp(v_i \mathfrak{x}_j)$ for $i, j \geq 1$. The elements $h_{11}, h_{21}, ..., h_{nd}$ constitute a minimal ordered set of topological generators for the uniform group G_0 and the global chart

$$G_0 \xrightarrow{\theta_{\mathbb{Q}_p}^{-1}} \oplus_j \oplus_i \mathbb{Z}_p v_i \mathfrak{x}_j \longrightarrow \mathbb{Z}_p^{nd}$$

induces expansions for the elements of $D(G_0, K)$ in the monomials $\mathbf{b}^\alpha = b_{11}^{\alpha_{11}} b_{21}^{\alpha_{21}} ... b_{nd}^{\alpha_{nd}}$, $\alpha \in \mathbb{N}_0^{nd}$ where $b_{ij} := h_{ij} - 1 \in \mathbb{Z}[G] \subseteq D(G_0, K)$ (section 3). Finally, let $\log(1 + X) := \sum_{k \geq 1} (-1)^{k-1} X^k / k \in \mathbb{Q}[[X]]$.

Lemma 5.2 *We have* $F_{ij} = \log(1 + b_{ij}) - v_i \log(1 + b_{1j})$.

Proof: Let $f \in C^{an}(G_0, K)$. It has a Mahler expansion of the form

$$(f \circ \theta_{\mathbb{Q}_p})(\sum_{ij} x_{ij} v_i \mathfrak{x}_j) = \sum_\alpha c_\alpha \binom{\mathbf{x}}{\alpha}$$

for all $\mathbf{x} := (x_{11}, x_{21}, ..., x_{nd}) \in \mathbb{Z}_p^{nd}$ with coefficients $c_\alpha \in K$. Since $\theta_{\mathbb{Q}_p}$ restricted to the direct summand $\mathbb{Q}_p v_i \mathfrak{x}_j$ equals exp we may explicitly calculate the values $(v_i \otimes \mathfrak{x}_j)(f)$ and $(1 \otimes v_i \mathfrak{x}_j)(f)$. Then a comparison of coefficients yields the claim. $\square$

Now let $p^{-1} < r < 1$, $r \in p^{\mathbb{Q}}$ and let $||.||_r$ denote a norm on $D(G_0, K)$ and $\|.\|_{\bar{r}}$ the quotient norm on $D(G, K)$. Consider the exact sequence

$$0 \longrightarrow I_r(G_0, K) \longrightarrow D_r(G_0, K) \longrightarrow D_r(G, K) \longrightarrow 0$$

of filtered $D_r(G_0, K)$-modules. By our assumptions ($r \in p^{\mathbb{Q}}$ and K discretely valued) the filtrations in question are essentially indexed by $\mathbb{Z}$ and so the functor $gr_r^{\cdot}$ is exact whence

$$gr_r^{\cdot} D_r(G, K) \simeq gr_r^{\cdot} D_r(G_0, K) / gr_r^{\cdot} I_r(G_0, K)$$

canonically. Recall that $gr^{\cdot} D_r(G_0, K) = (gr^{\cdot} K)[X_{11}, ..., X_{nd}]$ (Thm. 3.1) and so we aim at calculating $gr^{\cdot}_r I_r(G_0, K)$.

Abbreviate $D_r := D_r(G_0, K),\ I_r := I_r(G_0, K),\ D_{\bar{r}} := D_r(G, K)$ for the rest of this section.

We start by computing the principal symbols $\sigma(F_{ij})$ and then deduce that they generate $gr^{\cdot}_r I_r$. Recall that a real number $0 < s < 1$ is called *not critical* for the series $\log(1 + X)$ if the supremum

$$|\log(1 + X)|_s := \sup_{k \in \mathbb{N}} |\frac{1}{k} X^k|_s = \sup_{k \in \mathbb{N}} |\frac{1}{k}|\, s^k$$

is obtained at a single monomial of the series $\log(1 + X)$ [Ro, Def. 6.1.4]. This monomial is called *dominant*. According to [Ro, 6.1.6 Ex. 2], the critical radii for $\log(1 + X)$ are the increasing series of numbers

$$p^{-\frac{1}{p-1}} < p^{-\frac{1}{p^2-p}} < p^{-\frac{1}{p^3-p^2}} < ...$$

and

$$p^{j-1} p^{-\frac{1}{p-1}} < |\log(1 + x)| = |\frac{x^{p^j}}{p^j}| = p^j |x|^{p^j} < p^j p^{-\frac{1}{p-1}}$$

for

$$p^{-\frac{1}{p^j - p^{j-1}}} < |x| < p^{-\frac{1}{p^{j+1} - p^j}}.$$

We assume for the rest of this section that r^κ is not critical for $\log(1+X)$.

Lemma 5.3 *We have*

$$\sigma(F_{ij}) = \epsilon^{-h} X_{ij}^{p^h} - \bar{v}_i \epsilon^{-h} X_{1j}^{p^h} \in gr^{\cdot}_r D_r$$

with $h \in \mathbb{N}_0$ depending only on r^κ, $\bar{v}_i \in k$ is the residue class of v_i and $\epsilon := \sigma(p) \in gr^{\cdot} K$. For all $r^\kappa < p^{-\frac{1}{p-1}}$ one has $h = 0$ i.e. $\sigma(F_{ij}) = X_{ij} - \bar{v}_i X_{1j}$.

