INTERMEDIATE EXTENSIONS AND CRYSTALLINE DISTRIBUTION ALGEBRAS

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Abstract. Let G be a connected split reductive group over a complete discrete valuation ring of mixed characteristic. We use the theory of intermediate extensions due to Abe-Caro and arithmetic Beilinson-Bernstein localization to classify irreducible modules over the crystalline distribution algebra of G in terms of overconvergent isocrystals on locally closed subspaces in the flag variety of G. We treat the case of SL_2 as an example.

Contents

1. Introduction

Let $\mathfrak{o}$ denote a complete discrete valuation ring of mixed characteristic $(0, p)$, with fraction field L and perfect residue field k. Let G be a connected split reductive group over $\mathfrak{o}$ with L-Lie algebra $\mathfrak{g} = Lie(G) \otimes \mathbb{Q}$.

In [39] we have introduced and studied the crystalline distribution algebra $D^\dagger(\mathcal{G})$ associated to the p-adic completion $\mathcal{G}$ of G. It is a certain weak completion of the classical universal enveloping algebra $U(\mathfrak{g})$. The interest in the algebra $D^\dagger(\mathcal{G})$ comes at least from two sources. On the one hand, it has the universal property to act as global arithmetic differential operators (in the sense of Berthelot [6]) on any formal $\mathfrak{o}$-scheme which has a $\mathcal{G}$-action. On the other hand, $D^\dagger(\mathcal{G})$ is canonically isomorphic to Emerton's analytic distribution algebra $D^{an}(G^\circ)$ as introduced in [26]. Here, G° equals the rigid-analytic generic fibre of the formal completion of G along its unit section. Analytic distribution algebras are useful tools to study locally analytic representations p-adic Lie groups. Let $G(L)$ be the group of L-valued points of G and let $G(n)^\circ$ be the n-th rigid-analytic congruence subgroup of G (with $G(0)^\circ = G^\circ$). Any irreducible admissible locally analytic $G(L)$-representation V has an infinitesimal character θ and a level n. The latter equals the least natural number $n \geqslant 0$ such that $V_{G(n)^\circ-\mathrm{an}} \neq 0$, i.e. such that V contains a nonzero $G(n)^\circ$-analytic vector. The dual space $(V_{G(n)^\circ-\mathrm{an}})'$ is naturally a module over the ring $D^{an}(G(n)^\circ)$. Coherent modules over a central reduction like $D^{an}(G(n)^\circ)_\theta$ can be viewed as a p-adic local data, in analogy to classical Harish-Chandra modules for admissible representations of real-analytic Lie groups.

In this article, we only consider the simplest case: representations of level zero and with trivial infinitesimal character θ_0. We then propose to study the irreducible modules over the ring $D^\dagger(\mathcal{G})_{\theta_0}$. Our approach will be geometric through some crystalline version of localization, similar to the classical procedure of localizing $U(\mathfrak{g})$-modules. Recall that, in the classical setting of $U(\mathfrak{g})$-modules, a combination of the Beilinson-Bernstein localization theorem over the flag variety of $\mathfrak{g}$ together with the formalism of intermediate extensions [3, 13, 35] produces a geometric classification of many irreducible modules, namely those which localize to D-modules which are *holonomic*.

Let in the following $B \subset G$ be a Borel subgroup scheme. In [40] we have established an analogue of the Beilinson-Bernstein theorem for the sheaf of arithmetic differential operators $\mathscr{D}^\dagger_{\mathcal{P}}$ on the formal completion $\mathcal{P}$ of the flag scheme $P = G/B$: one has a canonical isomorphism $H^0(\mathcal{P}, \mathscr{D}^\dagger_{\mathcal{P}}) \simeq D^\dagger(\mathcal{G})_{\theta_0}$ and the global sections functor $H^0(\mathcal{P}, -)$ furnishes an equivalence between the category of coherent $\mathscr{D}^\dagger_{\mathcal{P}}$-modules and coherent $D^\dagger(\mathcal{G})_{\theta_0}$-modules. A quasi-inverse is given by the adjoint functor $\mathscr{L}oc(M) = \mathscr{D}^\dagger_{\mathcal{P}} \otimes_{D^\dagger(\mathcal{G})_{\theta_0}} M$. This allows to pass back and forth between modules over $D^\dagger(\mathcal{G})_{\theta_0}$ and sheaves on $\mathcal{P}$.

On the other hand, Abe-Caro have recently developed a theory of weights in p-adic cohomology [1]. On the way, they also developed a formalism of intermediate extensions for arithmetic $\mathscr{D}$-modules. Our aim is to use a combination of Abe-Caro's theory, specialized to the flag variety, and localization to obtain classification results for irreducible $D^\dagger(\mathcal{G})$-modules.

We emphasize straightaway that our results are arithmetic analogues of classical results on algebraic D-modules on the complex flag variety [35]. However, the arithmetic setting requires much more care, since the functors in question (direct images, intermediate extensions etc.) are *not* straightforward generalizations of the classical functors and much more subtle. Moreover, we consequently work in absence of Frobenius structures. Since many foundational results on arithmetic $\mathscr{D}$-modules and p-adic cohomology *do* contain Frobenius structures as a standard hypothesis, our level of generality requires several new arguments in many places.