Proof: Since r^κ is not critical there is a dominant monomial of $\log(1 + X)$ with respect to $|.|_{r^\kappa}$ whose index depends only on r^κ and is seen to be a p-power, say p^h. Using Lem. 5.2 the claim follows by direct calculation. $\square$

Corollary 5.4 *The $nd - d$ elements $F_{ij},\ i \neq 1$ are orthogonal in D_r i.e. one has for arbitrary $c_{ij} \in K$ that*

$$||\sum_{ij} c_{ij} F_{ij}||_r = \max_{ij} ||c_{ij} F_{ij}||_r.$$

Proof: We may assume (via leaving away possible summands) that $c_{ij} \neq 0$ for all ij and that $||c_{ij} F_{ij}||_r =: p^{-s},\ s \in \mathbb{R}$ is a constant for all ij. According to the above lemma the elements $\sigma(F_{ij})$ generate a free $gr^{\cdot} K$-submodule inside $gr^{\cdot}_r D_r$. We therefore get

$$0 \neq \sum_{ij} \sigma(c_{ij})\sigma(F_{ij}) = \sum_{ij} \sigma(c_{ij}F_{ij}) = \sum_{ij} c_{ij}F_{ij} \mod F_r^{s+}D_r$$

and so $||\sum_{ij} c_{ij}F_{ij}||_r = p^{-s}$. □

For the rest of this section write $\mathcal{F}$ for the set of elements $F_{ij},\ i \neq 1$. Put $s_0 :=$deg(F), some $F \in \mathcal{F}$. By Lem. 5.2 the value s_0 is independent of the choice of F.

Proposition 5.5 *For any given $s \in \mathbb{R}$ one has*

$$F_r^s I_r = \sum_{F \in \mathcal{F}} F\ F_r^{s-s_0} D_r$$

as additive groups. In particular, the graded ideal $gr^{\cdot}_r I_r$ is generated by the principal symbols $\sigma(F),\ F \in \mathcal{F}$.

Proof: Write $\tilde{F}_r^s I_r := \sum_{F\in\mathcal{F}} F\ F_r^{s-s_0} D_r$ and similarly for $s+$ instead of s. The other inclusion being trivial we prove $F_r^s I_r \subseteq \tilde{F}_r^s I_r$ for all s. This reduces to show

$$\tilde{F}_r^s I_r \cap F_r^{s+} I_r \subseteq \tilde{F}_r^{s+} I_r. \tag{4}$$

Indeed, if $\lambda \in F_r^s I_r$, let $s' \leq s$ be a number such that $\lambda \in \tilde{F}_r^{s'} I_r$. Since $r \in p^{\mathbb{Q}}$ and K is discretely valued the filtration $F_r^{\cdot} I_r$ is essentially indexed by $\mathbb{Z}$. Using (4) we see that we may choose $s' = s$.

So let us prove (4). Let $\lambda \in \tilde{F}_r^s I_r \cap F_r^{s+} I_r$. Of all representations $\lambda = \sum_k F_k \lambda_k \in F_r^{s+} I_r$ with $F_k \in \mathcal{F}$, $F_k\lambda_k \in F_r^s I_r$ take one with minimal number t of nonzero summands $F_k\lambda_k$. It suffices to show that if $t > 1$ then, modulo a term T in $\tilde{F}_r^{s+} I_r$, there is a representation with $t-1$ nonzero summands. Putting summands contained in $F_r^{s+} I_r$ into T we may assume that $\deg(F_k\lambda_k) = s$ for all k. Then the hypothesis implies $\sum_k \sigma(F_k)\sigma(\lambda_k) = 0$. Using Lem. 5.3 and [Ka], 3.1, Ex. 12 (c) it easily follows that $\sigma(\lambda_k)$ is contained in the ideal generated by all $\sigma(F_{k'}),\ k' \neq k$. We now assume $t = 2$ (the general case follows in exactly the same fashion but with more notation). The element $\sigma(\lambda_1)$ thus lies in the ideal generated by $\sigma(F_2)$ whence $\lambda_1 = F_2\lambda_2' + R$ with $\lambda_2', R \in D_r$ and $||R||_r < ||\lambda_1||_r$. Hence

$$\lambda = F_1(F_2\lambda_2' + R) + F_2\lambda_2.$$

Now $||F_1F_2 - F_2F_1||_r < ||F_1F_2||_r$ since $gr^{\cdot}_r D_r$ is commutative. One the other hand, the $nd-d$ elements in $\mathcal{F}$ generate (over L) the kernel of the Lie algebra map $L \otimes_{\mathbb{Q}_p} \mathfrak{g}_{\mathbb{Q}_p} \to \mathfrak{g}_L,\ a \otimes \mathfrak{x} \mapsto a\mathfrak{x}$. It follows that $F_1F_2 - F_2F_1 = \sum_{F\in\mathcal{F}} Fc_F$ with $c_F \in L$ and so, together with Cor. 5.4, we obtain $|c_F| < ||F_1||_r$ for all

$F \in \mathcal{F}$. All in all we have $\lambda \equiv F_2(F_1\lambda_2' + \lambda_2)$ modulo terms in $\tilde{F}_r^{s+} I_r$ and $F_2(F_1\lambda_2' + \lambda_2) \in F_r^s I_r$. $\square$

Summarizing the results obtained so far yields the

Proposition 5.6 *There is an isomorphism of $gr^{\cdot} K$-algebras*

$$gr^{\cdot}_r D_{\bar{r}} \xrightarrow{\sim} (gr^{\cdot} K)[X_{11}, ..., X_{nd}]/(\{X_{ij}^{p^h} - \bar{v}_i X_{1j}^{p^h}\}_{i \geq 2, j \geq 1})$$

where $h \in \mathbb{N}_0$ depends only on r^κ. If $r^\kappa < p^{-\frac{1}{p-1}}$ then $h = 0$ and

$$gr^{\cdot}_r D_{\bar{r}} \simeq (gr^{\cdot} K)[X_{11}, ..., X_{1d}]$$

as $gr^{\cdot} K$-algebras where the isomorphism is obtained by the first map composed with the algebra homomorphism induced by $X_{ij} \mapsto \bar{v}_i X_{1j}$ for all $i \geq 2, j \geq 1$. In this case the norm $\|.\|_{\bar{r}}$ is multiplicative and $D_{\bar{r}}$ is an integral domain.

Since $D(G, K)$ embeds in any completion $D_{\bar{r}}$ there is the

Corollary 5.7 *The ring $D(G, K)$ is an integral domain.*

This generalizes the case $G := \mathfrak{o}$ appearing in [ST4], Cor. 3.7.

Recall that D_r equals a noncommutative power series ring in the elements $\mathbf{b}^\alpha$, $\alpha \in \mathbb{N}_0^{nd}$ where $b_{ij} = h_{ij} - 1$. Viewing the elements $\mathbf{b}^\alpha$ in $D_{\bar{r}}$ via $K[G] \subseteq D_{\bar{r}}$ we may consider their principal symbols inside $gr^{\cdot}_r D_{\bar{r}}$. Tracing through the definitions of the maps involved yields the

Corollary 5.8 *Let $r^\kappa < p^{-\frac{1}{p-1}}$. The isomorphism*

$$gr^{\cdot}_r D_{\bar{r}} \to (gr^{\cdot} K)[X_{11}, ..., X_{1d}]$$

maps $\sigma(b_{1j}) \mapsto X_{1j}$.

For an index $\beta \in \mathbb{N}_0^d$ (consisting of d components!) let us agree that $\mathbf{b}^\beta$ denotes the element $b_{11}^{\beta_1} ... b_{1d}^{\beta_d} \in K[G]$.

Proposition 5.9 *Let $r^\kappa < p^{-\frac{1}{p-1}}$. Every $\lambda \in D_{\bar{r}}$ has a convergent expansion*

$$\lambda = \sum_{\beta \in \mathbb{N}_0^d} d_\beta \mathbf{b}^\beta$$

with uniquely determined $d_\beta \in K$. Furthermore,

$$||\lambda||_{\bar{r}} = \sup_\beta ||d_\beta \mathbf{b}^\beta||_{\bar{r}} = \sup_\beta |d_\beta| r^{\kappa |\beta|}.$$

Proof: The norm $\|.\|_{\bar{r}}$ is multiplicative and so $gr^{\cdot}_r D_{\bar{r}}$ is freely generated as $gr^{\cdot} K$-module by the symbols $\{\sigma(\mathbf{b}^\beta)\}_{\beta\in\mathbb{N}_0^d}$ according to Cor. 5.8. This implies that the monomials $\mathbf{b}^\beta$ generate a dense K-subspace in $D_{\bar{r}}$ such that $||\lambda||_{\bar{r}} = \max_\beta ||d_\beta \mathbf{b}^\beta||_{\bar{r}}$ for any finite sum $\lambda = \sum_\beta d_\beta \mathbf{b}^\beta$ out of this submodule. Finally, Cor. 5.8 yields that $\sigma(b_{1j}) \in gr^{\cdot}_r D_r$ is mapped to $\sigma(b_{1j}) \in gr^{\cdot}_r D_{\bar{r}}$ by the canonical map $gr^{\cdot}_r D_r \to gr^{\cdot}_r D_{\bar{r}}$ whence $||b_{1j}||_r = ||b_{1j}||_{\bar{r}}$. $\square$

Remark: When $r^\kappa > p^{-\frac{1}{p-1}}$ it is easy to see that $gr^{\cdot}_r D_{\bar{r}}$ has nonzero nilpotents, in contrast to Cor. 5.8.

6 Lower p-series subalgebras

In this section we prepare the proof of the main result.

To begin with let G be an arbitrary uniform locally $\mathbb{Q}_p$-analytic group of dimension $\dim_{\mathbb{Q}_p} G = d$. Fix a sequence of topological generators $h_1, ..., h_d$. Let

$$G = G^0 \supseteq G^1 \supseteq G^2 \supseteq ...$$

be its lower p-series. By functoriality we obtain a series of subalgebras

$$D(G,K) = D(G^0,K) \supseteq D(G^1,K) \supseteq D(G^2,K) \supseteq ...$$

where the inclusions are topological embeddings with respect to Fréchet topologies ([Ko], Prop. 1.1.3). If $n \geq m$ then $D(G^m,K)$ is a finitely generated free (left or right) module over $D(G^n,K)$ on a basis any set of coset representatives for G^m/G^n. Since each G^m is a uniform pro-p group, generated by $h_1^{p^m}, ..., h_d^{p^m}$, the algebra $D(G^m,K)$ is Fréchet-Stein and equals a noncommutative power series ring in the elements $h_1^{p^m}-1, ..., h_d^{p^m}-1$. We denote the induced set of norms on $D(G^m,K)$ by $||.||_r^{(m)}$, $p^{-1} < 1 < 1$, $r \in p^{\mathbb{Q}}$. Abbreviate $||.||_r := ||.||_r^{(0)}$.

For clarity we recall two simple definitions. If I denotes a countable index set a family of pairwise different nonzero elements $(v_i)_{i\in I}$ in a normed K-vectorspace $(V, ||.||)$ is called *orthogonal* in V if one has $||v|| = \max_I |c_i|\,||v_i||$ for any convergent series $v = \sum_I c_i v_i$, $c_i \in K$. The family is called an *orthogonal basis* if any element of V can be written as such a convergent series.

Lemma 6.1 *Let $\Lambda = \{\lambda_i\}_{i\in I}$ be a countable family of pairwise different nonzero elements of $D_r(G,K)$. For any $\lambda_i \in \Lambda$ expand $\lambda_i = \sum_{\alpha\in\mathbb{N}_0^d} d_{i,\alpha}\mathbf{b}^\alpha$ and choose $\gamma_i \in \mathbb{N}_0^d$ with $||\lambda_i||_r = ||d_{i,\gamma_i}\mathbf{b}^{\gamma_i}||_r$. Suppose that for each λ_i the index γ_i is uniquely determined. If the map $\iota : \Lambda \to \mathbb{N}_0^d$, $\lambda_i \mapsto \gamma_i$ is injective resp. bijective the system Λ is orthogonal resp. an orthogonal basis in $D_r(G,K)$.*

Proof: This follows easily from [F], Lem. 1.4.1/2. □

Proposition 6.2 *Fix $m \geq 1$ in $\mathbb{N}$ and suppose $r^{\kappa p^{m-1}(p-1)} > p^{-1}$. Then $\|.\|_r$ restricts to $\|.\|_{r'}^{(m)}$ on the subring $D(G^m, K) \subseteq D(G, K)$ where $r' = r^{p^m}$.*