To be more precise, we introduce some more notation. A nonzero $D^\dagger(\mathcal{G})_{\theta_0}$-module M is called *geometrically overholonomic*, if its localization $\mathscr{L}oc(M)$ is overholonomic after any base change. We then consider the parameter set of *pairs* $(Y,\mathcal{E})$ where $Y \subseteq \mathcal{P}_s$ is a connected smooth locally closed subvariety of the special fibre $\mathcal{P}_s$ with Zariski closure X, and $\mathcal{E}$ is an irreducible overconvergent isocrystal on the couple $\mathbb{Y} = (Y,X)$, which is overholonomic after any base change.[1] Two pairs are equivalent $(Y,\mathcal{E}) \sim (Y',\mathcal{E}')$ if $X = X'$ and the two isocrystals $\mathcal{E},\mathcal{E}'$ coincide on an open dense subset of X. Given such a pair $(Y,\mathcal{E})$ we put

$$\mathcal{L}(Y,\mathcal{E}) := v_{!+}(\mathcal{E})$$

where $v : \mathbb{Y} \to \mathbb{P} = (\mathcal{P}_s,\mathcal{P}_s)$ is the immersion of couples associated with Y and $v_{!+}$ is its arithmetic intermediate extension functor. We then have, cf. Thm. 4.2.3:

Theorem 1. *The correspondence* $(Y,\mathcal{E}) \mapsto H^0(\mathcal{P},\mathcal{L}(Y,\mathcal{E}))$ *induces a bijection*

$$\{\textit{pairs } (Y,\mathcal{E})\}/\!\sim\ \xrightarrow{\simeq} \{\textit{irreducible geometrically overholonomic } D^\dagger(\mathcal{G})_{\theta_0}\textit{-modules}\}/\!\simeq$$

For example, each couple $\mathbb{Y}$ is equipped with the constant overconvergent isocrystal $\mathcal{O}_{\mathbb{Y}}$. If Z is a divisor in $\mathcal{P}_s$ and $\mathcal{U} = \mathcal{P}\backslash Z$ with $\mathbb{Y} = (\mathcal{U}_s,\mathcal{P}_s)$, then $\mathcal{O}_{\mathbb{Y}} = \mathcal{O}_{\mathcal{P},\mathbb{Q}}({}^\dagger Z)$ equals functions on $\mathcal{U}$ with overconvergent singularities along Z. In general, if Y admits a formal lift with connected rigid-analytic generic fibre, then $\mathcal{O}_{\mathbb{Y}}$ is irreducible and corresponds therefore to an irreducible overholonomic $D^\dagger(\mathcal{G})_{\theta_0}$-module.

In general, we expect that many $D^\dagger(\mathcal{G})_{\theta_0}$-modules, in particular those which come from admissible $G(L)$-representations, are in fact geometrically overholonomic. As an example, we treat the case of highest weight modules (but there are many more, already in dimension one, cf. Theorem 3 below). We show that the central block of the classical BGG category $\mathcal{O}_0$ embeds, via the base change $U(\mathfrak{g}) \to D^\dagger(\mathcal{G})$, fully faithfully into the category of coherent $D^\dagger(\mathcal{G})$-modules (cf. Thm. 5.1.7). It is well-known that the irreducible modules in $\mathcal{O}_0$ are parametrized by the Weyl group elements $w \in W$ via $L(w) := L(-w(\rho)-\rho)$ where ρ denotes half the sum over the positive roots and where $L(-w(\rho)-\rho)$ denotes

[1] This extra condition is automatic, if $\mathcal{E}$ has a Frobenius structure.

the unique irreducible quotient of the Verma module with highest weight $-w(\rho) - \rho$. We write

$$L^\dagger(w) := D^\dagger(\mathcal{G}) \otimes_{U(\mathfrak{g})} L(w)$$

for its crystalline counterpart. On the other hand, let

$$Y_w := BwB/B \subset P = G/B$$

be the Bruhat cell in P associated with $w \in W$ and let X_w be its Zariski-closure, a Schubert scheme. We have the couple $\mathbb{Y}_w = (Y_{w,s}, X_{w,s})$ and the immersion $v : \mathbb{Y}_w \to \mathbb{P}$. Our second main result is the following, cf. Thm. 5.1.9:

Theorem 2. *Let $w \in W$. One has a canonical $\mathscr{D}^\dagger_{\mathcal{P}}$-linear isomorphism*

$$\mathscr{L}oc(L^\dagger(w)) \simeq v_{!+}(\mathcal{O}_{\mathbb{Y}_w}).$$

The $D^\dagger(\mathcal{G})$-module $L^\dagger(w)$ is geometrically overholonomic.

In the final section, we discuss in more detail the example $G = \mathrm{SL}_2$. In this case, P equals the projective line over $\mathfrak{o}$ and we show that any irreducible coherent $\mathscr{D}^\dagger_{\mathcal{P}}$-module is in fact holonomic. This implies that any irreducible $D^\dagger(\mathcal{G})_{\theta_0}$-module is geometrically holonomic (i.e. localizes to a holonomic module). We explain the difference between holonomic and overholonomic in this case. In this case, theorem 1 gives a classification in terms of isocrystals on either a closed point y of $\mathbb{P}^1_k$ or an open complement Y of finitely many closed points $Z = \{y_1, ..., y_n\}$ of $\mathbb{P}^1_k$. In the first case, the point is a complete invariant. For example, the point $y = \infty$ corresponds to $L^\dagger(-2\rho)$. In the second case, the empty divisor $Z = \varnothing$ corresponds to the trivial representation. For a non-empty Z, we may suppose that all its points $y_1, ..., y_n$ are k-rational with $y_1 = \infty$. There are then two extreme cases