Proof: Abbreviate $R := D(G, K)$, $R^{(m)} := D(G^m, K)$ and use induction on m. Let $m = 1$. Put $b_i' := h_i^p - 1$, $\mathbf{b}'^\alpha := b_1'^{\alpha_1} ... b_d'^{\alpha_d}$, $\alpha \in \mathbb{N}_0^d$ and consider an arbitrary element

$$\lambda = \sum_{\alpha \in \mathbb{N}_0^d} d_\alpha \mathbf{b}'^\alpha$$

in $R^{(1)}$. To verify that $\|.\|_r$ restricts to $\|.\|_{r^p}^{(1)}$ on $R^{(1)}$ amounts to show, by definition of the norms $||.||_s^{(1)}$, that $\|\lambda\|_r = \sup_\alpha |d_\alpha| r^{p\kappa|\alpha|}$. To do this we first compute the norm $\|\mathbf{b}'^\alpha\|_r$. We have

$$b_i' = h_i^p - 1 = (b_i + 1)^p - 1 = b_i^p + \sum_{k=1,...,p-1} \binom{p}{k} b_i^k.$$

By $r^{\kappa(p-1)} > p^{-1}$ it follows

$$\|b_i^p\|_r = r^{\kappa p} > p^{-1} r^\kappa \geq p^{-1} r^{\kappa k} = |\binom{p}{k}| r^{\kappa k} = \|\binom{p}{k} b_i^k\|_r$$

for all $k = 1, ..., p-1$ and so $\|b_i'\|_r = r^{\kappa p}$. Hence, $\|\mathbf{b}'^\alpha\|_r = \prod_i \|b_i'\|_r^{\alpha_i} = r^{p\kappa|\alpha|}$.

Since the Fréchet topology on $R^{(1)}$ is stronger than the induced $||.||_r$-topology we may view $\lambda = \sum_\alpha d_\alpha \mathbf{b}'^\alpha$ as a convergent series in the K-Banach space $(D_r(G, K), ||.||_r)$. Then we are reduced to show that the family of pairwise different nonzero elements $\Lambda := \{\mathbf{b}'^\alpha\}_{\alpha \in \mathbb{N}_0^d}$ is orthogonal in $(D_r(G, K), ||.||_r)$. But this follows from Lem. 6.1: by our calculations γ_α is uniquely determined and equals $p\alpha$. Hence, $\iota : \mathbf{b}'^\alpha \mapsto p\alpha$ is injective.

Now let $m > 1$ and assume that the result holds true for all numbers strictly smaller than m and that $r^{\kappa p^{m-1}(p-1)} > p^{-1}$. Since the latter implies $r^{\kappa p^{m-2}(p-1)} > p^{-1}$, the induction hypothesis shows that $\|.\|_r$ on R restricts to $\|.\|_{r^{p^{m-1}}}^{(m-1)}$ on $R^{(m-1)}$.

Now G^m appears also as first step in the lower p-series of G^{m-1}. Hence, the induction hypothesis applies to these two groups: for $0 < s < 1$, every $\|.\|_s^{(m-1)}$ on $R^{(m-1)}$ restricts to $\|.\|_{s^p}^{(m)}$ on $R^{(m)}$ as long as $s^{\kappa(p-1)} > p^{-1}$. We choose $s = r^{p^{m-1}}$. Then $s^{\kappa(p-1)} = r^{\kappa p^{m-1}(p-1)} > p^{-1}$ and so $\|.\|_{r^{p^{m-1}}}^{(m-1)}$ restricts to $\|.\|_{r^{p^m}}^{(m)}$ on $R^{(m)}$. □

Denote by $D_{(r)}(G^m, K) \subseteq D_r(G, K)$ the norm closure of $D(G^m, K)$ inside the Banach algebra $D_r(G, K)$.

Lemma 6.3 *Assume* $r^{\kappa p^{m-1}(p-1)} > p^{-1}$*. Then* $D_r(G,K)$ *is a finite free (left or right)* $D_{(r)}(G^m,K)$*-module on a basis any system of coset representatives for* G/G^m*.*

Proof: Since the argument for right modules is the same we only prove the statement concerning left modules.

Put $R^{(m)} := D(G^m,K)$, $R_r^{(m)} := D_{(r)}(G^m,K)$, $R := R^{(0)}$, $R_r := R_r^{(0)}$. We again proceed via induction on m. Let $m = 1$ and $r^{\kappa(p-1)} > p^{-1}$.

As in the preceding proof put $b_i' := h_i^p - 1$ for all i. The elements $\mathbf{h}^\beta := h_1^{\beta_1} \cdots h_d^{\beta_d}$ with $\beta \in \{\alpha \in \mathbb{N}_0^d,\ \alpha_i < p\ \forall i\} =: \mathbb{N}_{0,<p}^d$ constitute a system $\mathcal{R}'$ of representatives for the cosets in G/G^1. Furthermore, the family of pairwise different nonzero elements

$$\Lambda := \{\mathbf{b}'^{\alpha}\,\mathbf{b}^{\beta}\}_{(\alpha,\beta)\in\mathbb{N}_0^d\times\mathbb{N}_{0,<p}^d}$$

is an orthogonal basis for $(R_r, ||.||_r)$. The latter follows from Lem. 6.1 in a way similar to the preceding proof observing that here, ι is given by the bijection $\mathbf{b}'^{\alpha}\,\mathbf{b}^{\beta} \mapsto p\alpha + \beta$. Since the elements $\mathbf{b}'^{\alpha}$, $\alpha \in \mathbb{N}_0^d$ constitute a topological K-basis for $R_r^{(1)}$ the definition of an orthogonal basis implies that R_r is a finite and free left $R_r^{(1)}$-module on the finite basis $\mathbf{b}^\beta$, $\beta \in \mathbb{N}_{0,<p}^d$. Mapping $\mathbf{b}^\beta \mapsto \mathbf{h}^\beta$ induces a left $R_r^{(1)}$-module isomorphism of R_r onto itself. The latter is thus finite free over $R_r^{(1)}$ on the basis $\mathcal{R}'$ and hence on any basis consisting of coset representatives for G/G^1.

Now let $m > 1$ and assume that the result holds true for all numbers strictly smaller than m. By the induction hypothesis we can assume that

$$R_r = \oplus_{h\in\mathcal{R}'}\, R_r^{(m-1)}\, h$$

where $\mathcal{R}'$ is a system of representatives for the cosets in G/G^{m-1}. But G^m appears also as first step in the lower p-series of G^{m-1} and so the induction hypothesis applies to these two groups as well: if the index $p^{-1} < s < 1$ in $p^{\mathbb{Q}}$ satisfies $s^{\kappa(p-1)} > p^{-1}$ then one has

$$R_{[s]}^{(m-1)} = \oplus_{h\in\mathcal{R}''} R_{[s]}^{(m)} h$$

where $\mathcal{R}''$ is a system of representatives for G^{m-1}/G^m, $R_{[s]}^{(m-1)}$ denotes the completion of $R^{(m-1)}$ via the norm $||.||_s^{(m-1)}$ and $R_{[s]}^{(m)}$ is the closure of $R^{(m)}$ inside this completion.

Now choose $s = r^{p^{m-1}}$. Then we obtain on the one hand $s^{\kappa(p-1)} = r^{\kappa p^{m-1}(p-1)} > p^{-1}$. On the other hand Prop. 6.2 implies that $||.||_r$ restricts on $R^{(m-1)}$ to the norm $||.||_{r^{p^{m-1}}}^{(m-1)}$. Hence

$$R_{[r^{p^{m-1}}]}^{(m-1)} = R_r^{(m-1)}, \quad R_{[r^{p^{m-1}}]}^{(m)} = R_r^{(m)}$$

and the proof is complete. $\square$

Now assume that G is a locally L-analytic group such that its scalar restriction G_0 is uniform. Endowing each G^m with the locally L-analytic structure as an open subgroup of G we obtain, again by functoriality, a series $(D(G^m, K))_m$ of Fréchet-Stein algebras the transition maps being topological embeddings. Write $\|.\|_{\bar{r}}^{(m)}$ for the set of quotient norms on $D(G^m, K)$. Again we abbreviate $\|.\|_{\bar{r}} := \|.\|_{\bar{r}}^{(0)}$. Moreover, let us write $res(.)$ resp. $q(.)$ for restricting a norm (from $D(G_0, K)$ or $D(G, K)$ to a subalgebra) resp. forming the quotient norm (with respect to any of the maps $D(G_0^m, K) \to D(G^m, K)$).

Corollary 6.4 *Assume $r^{\kappa p^{m-1}(p-1)} > p^{-1}$ and let $||.||_r$ be given on $D(G_0, K)$. The two norms $res(\|.\|_{\bar{r}})$ and $q \circ res(||.||_r)$ induce the same topology on $D(G^m, K)$. Writing $D_{(r)}(G^m, K)$ for the completion $D_r(G, K)$ becomes a finite and free (left or right) module over $D_{(r)}(G^m, K)$ with a basis any set of coset representatives for G/G^m.*

Proof: Let $\mathcal{R}$ be a system of coset representatives for G/G^m. By the preceding lemma $D_r(G_0, K)$ is a finite free Banach module over the noetherian Banach algebra $D_{(r)}(G_0^m, K)$. By [BGR], 3.7.3/3 the $||.||_r$-topology on $D(G_0, K)$ equals the direct sum topology with respect to $D(G_0, K) = \oplus_{h \in \mathcal{R}} D(G_0^m, K)h$ (where $D(G_0^m, K)$ carries the $res(||.||_r)$-topology). Furthermore, one has
$I(G_0, K) = \oplus_{h \in \mathcal{R}} I(G_0^m, K)h$ as a direct consequence of the definitions. Passing to completions and then to quotients yields the assertions. $\square$

7 Regularity

In this section we prove the main result. We start by recalling the notion of an Auslander regular ring (cf. [LVO], chap. III.)

Let R be an arbitrary associative unital ring. For any (left or right) R-module N the *grade* $j_R(N)$ is defined to be either the smallest integer l such that $\mathrm{Ext}_R^l(N, R) \neq 0$ or ∞. A left and right noetherian regular ring R is called *Auslander regular* if every finitely generated left or right R-module N satisfies the Auslander condition (AC): for any $l \geq 0$ and any R-submodule $L \subseteq \mathrm{Ext}_R^l(N, R)$ one has $j_R(L) \geq l$.

A commutative noetherian regular ring is Auslander regular ([LVO], III.2.4.3). A complete filtered ring whose graded ring is Auslander regular of global dimension d is Auslander regular and has global dimension $\leq d$ ([LVO], II.2.2.1, II.3.1.4, III.2.2.5). We deduce

Proposition 7.1 *Let G be a locally L-analytic group of dimension d that is uniform and satifies* (L). *If $r^\kappa < p^{-\frac{1}{p-1}}$ then $D_r(G,K)$ is an Auslander regular ring of global dimension $\leq d$.*

Proof: According to Prop. 5.6 the graded ring $gr^{\cdot}_r\, D_r(G,K)$ equals a polynomial ring over $gr^{\cdot} K$ in d variables. Since $gr^{\cdot} K$ equals Laurent polynomials in one variable we obtain, by the preceding remarks, the result with the bound $d+1$ on the global dimension. Using Prop. 5.9 we may deduce, by a computation similar to [ST5], (proof of) Lem. 4.8 that $gr^{\cdot}_r\, F^0_r D_r(G,K)$ is isomorphic to a polynomial ring $k[X_0,...,X_d]$ over the residue field k of K. Hence, the global dimension of $F^0_r D_r(G,K)$ is $\leq d+1$ by the above remarks. On the other hand, multiplication by p is zero on any simple module over $F^0_r D_r(G,K)$ ([LVO], I.3.5.5) whence $D_r(G,K) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} F^0_r D_r(G,K)$ has global dimension $\leq d$ ([MCR], 7.4.3/4). $\square$

We recall the following facts from general ring theory.