$$Y = \mathbb{A}^1_k \quad \text{and} \quad Y = \mathbb{P}^1_k \backslash \mathbb{P}^1(k),$$

the affine line and Drinfeld's upper half plane, respectively. We illustrate the two by means of two "new" examples. In the case $Y = \mathbb{A}^1_k$ we assume that L contains the p-th roots of unity μ_p and we choose an element $\pi \in \mathfrak{o}$ with

$$\mathrm{ord}_p(\pi) = 1/(p-1).$$

We let $\mathscr{L}_\pi$ be the coherent $\mathscr{D}^\dagger_{\mathcal{P}}$-module defined by the *Dwork overconvergent F-isocrystal* on $\mathbb{Y}$ associated with π. On the other hand, we let $\mathfrak{n} = L.e$ be the nilpotent radical of $Lie(B)$, where $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Let $\eta : \mathfrak{n} \to L$ be a nonzero character and consider Kostant's *standard Whittaker module*

$$W_{\theta_0,\eta} := U(\mathfrak{g}) \otimes_{Z(\mathfrak{g}) \otimes U(\mathfrak{n})} L_{\theta_0,\eta}$$

with character η and infinitesimal character θ_0 , cf. [42, (3.6.1)] for its original definition over the complex numbers. It is an irreducible $U(\mathfrak{g})$-module [11, Lem. 5.3], but *not* a highest weight module, i.e. it does not lie in $\mathcal{O}_0$. We write

$$W^\dagger_{\theta_0,\eta} := D^\dagger(\mathcal{G}) \otimes_{U(\mathfrak{g})} W_{\theta_0,\eta}$$

for its crystalline counterpart. Our third main result is the following, cf. 5.2.3:

Theorem 3. *Let $\eta(e) := \pi$. There is a canonical $\mathscr{D}^\dagger_{\mathcal{P}}$-linear isomorphism*

$$\mathscr{L}oc(W^\dagger_{\theta_0,\eta}) \xrightarrow{\simeq} \mathscr{L}_\pi.$$

The crystalline Whittaker module $W^\dagger_{\theta_0,\eta}$ is geometrically overholonomic.

The theorem shows, in particular, that the Dwork isocrystal $\mathscr{L}_\pi$ is *algebraic* in the sense that it comes from an algebraic $\mathscr{D}_{\mathbb{P}^1_L}$-module, namely $\mathrm{Loc}(W_{\theta_0,\eta})$, by extension of scalars $\overline{\mathscr{D}}_{\mathbb{P}^1_L} \to \mathscr{D}^\dagger_{\mathcal{P}}$. However, the holonomic $\mathscr{D}_{\mathbb{P}^1_L}$-module $\mathrm{Loc}(W_{\theta_0,\eta})$ is not regular, but has an irregular singularity at infinity.

We discuss an example in the Drinfeld case, where $Y = \mathbb{P}^1_k \backslash \mathbb{P}^1(k)$. We identify $k = \mathbb{F}_q$. We assume that L contains the cyclic group μ_{q+1} of $(q+1)$-th roots of unity. The space Y admits a distinguished unramified Galois covering $u : Y' \to Y$ with Galois group μ_{q+1}, given by the so-called *Drinfeld curve*

$$Y' = \Big\{(x,y) \in \mathbb{A}^2_k \mid xy^q - x^q y = 1\Big\}.$$

The latter admits a smooth and projective compactification $\overline{Y'}$. The covering map u extends to a smooth and tamely ramified morphism $u : \overline{Y'} \to \mathbb{P}^1_k$ which maps the boundary bijectively to $Z = \mathbb{P}^1(k)$. We denote by $u : \mathbb{Y}' \to \mathbb{Y}$ the morphism of couples induced by u in this situation and we let

$$\mathcal{E} = \mathbb{R}^\bullet u_{\mathrm{rig},*}\mathcal{O}_{\mathbb{Y}'}$$

be the relative rigid cohomology sheaf. Using results of Grosse-Klönne [29], we show that $\mathcal{E}$ admits an isotypic decomposition into overconvergent F-isocrystals $\mathcal{E}(j)$ on $\mathbb{Y}$ of rank one. In particular, each pair $(Y, \mathcal{E}(j))$ corresponds in the classification of theorem 1 to an irreducible geometrically overholonomic $D^\dagger(\mathcal{G})_{\theta_0}$-module $H^0(\mathcal{P}, v_{!+}\mathcal{E}(j))$.

We do not know whether the modules $H^0(\mathcal{P}, v_{!+}\mathcal{E}(j))$ are *algebraic*, in the sense that they arise by base change from irreducible $U(\mathfrak{g})$-modules. If algebraic, to which class do they belong? We recall that irreducible $U(\mathfrak{g})$-modules fall into three classes: highest weight modules, Whittaker modules and a third class whose objects (with a fixed central character) are in bijective correspondence with similarity classes of irreducible elements of a certain localization of the first Weyl algebra [11]. We plan to come back to these questions in future work.

We would like to warmly thank Daniel Caro for his quick and precise answers to our questions on overholonomic modules and on several other related problems for arithmetic D-modules.

Notations and Conventions. In this article, $\mathfrak{o}$ denotes a complete discrete valuation ring of mixed characteristic $(0,p)$. We let L be its fraction field and k its residue field, which is assumed to be perfect. We suppose that there exists a lifting of the Frobenius of k to $\mathfrak{o}$. We denote by ϖ a uniformizer of $\mathfrak{o}$. All formal schemes $\mathfrak{X}$ over $\mathfrak{o}$ are assumed to be locally noetherian and such that $\varpi\mathcal{O}_{\mathfrak{X}}$ is an ideal of definition. Without further mentioning, all occuring modules will be left modules.

2. Overholonomic modules and intermediate extensions

For a smooth formal $\mathfrak{o}$-scheme $\mathfrak{X}$ we denote by $\mathscr{D}^{\dagger}_{\mathfrak{X}}$ the sheaf of arithmetic differential operators on $\mathfrak{X}$ (with p inverted). We refer to [6] for the basic features of the category of $\mathscr{D}^{\dagger}_{\mathfrak{X}}$-modules.

2.1. **Overholonomic modules.** We introduce the framework of overholonomic complexes of arithmetic $\mathscr{D}$-modules (without Frobenius structure), stable after any base change following Caro [20], Abe-Caro [1], [2].

Recall that a *variety* over some field k is a reduced, separated, k-scheme of finite type.
A *frame* $(Y, X, \mathcal{P})$ is the data consisting of a separated and smooth formal scheme $\mathcal{P}$ over $\mathfrak{o}$, a closed subvariety X of its special fibre $\mathcal{P}_s$, and an open subscheme Y of X. A *morphism* between two such frames is the data $u = (b, a, f)$ consisting of morphisms $b : Y' \to Y, a : X' \to X, f : \mathcal{P}' \to \mathcal{P}$ such that f induces b and a. A *l.p. frame* $(Y, X, \mathcal{P}, \mathcal{Q})$ is the data of a proper and smooth formal scheme $\mathcal{Q}$ over $\mathfrak{o}$, an open formal subscheme $\mathcal{P} \subset \mathcal{Q}$ such that $(Y, X, \mathcal{P})$ is a frame. A *morphism* of l.p frames is defined in analogy to a morphisms of frames. It is called *complete* if the morphism $a : X' \to X$ is proper.
A *couple* $\mathbb{Y}$ is the data (Y, X) consisting of a k-variety X and an open subscheme $Y \subset X$ such that there exists a l.p. frame of the form $(Y, X, \mathcal{P}, \mathcal{Q})$. A *morphism of couples* is the data $u = (b, a)$ consisting of morphisms $b : Y' \to Y, a : X' \to X$ such that b is induced by a. It is called *complete* if a is proper. Let P be a property of morphisms of schemes. One says that u is c-P if u is complete and b satisfies P. For all this, cf. [1, 1.1].

Denote by $\mathcal{P}$ a smooth and proper formal scheme over $\mathfrak{o}$.

Let $\mathcal{E}$ be a complex of overholonomic $\mathscr{D}^{\dagger}_{\mathcal{P}}$-modules, as introduced by Caro [16, 3.1]. Following [2, 1], we say that $\mathcal{E}$ is overholonomic after any base change if the following is true. For any morphism $k \to k'$ of perfect fields, denote $\mathfrak{o}' = \mathfrak{o} \otimes_{W(k)} W(k')$ (where $W(k)$, resp. $W(k')$ is the Witt vector ring of k, resp. k'), f the canonical morphism $\mathcal{P}' = \mathcal{P} \times_{\operatorname{Spf} \mathfrak{o}} \operatorname{Spf} \mathfrak{o}' \to \mathcal{P}$. Then $\mathcal{E}$ is *stable by any base change* if for any such morphism $k \to k'$, $f^*\mathcal{E}$ is a complex of overholonomic modules. This category is stable by all cohomological operations except (maybe) by tensor product.
We denote by $D^b_{\mathrm{ovhol}}(\mathscr{D}^{\dagger}_{\mathcal{P}})$ the triangulated category of complexes of overholonomic $\mathscr{D}^{\dagger}_{\mathcal{P}}$-modules. Note that by [20, Exemple 3.2.2], an element of the triangulated category of complexes of overholonomic $\mathscr{D}^{\dagger}_{\mathcal{P}}$ endowed with a Frobenius structure is stable by any base change. In particular, the category $F\text{-}D^b_{\mathrm{ovhol}}(\mathscr{D}^{\dagger}_{\mathcal{P}})$ is the same as the usual one introduced in [1] (without any base change condition).
Let Z be a closed subset of $\mathcal{P}_s$, the special fibre of $\mathcal{P}$. There are two functors $\mathbb{R}\Gamma^{\dagger}_Z$ and $({}^{\dagger}Z)$ defined on $D^b_{\mathrm{ovhol}}(\mathscr{D}^{\dagger}_{\mathcal{P}})$ giving rise to a localization triangle

$$\mathbb{R}\Gamma^{\dagger}_Z(\mathcal{E}) \to \mathcal{E} \to ({}^{\dagger}Z)\mathcal{E} \xrightarrow{+1}$$

for $\mathcal{E} \in D^b_{\mathrm{ovhol}}(\mathscr{D}^{\dagger}_{\mathcal{P}})$, cf. [1, 1.1.5].