Lemma 7.2 *Let $R_0 \subseteq R_1$ be an extension of unital rings. Suppose there are units $b_1 = 1, b_2, ..., b_t \in R_1^\times$ which form a basis of R_1 as (left and right) R_0-module and which satisfy:*

1. *$b_i R_0 = R_0 b_i$ for any $1 \leq i \leq t$,*
2. *for any $1 \leq i,j \leq t$ there is $1 \leq k \leq t$ such that $b_i b_j \in b_k R_0$,*
3. *for any $1 \leq i \leq t$ there is $1 \leq l \leq t$ such that $b_i^{-1} \in b_l R_0$.*

Suppose t is invertible in R_0.

Then: R_0 is noetherian if and only if R_1 is noetherian. In this case both rings have the same global dimension.

Proof: E.g. [ST5], Lem. 8.8. $\square$

Corollary 7.3 *Keeping the assumptions R_0 is an Auslander regular ring if and only if this holds true for R_1.*

Proof: Assume that R_0 is Auslander regular. Let N be a finitely generated left or right R_1-module. We have a group isomorphism

$$\mathrm{Ext}^*_{R_1}(N, R_1) \xrightarrow{\sim} \mathrm{Ext}^*_{R_0}(N, R_0). \tag{5}$$

Indeed, using a projective resolution of N by finitely generated free R_1-modules this reduces to $* = 0$ and $N = R_1$. Denoting by $l : R_1 \to R_0$ the projection onto the b_1-component the map $\Phi \mapsto l \circ \Phi$ is the desired bijection. Now let $L \subseteq \mathrm{Ext}^l_{R_1}(N, R_1)$ be any R_1-submodule. Since R_1 is noetherian and N is finitely generated so is $\mathrm{Ext}^l_{R_1}(N, R_1)$ and L. Consider L as R_0-module. Then it is finitely generated and so from $L \subseteq \mathrm{Ext}^l_{R_0}(N, R_0)$ we deduce by (AC) for the finitely generated R_0-module N that $j_{R_0}(L) \geq l$. But this implies $j_{R_1}(L) \geq l$ by (5).

Conversely, suppose that R_1 is Auslander regular. Let N be a finitely generated left R_0-module and let $L \subseteq \mathrm{Ext}^l_{R_0}(N, R_0)$ be any right R_0-module. It is finitely generated by the same argument as above. Put $N_1 := R_1 \otimes_{R_0} N$ whence

$$L \otimes_{R_0} R_1 \subseteq \mathrm{Ext}^l_{R_0}(N, R_0) \otimes_{R_0} R_1 = \mathrm{Ext}^l_{R_1}(N_1, R_1).$$

By (AC) for N_1 we have $j_{R_1}(L \otimes_{R_0} R_1) \geq l$ and so $\mathrm{Ext}^k_{R_1}(L \otimes_{R_0} R_1, R_1) = 0$ for all $k < l$. One has

$$\mathrm{Ext}^k_{R_1}(L \otimes_{R_0} R_1, R_1) = R_1 \otimes_{R_0} \mathrm{Ext}^k_{R_0}(L, R_0)$$

as left R_1-modules and so $\mathrm{Ext}^k_{R_0}(L, R_0) = 0$ for all $k < l$ by faithful flatness of R_1. By definition we obtain $j_{R_0}(L) \geq l$. The proof for right modules being the same this shows that R_0 is Auslander regular. $\square$

Consider a number $p^{-1} < s < 1$ in $p^{\mathbb{Q}}$ such that $s^{\kappa} < p^{-\frac{1}{p-1}}$. Define S to be the set of all positive real p-power roots of such numbers. Clearly, $S \subseteq (p^{-1}, 1) \cap p^{\mathbb{Q}}$ and, most importantly, S contains a sequence $s \uparrow 1$.

Turning back to our compact locally L-analytic group G which is uniform satisfying (L) recall that we have a filtration of $D(G, K)$ by Fréchet-Stein subalgebras

$$D(G, K) \supseteq D(G^1, K) \subseteq D(G^2, K) \supseteq ...$$

coming via functoriality from the lower p-series $(G^m)_m$ of G.

It is clear from the definition that the subfamily of norms $||.||^{(m)}_{\bar{r}}$, $r \in S$ gives each $D(G^m, K)$ a Fréchet-Stein structure.

Lemma 7.4 *Let $r \in S$ and choose $p^{-1} < s < 1$, $s^{\kappa} < p^{-\frac{1}{p-1}}$ such that $r^{p^m} = s$ for some $m \in \mathbb{N}_0$. The norm $res(\|.\|_{\bar{r}})$ induces on $D(G^m, K)$ the same topology as the norm $||.||^{(m)}_{\bar{s}}$. Furthermore, $D_r(G, K)$ is a finite and free (left or right) module over the subring $D_s(G^m, K)$ on a basis any set of coset representatives for G/G^m. In particular, $D_r(G, K)$ is Auslander regular if and only if this is true for $D_s(G^m, K)$ and both rings have the same global dimension.*

Proof: The last two claims follow from the second according to Lem. 7.2 and Cor. 7.3.

Now $p^{-1} < s = r^{p^m}$ implies $p^{-1} < r^{\kappa p^{m-1}(p-1)}$. Thus, Cor. 6.4 yields that $res(\|.\|_{\bar{r}})$ induces the same topology as $q \circ res(||.||_r)$ on the subring $D(G^m, K) \subseteq D(G, K)$. Moreover, $D_r(G, K)$ is a finite free module over the closure $D_{(r)}(G^m, K)$ with basis any set of coset representatives for G/G^m. But, according to Prop. 6.2 $res(||.||_r) = ||.||^{(m)}_{r^{p^m}}$ on $D(G^m_0, K)$.