Let now $\mathbb{Y} = (Y, X)$ be a couple such that $(Y, X, \mathcal{P}, \mathcal{P})$ is a l.p. frame. By abuse of notation, we will sometimes denote the frame $(Y, X, \mathcal{P})$ (or even the l.p. frame $(Y, X, \mathcal{P}, \mathcal{P})$) by $\mathbb{Y}$, too. This should not cause confusion. Let

$$Z := X \backslash Y \quad \text{and} \quad \mathcal{U} := \mathcal{P} \backslash Z.$$

For $\mathcal{E} \in D^b_{\mathrm{ovhol}}(\mathscr{D}^\dagger_{\mathcal{P}})$ one sets

$$\mathbb{R}\Gamma^\dagger_Y(\mathcal{E}) := \mathbb{R}\Gamma^\dagger_X \circ ({}^\dagger Z)(\mathcal{E}).$$

The category $D^b_{\mathrm{ovhol}}(\mathbb{Y}/L)$ of overholonomic complexes on $\mathbb{Y}$, stable by any base change, is defined to be the full subcategory of $D^b_{\mathrm{ovhol}}(\mathscr{D}^\dagger_{\mathcal{P}})$ formed by objects $\mathcal{E}$ such that there is an isomorphism $\mathcal{E} \xrightarrow{\simeq} \mathbb{R}\Gamma^\dagger_Y(\mathcal{E})$ [1, 1.1.5].

The couple $\mathbb{P}$ is obtained by taking $Y = X = \mathcal{P}_s$ and then $D^b_{\mathrm{ovhol}}(\mathbb{P}/L) = D^b_{\mathrm{ovhol}}(\mathscr{D}^\dagger_{\mathcal{P}})$.

We shall make use of the following two lemmas later on.

Lemma 2.1.1. *Let α be a morphism in $D^b_{\mathrm{ovhol}}(\mathbb{Y}/L)$. Then α is an isomorphism in $D^b_{\mathrm{ovhol}}(\mathbb{Y}/L)$ if and only if $\alpha_{|\mathcal{U}}$ is an isomorphism in $D^b(D^\dagger_{\mathcal{U}})$.*

Proof. This is [1, 1.2.3]. □

Lemma 2.1.2. *Let $\mathcal{Q}$ be a smooth formal $\mathfrak{o}$-scheme, $\mathcal{E} \in D^b_{\mathrm{coh}}(\mathscr{D}^\dagger_{\mathcal{Q}})$, X and T two closed subsets of $\mathcal{Q}$ and $S := T \bigcap X$. Assume that $\mathcal{E}_{|\mathcal{Q}\backslash T}$ has support in $X \backslash T$, then we have the following isomorphism $\mathbb{R}\Gamma^\dagger_X({}^\dagger S)(\mathcal{E}) \simeq ({}^\dagger T)\mathcal{E}$.*

Proof. First remark that the canonical morphism $\mathbb{R}\Gamma^\dagger_X({}^\dagger T)\mathcal{E} \to ({}^\dagger T)\mathcal{E}$ is an isomorphism outside T by hypothesis, between two coherent $\mathscr{D}^\dagger_{\mathcal{Q}}({}^\dagger T)$-modules, so that it is an isomorphism and the module $({}^\dagger T)\mathcal{E}$ has support in X. Then we may apply the Mayer-Vietoris exact sequence [14, Théorème 2.2.16] and obtain an exact triangle

$$({}^\dagger S)\mathcal{E} \to ({}^\dagger T)\mathcal{E} \bigoplus ({}^\dagger X)\mathcal{E} \to ({}^\dagger (T \bigcup X))\mathcal{E} \xrightarrow{+1}.$$

The result follows then from $\mathbb{R}\Gamma^\dagger_X({}^\dagger X)\mathcal{E} = \mathbb{R}\Gamma^\dagger_X({}^\dagger (T \bigcup X))\mathcal{E} = 0$. □

As a next step, recall that there is a canonical t-structure on $D^b_{\mathrm{ovhol}}(\mathbb{Y}/L)$, cf. [1, 1.2]. First of all, $D^{\geqslant 0}_{\mathrm{ovhol}}(\mathbb{Y}/L)$ is defined to be the strictly full subcategory of objects $\mathcal{E} \in D^b_{\mathrm{ovhol}}(\mathbb{Y}/L)$ such that

$$\mathcal{E}_{|\mathcal{U}} \in D^{\geqslant 0}(D^\dagger_{\mathcal{U}})$$

(analogously for $\leqslant 0$). The truncation functors relative to the couple $\mathbb{Y}$ are defined to be

$$\tau^{\mathbb{Y}}_{\geqslant 0} = ({}^\dagger Z) \circ \tau_{\geqslant 0} \quad \text{resp.} \quad \tau^{\mathbb{Y}}_{\leqslant 0} = ({}^\dagger Z) \circ \tau_{\leqslant 0},$$

where $\tau_{\geqslant 0}$ resp. $\tau_{\leqslant 0}$ are the usual truncation functors. The functors $\tau^{\mathbb{Y}}_{\geqslant 0}$ and $\tau^{\mathbb{Y}}_{\leqslant 0}$ define a t-structure on $D^b_{\mathrm{ovhol}}(\mathbb{Y}/L)$ whose heart is denoted by $\mathrm{Ovhol}(\mathbb{Y}/L)$ [1, 1.2.9]. As the truncation functors commute with base change, the category $\mathrm{Ovhol}(\mathbb{Y}/L)$ is stable by any base change.

Main examples: (i) In the case where $\mathbb{Y} = \mathbb{P}$, the category $\mathrm{Ovhol}(\mathbb{P}/L)$ is the category of overholonomic arithmetic $\mathscr{D}^\dagger_{\mathcal{P}}$-modules on $\mathcal{P}$, stable by any base change.
(ii) If Z is a divisor in $\mathcal{P}_s$ with open complement $Y = \mathcal{P}_s \backslash Z$ and $\mathbb{Y} = (Y, \mathcal{P}_s, \mathcal{P})$, then $\mathrm{Ovhol}(\mathbb{Y}/L)$ is the category of overholonomic $\mathscr{D}^\dagger_{\mathcal{P}}({}^\dagger Z)$-modules, stable by any base change.

Definition 2.1.3. Let $\mathbb{Y} = (Y, X, \mathcal{P})$ be a frame and let $Z = X\backslash Y$. If Y is smooth, and if there exists a divisor T of $\mathcal{P}_s$ such that

$$Z = X \bigcap T,$$

we say that $\mathbb{Y}$ is *smooth outside of* T. In this case, we let $\mathcal{U}_T := \mathcal{P}\backslash T$.

For the rest of this subsection, we let $\mathbb{Y} = (Y, X, \mathcal{P})$ be a frame which is smooth outside T, for some divisor $T \subset \mathcal{P}_s$.

In this case, the category of coherent $\mathscr{D}^\dagger_{\mathcal{P}}({}^\dagger Z)$-modules contains a full subcategory, denoted by Caro

$$\mathrm{Isoc}^{\dagger\dagger}(\mathbb{Y}/L),$$

which is equivalent to the category of overconvergent isocrystals on Y, overconvergent along Z, the equivalence being given by a certain specialization functor [1, 1.2.14]. This category does not depend on the choice of the ambient formal scheme $\mathcal{P}$. Since we work here with modules without Frobenius structure, we consider from now on the full subcategory of objects of $\mathrm{Isoc}^{\dagger\dagger}(\mathbb{Y}/L)$, belonging to $\mathrm{Ovhol}(\mathbb{Y}/L)$, thus consisting of objects which are overholonomic after any base change.
To avoid too many notations, let us keep the notation $\mathrm{Isoc}^{\dagger\dagger}(\mathbb{Y}/L)$ for this category. The following result is due to Caro.

Theorem 2.1.4. *(Caro) Let $\mathbb{Y} = (Y, X, \mathcal{P})$ be smooth outside a divisor $T \subset \mathcal{P}_s$. Let $\mathcal{E}$ be a coherent $\mathscr{D}^\dagger_{\mathcal{P}}$-module, that is an overholonomic $\mathscr{D}^\dagger_{\mathcal{P}}({}^\dagger T)$-module, stable by any base change (see* [14] *and* [20]*), and such that $\mathcal{E}_{|\mathcal{U}_T} \in \mathrm{im}(\mathrm{sp}_{Y\hookrightarrow \mathcal{U}_T,+})$, which means that*

(i) the module $\mathcal{E}_{|\mathcal{U}_T}$ has support in Y,
(ii) for any affine open $\mathcal{V} \subset \mathcal{U}_T$, for any smooth formal scheme $\mathfrak{Y}$ lifting $Y \bigcap \mathcal{V}$ and for any lifting $v : \mathfrak{Y} \hookrightarrow \mathcal{V}$ of the closed immersion $Y \bigcap \mathcal{V} \hookrightarrow \mathcal{V}$, the module $v^!\mathcal{E}_{|\mathcal{V}}$ is coherent over $\mathcal{O}_{\mathfrak{Y}}$.

Then $\mathcal{E} \in \mathrm{Isoc}^{\dagger\dagger}(\mathbb{Y}/L)$.

Proof. This is [18, Corollaire 3.5.10] and [15, Théorème 2.5.10]. □

In the situation of the theorem, there is the following equivalent characterisation of the objects in the subcategory $\mathrm{im}(\mathrm{sp}_{Y\hookrightarrow \mathcal{U},+})$, which we will use later on.

Proposition 2.1.5. *Let $\mathbb{Y} = (Y, X, \mathcal{P})$ be smooth outside a divisor $T \subset \mathcal{P}_s$ and let $\mathcal{E}$ be a coherent $\mathscr{D}^\dagger_{\mathcal{U}_T}$-module. Then $\mathcal{E} \in \mathrm{im}(\mathrm{sp}_{Y\hookrightarrow \mathcal{U}_T,+})$ if and only if*

(i) the module $\mathcal{E}_{|\mathcal{U}_T}$ has support in Y and
(ii) there exist affine opens $(\mathcal{U}_i)_{i\in I}$ of $\mathcal{U}_T$, such that

(a) $(Y_i := Y \bigcap \mathcal{U}_i)_{i\in I}$ *is a Zariski cover of* Y,
(b) *for each* i, *there exists a smooth formal affine scheme* $\mathfrak{Y}_i$ *lifting* Y_i *and a lifting* $u_i : \mathfrak{Y}_i \hookrightarrow \mathcal{U}_i$ *of the closed immersion* $Y_i \hookrightarrow \mathcal{U}_i$, *such that* $u_i^!\mathcal{E}_{|\mathcal{U}_i}$ *is a coherent* $\mathcal{O}_{\mathfrak{Y}_i}$*-module.*