Thus, $res(\|.\|_{\bar{r}})$ induces in fact the same topology as $q(||.||^{(m)}_{r^{p^m}}) = ||.||^{(m)}_{\bar{s}}$ on $D(G^m, K)$ whence $D_{(r)}(G^m, K) = D_s(G^m, K)$. □

Proposition 7.5 *Consider the usual family* $\|.\|_{\bar{r}}$*,* $p^{-1} < r < 1$*,* $r \in p^{\mathbb{Q}}$ *of quotient norms on* $D(G, K)$*. If* $r \in S$ *then* $D_r(G, K)$ *is Auslander regular and of global dimension* $\leq d$*.*

Proof: Let $r \in S$. Choose $m \in \mathbb{N}_0$ such that $r^{p^m} =: s$ satisfies $p^{-1} < s < 1$ in $p^{\mathbb{Q}}$ with $s^{\kappa} < p^{-\frac{1}{p-1}}$. Then $D_s(G^m, K)$ is an Auslander regular ring of global dimension $\leq d$ according to Prop. 7.1. Thus, by the preceding lemma the same is true for $D_r(G, K)$. □

Theorem 7.6 *Let* G *be a compact locally* L*-analytic group of dimension* d*. Then* $D(G, K)$ *admits a two-sided* K*-Fréchet-Stein structure consisting of Auslander regular Banach algebras of global dimension* $\leq d$*.*

Proof: Choose an open normal subgroup H which is uniform and satisfies (L). Consider the set of norms $||.||_r$, $r \in S$ on $D(H_0, K)$ and let $q(||.||_r)$, $r \in S$ be the family of quotient norms on $D(H, K)$. Choose a system $\mathcal{R}$ of representatives g_i containing 1 for the cosets in G/H and use the decomposition $D(G_0, K) = \oplus_i D(H_0, K)g_i$ to define on $D(G_0, K)$ for each $r \in S$ the maximum norm:

$$|| \sum_i \lambda_i \, g_i ||_r := \max_i \, ||\lambda_i||_r.$$

As explained in section 3 the corresponding completions $D_r(G_0, K)$ resp. their quotients $D_r(G, K)$ constitute a two-sided K-Fréchet-Stein structure on $D(G_0, K)$ resp. $D(G, K)$. Now a direct calculation gives that such a quotient norm $\|.\|_{\bar{r}}$ on $D(G, K)$ restricts to $q(||.||_r)$ on the subring $D(H, K)$. Hence, the ring extension $D_r(H, K) \subseteq D_r(G, K)$ satisfies the assumptions of Cor. 7.3. But the preceding proposition yields that $D_r(H, K)$ is Auslander regular of global dimension $\leq d$. □

8 Dimension and duality

In this last section G denotes an *arbitrary* locally L-analytic group. We indicate two applications of our results.

For any compact open subgroup $H \subseteq G$ the algebra $D(H, K)$ is endowed with the two-sided Fréchet-Stein structure exhibited in the last theorem. It then satisfies the axiom (DIM) formulated in [ST5], Sect. 8. In [loc.cit.], Sect. 6 the authors introduce the abelian category $\mathcal{C}_G$ of *coadmissible* (left) $D(G,K)$-modules. We do not recall the precise definition here. As an immediate consequence of (DIM) the codimension theory, as developed in [loc.cit.] over $\mathbb{Q}_p$, is now available on $\mathcal{C}_G$. In particular, any nonzero $M \in \mathcal{C}_G$ has a well-defined codimension $j_{D(H,K)}(M)$, independent of the choice of H and bounded above by $\dim_L G$. A coadmissible module which is either zero or has maximal codimension equal to $\dim_L G$ is called *zero-dimensional.*

As an application we prove that coadmissible modules coming from smooth or, more general, $U(\mathfrak{g})$-finite G-representations (as studied in [ST1]) are zero-dimensional. These are natural generalizations of the theorems [ST5], 8.12 and 8.15. For the basic properties of $\mathcal{C}_G$ and its codimension which will be used in the following we refer to [loc.cit.].

Proposition 8.1 *Consider a d-dimensional locally L-analytic group H which is uniform and satisfies* (L) *with corresponding L-basis $\mathfrak{x}_1, ..., \mathfrak{x}_d$ of $\mathfrak{g}_L$, the Lie algebra of H. Let $\lambda_1, ..., \lambda_d$ be elements of $U(\mathfrak{g}_L)$, the enveloping algebra, such that $\lambda_j = P_j(\mathfrak{x}_j)$ where P_j is a nonzero polynomial in $L[X]$ with $P_j(0) = 0$. Denote by J the left ideal of $D(H,K)$ generated by the λ_j. The coadmissible module $D(H,K)/J$ is zero-dimensional.*

Proof: Since J is finitely generated the $D(H,K)$-modules J and $D(H,K)/J$ are coadmissible. It thus suffices to fix $r \in S$ and prove

$$j_{D_r(H,K)}\left(D_r(H,K)/D_r(H,K)J\right) \geq d.$$

By Lem. 7.4 we have with $R_1 := D_r(H,K)$, $R_0 := D_s(H^m,K)$ that R_1 is free as R_0-module on the finite basis $\mathcal{R}$. Here, $m \in \mathbb{N}_0$ is appropriately chosen, $s = r^{p^m}$ and $\mathcal{R}$ is a finite system of representatives for the cosets in H/H^m. According to (the proof of) Cor. 4.5 the uniform group H^m satisfies (L) with the L-basis $p^m\mathfrak{x}_j$. Let $h_{11}, ..., h_{nd}$ denote the induced minimal ordered generating system for H^m and put as usual $b_j := h_{1j} - 1$. According to Prop. 5.9 the elements $\mathbf{b}^\alpha$, $\alpha \in \mathbb{N}_0^d$ are a topological K-basis for the Banach space R_0 and moreover, orthogonal with respect to $||.||_{\bar{s}}^{(m)}$. As graded ring we have $gr^{\cdot}_s R_0 = (gr^{\cdot} K)[\sigma(b_1), ..., \sigma(b_d)]$ where σ denotes the principal symbol map.

Since $H^m \subseteq H$ is open we may consider the left ideal J_0 generated by $\lambda_1, ..., \lambda_d$ inside R_0. Abbreviate $J_1 := D_r(H,K)J$ and use faithful flatness of R_1 over R_0 to obtain $j_{R_1}(R_1/J_1) = j_{R_0}(R_0/J_0)$. Since R_0 is Zariski with Auslander regular graded ring the right-hand side equals $j_{gr^{\cdot}_s R_0}(gr^{\cdot}_s R_0/gr^{\cdot}_s J_0)$ according to [LVO], III.2.5.2. By standard commutative algebra the commutative noetherian regular and catenary domain $gr^{\cdot}_s R_0$ is Cohen-Macaulay with respect to Krull dimension (on the category of finitely generated modules) ([BH], Cor. 3.5.11). Since $gr^{\cdot}_s R_0$ has Krull dimension $d+1$ it therefore suffices to show that the finitely generated module $gr^{\cdot}_s R_0/gr^{\cdot}_s J_0$ has Krull dimension ≤ 1.