Proof. Let us start with maps of smooth affine formal schemes as in the statement

$$u_i : \mathfrak{Y}_i \to \mathcal{U}_i$$

with $\mathfrak{Y}_i$ a smooth lifting of Y_i. Let us consider an affine open $\mathcal{V}$ and a lifting $v : \mathfrak{Y} \hookrightarrow \mathcal{V}$ of the closed immersion $Y \bigcap \mathcal{V} \hookrightarrow \mathcal{V}$ as in (ii) of 2.1.4. We have to show that $v^!\mathcal{E}_{|\mathcal{V}}$ is a coherent $\mathcal{O}_{\mathfrak{Y}}$-module. This is a local question on $\mathfrak{Y}$. We put $\mathcal{V}_i := \mathcal{U}_i \bigcap \mathcal{V}$ and may thus work over each

$$\mathfrak{Z}_i := \mathfrak{Y} \bigcap \mathcal{V}_i.$$

Let $v_i := v|_{\mathfrak{Z}_i} : \mathfrak{Z}_i \hookrightarrow \mathcal{V}_i$ be the restriction of the map v and put $\mathcal{E}_i := \mathcal{E}_{|\mathcal{V}_i}$. We shall also need the restrictions $u_i := u_i|_{\mathfrak{Y}_i \cap \mathcal{V}} : \mathfrak{Y}_i \bigcap \mathcal{V} \hookrightarrow \mathcal{V}_i$. The maps v_i and u_i are thus liftings of the closed immersion $Y_i := Y \bigcap \mathcal{V}_i \hookrightarrow \mathcal{V}_i$. Since $\mathfrak{Z}_i$ and $\mathfrak{Y}_i$ are affine, there exists, by formal smoothness, an isomorphism $a : \mathfrak{Z}_i \simeq \mathfrak{Y}_i \bigcap \mathcal{V}$ of $\mathfrak{o}$-formal schemes; The following diagram of formal schemes is not necessarily commutative but induces a commutative diagram of special fibers

$$\begin{tikzcd} \mathcal{V}_i \arrow[r, equal] & \mathcal{V}_i \\ \mathfrak{Z}_i \arrow[u, hook, "v_i"] \arrow[r, "a", "\sim"'] & \mathfrak{Y}_i \bigcap \mathcal{V}. \arrow[u, hook, "u_i"'] \end{tikzcd}$$

We can thus apply [15, Proposition 2.2.2] and we see that there is an isomorphism of functors $v_{i+}a^! \simeq u_{i+}$. Applying this to $u_i^!\mathcal{E}_i$ and using Berthelot-Kashiwara theorem for the closed immersion $\mathfrak{Y}_i \bigcap \mathcal{V} \hookrightarrow \mathcal{V}_i$ we find that $v_{i+}a^!u_i^!\mathcal{E}_i \simeq \mathcal{E}_i$. Using Berthelot-Kashiwara theorem for the closed immersion $\mathfrak{Z}_i \hookrightarrow \mathcal{V}_i$, we get that $v_i^!\mathcal{E}_i \simeq a^!u_i^!\mathcal{E}_i$. By hypothesis, $u_i^!\mathcal{E}_i$ is a coherent $\mathcal{O}_{\mathfrak{Y}_i \bigcap \mathcal{V}}$-module, so that $v_i^!\mathcal{E}_i$ is a coherent $\mathcal{O}_{\mathfrak{Z}_i}$-module as well, which proves our claim. ☐

2.2. **Intermediate extensions.** We keep the notation of the previous subsection. We introduce the intermediate extension functor for arithmetic $\mathscr{D}$-modules following Abe-Caro [1]. Note that this part of Abe-Caro's article does not make use of the fact that the residue field k is finite, as they state at the beginning of their paper. All their results remain thus true when we consider the category of modules, which are overholonomic after any base change.

Let

$$u : \mathbb{Y} \longrightarrow \mathbb{Y}'$$

be a complete morphism of couples. There is a canonical homomorphism

$$\theta_{u,\mathcal{E}} : u_!\mathcal{E} \longrightarrow u_+\mathcal{E}$$

for any complex $\mathcal{E} \in D^b_{\mathrm{ovhol}}(\mathbb{Y})$, cf. [1, 1.3.4]. The morphism is compatible with composition in the following sense: if $w = u_2 \circ u_1$, where u_1 and u_2 are c-complete morphisms of couples, then

$$\begin{tikzcd} u_{2!} \circ u_{1!} \arrow[r, "u_{2!}(\theta_{u_1})"] \arrow[rr, bend right, "\theta_{u_2 \circ u_1}"'] & u_{2!} \circ u_{1+} \arrow[r, "\theta_{u_2}(u_{1+})"] & u_{2+} \circ u_{1+} \end{tikzcd} \tag{2.2.1}$$

by [1, Prop. 1.3.7]. We denote by an exponent $(-)^0 = \mathcal{H}^0_t(-)$ the application of the first cohomology sheaf $\mathcal{H}^0_t = \tau^{\mathbb{Y}}_{\leqslant 0}\tau^{\mathbb{Y}}_{\geqslant 0}$ relative to the t-structure on $D^b_{\mathrm{ovhol}}(\mathbb{Y}/L)$ (and similar for $\mathbb{Y}'$). If u is a c-immersion, and if $\mathcal{E} \in \mathrm{Ovhol}(\mathbb{Y}/L)$, then the *intermediate extension of* $\mathcal{E}$ *on* $\mathbb{Y}'$ is defined to be

$$u_{!+}(\mathcal{E}) := \mathrm{im}(\theta^0_{u,\mathcal{E}} : u^0_!\mathcal{E} \to u^0_+\mathcal{E}).$$