Now identifying the Lie algebras of H^m and H_0^m over $\mathbb{Q}_p$ we have $p^m\mathfrak{x}_j = \log(1+b_j)$ in R_0. In $gr_s^{\cdot} R_0$ we obtain $\sigma(\log(1+b_j)) = \sigma(b_j)$ by definition of the norm $||.||_{\bar{s}}^{(m)}$ and the fact that $s^\kappa < p^{-\frac{1}{p-1}}$. Since $\sigma(p^m) \in gr^{\cdot} L$ is a unit and since $P_j(0) = 0$ we obtain

$$\sigma(\lambda_j) = \sigma(P_j(\mathfrak{x}_j)) = P_j'(\sigma(b_j))$$

with some nonconstant polynomial $P_j' \in (gr^{\cdot} L)[X]$. Since $\sigma(\lambda_j) \in gr_s^{\cdot} J_0$ we have a surjection

$$(gr^{\cdot} K)[\sigma(b_1)]/(\sigma(\lambda_1)) \otimes_{gr^{\cdot} K} \dots \otimes_{gr^{\cdot} K} (gr^{\cdot} K)[\sigma(b_d)]/(\sigma(\lambda_d)) \to gr_s^{\cdot} R_0/gr_s^{\cdot} J_0.$$

Each $(gr^{\cdot} K)[\sigma(b_j)]/(\sigma(\lambda_j))$ is finitely generated as a module over the one-dimensional ring $gr^{\cdot} K$ and hence, so is $gr_s^{\cdot} R_0/gr_s^{\cdot} J_0$. Then $gr_s^{\cdot} R_0/gr_s^{\cdot} J_0$ has Krull dimension ≤ 1. □

Theorem 8.2 *Let $M \in \mathcal{C}_G$. If the action of the universal enveloping algebra $U(\mathfrak{g}_L)$ is locally finite, i.e., if $U(\mathfrak{g}_L)x$, for any $x \in M$, is a finite dimensional L-vectorspace then M is zero-dimensional.*

Proof: Fix an open subgroup $H \subseteq G$ that is uniform satisfying (L) and let $x \in M$. It suffices to check that the cyclic module $D(H,K)x \in \mathcal{C}_H$ is zero-dimensional. Let $\mathfrak{x}_1, ..., \mathfrak{x}_d$ be a basis of $\mathfrak{g}_L$ realising condition (L) for H. Write $D(H,K)x = D(H,K)/J_1$ with some left ideal J_1 of $D(H,K)$ and let J_0 be the kernel ideal of $U(\mathfrak{g}_L) \to U(\mathfrak{g}_L)x$. Then $J_0 \subseteq J_1$ and the left ideal J_0 of $U(\mathfrak{g}_L)$ has, by assumption, finite codimension in $U(\mathfrak{g}_L)$. Hence, there is a left ideal $J \subseteq J_1$ in $D(H,K)$ satisfying the assumptions of the preceding proposition. Thus $D(H,K)/J$ is zero-dimensional and hence, so is its quotient $D(H,K)/J_1$. □

A smooth G-representation V is called *admissible-smooth* if, for any compact open subgroup $H \subseteq G$, the vector subspace V^H of H-invariant vectors in V is finite dimensional. An admissible-smooth G-representation equipped with the finest locally convex topology is admissible in the sense that its strong dual lies in $\mathcal{C}_G$ ([ST5], Thm. 6.6).

Corollary 8.3 *If V is an admissible-smooth G-representation then the corresponding coadmissible $D(G,K)$-module is zero-dimensional.*

Proof: By smoothness the derived Lie algebra action on V is trivial. The corresponding coadmissible module M is the strong dual V_b' with $D(G,K)$-action induced by the contragredient G-action. Thus $U(\mathfrak{g}_L)x = Lx$ for all $x \in M$ and the preceding theorem applies. □

Theorem 8.4 *Let G be compact and $M \in \mathcal{C}_G$. If the K-vectorspace*

$$M_r := D_r(G, K) \otimes_{D(G,K)} M$$

is finite dimensional for all $r \in S$ then M is zero-dimensional.

Proof: Let $H \subseteq G$ be an open uniform subgroup satisfying (L). Letting $x \in M$ it suffices to prove that $D(H, K)x \in \mathcal{C}_H$ is zero-dimensional. Write $D(H, K)x = D(H, K)/J$, J some left ideal and fix $r \in S$. Since $D_r(H, K)$ is flat over $D(H, K)$ according to [ST5], Remark 3.2 we have an inclusion of $D_r(H, K)$-modules

$$D_r(H, K)/D_r(H, K)J = D_r(H, K) \otimes_{D(H,K)} D(H, K)x \longrightarrow M_r$$

and hence the left-hand side is finite-dimensional over K. If $\mathfrak{g}_L$ denotes the Lie algebra the left ideal $J_0 := (K \otimes_L U(\mathfrak{g}_L)) \cap (D_r(H, K)J)$ has therefore finite K-codimension in $K \otimes_L U(\mathfrak{g}_L)$. As in the proof of the preceding theorem $D(H, K)/D(H, K)J_0$ and also $D(H, K)/J$ are zero-dimensional. $\square$

As a second consequence let us remark that the duality theory for admissible locally analytic representations (in the sense of [ST6]) is now completely available over the base field L. Thus, also for locally L-analytic groups G, the duality functor (defined on the bounded derived category of $D(G, K)$-modules with coadmissible cohomology) is an anti-involution. Due to the presence of the codimension the category $\mathcal{C}_G$ is filtered by abelian subquotient categories and the functor is computed as a particular Ext-group on each of them.

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