Note that if u is a c-affine immersion, then u_+ and $u_!$ are t-exact by [1, Remark 1.4.2], so that the definition simplifies to

$$u_{!+}(\mathcal{E}) = \mathrm{im}(\theta_{u,\mathcal{E}} : u_!\mathcal{E} \to u_+\mathcal{E}).$$

Remark: There are also versions with Frobenius F-$\mathrm{Isoc}^{\dagger\dagger}(\mathbb{Y}/L)$ and F-$\mathrm{Ovhol}(\mathbb{Y}/L)$ [1, 1.2.13/14], which we will occasionally make use of (e.g. in subsection 3.4). The category F-$\mathrm{Isoc}^{\dagger\dagger}(\mathbb{Y}/L)$ is a full subcategory of F-$\mathrm{Ovhol}(\mathbb{Y}/L)$. The intermediate extension functor preserves Frobenius structures [1, 1.4.1]. Recall that in this case, the base change condition on overholonomic modules is automatically satisfied.

2.3. **A classification result.** We keep the notation of the previous subsections. In particular, $\mathcal{P}$ still denotes a smooth and proper formal scheme over $\mathfrak{o}$. Our goal here is to classify the $\mathscr{D}^\dagger_{\mathcal{P}}$-modules overholonomic after any base change, which are irreducible, up to isomorphism. Although this is in analogy to the classical setting of algebraic D-modules on complex varieties [35, 3.4], this requires some care, since the functors in questions (direct images, intermediate extensions etc.) are *not* straightforward generalizations of the classical functors.

We will only consider couples that arise from a smooth locally closed subvariety $Y \subseteq \mathcal{P}_s$ by taking its Zariski closure $X = \overline{Y}$ in $\mathcal{P}_s$. Then $(Y, X, \mathcal{P})$ is a frame and $(Y, X, \mathcal{P}, \mathcal{P})$ is a l.p. frame and $\mathbb{Y} = (Y, X)$ is a couple. By abuse of notation, we will sometimes denote the frame $(Y, X, \mathcal{P})$ (or even the l.p. frame $(Y, X, \mathcal{P}, \mathcal{P})$) by $\mathbb{Y}$, too. This should not cause confusion.

For such a couple $\mathbb{Y} = (Y, X)$, we consider the corresponding c-locally closed immersion

$$v : \mathbb{Y} \longrightarrow \mathbb{P}.$$

The associated intermediate extension functor between categories of overholonomic modules, stable by any base change,

$$v_{!+} : \mathrm{Ovhol}(\mathbb{Y}/L) \longrightarrow \mathrm{Ovhol}(\mathbb{P}/L),$$

is given by

$$v_{!+}(\mathcal{E}) := \operatorname{im}\big(\theta^0_{v,\mathcal{E}} : v^0_!\mathcal{E} \longrightarrow v^0_+\mathcal{E}\big).$$

Let us suppose for a moment that $Y \subseteq \mathcal{P}_s$ is *closed* and lifts to a closed immersion $\mathfrak{Y} \subseteq \mathcal{P}$ between $\mathfrak{o}$-smooth closed formal schemes. Then $\mathrm{Ovhol}(\mathbb{Y}/L)$ identifies with the category of overholonomic $\mathscr{D}^\dagger_{\mathcal{Y}}$-modules and the functor $v_{!+}$ coincides with the direct image functor appearing in the Berthelot-Kashiwara equivalence [9]. By the latter equivalence, the functor $v_{!+}$ induces a bijection between the (isomorphism classes of) irreducible $\mathscr{D}^\dagger_{\mathcal{Y}}$-modules and irreducible $\mathscr{D}^\dagger_{\mathcal{P}}$-modules supported on Y.

The case of a closed immersion generalizes as follows. Recall that $\mathrm{Ovhol}(\mathbb{Y}/L)$ denotes the category of overholonomic modules on $\mathbb{Y}$ stable by any base change.

Lemma 2.3.1. *Let $\mathcal{M} \in$ Ovhol$(\mathbb{P}/L)$. There is an open dense smooth subscheme $U \subset \mathcal{P}_s$ with the property: if u: $\mathbb{U} = (\mathcal{U}, \mathcal{P}_s, \mathcal{P}) \to \mathbb{P}$ denotes the corresponding c-open immersion, then $u^!\mathcal{M}$ is an overconvergent isocrystal on $\mathbb{U}$, which is overholonomic after any base change.*

Proof. The module $\mathcal{M}$ is overcoherent with finite extraordinary fibers, and by [17, Théorème 3.7], there exists a divisor T such that $({}^\dagger T)\mathcal{M}$ is an overconvergent isocrystal along T. Denote $\mathcal{U} = \mathcal{P}\backslash T$, and u: $\mathbb{U} \hookrightarrow \mathbb{P}$ the c-immersion of triples, then $u^!\mathcal{M} = ({}^\dagger T)\mathcal{M}$ is an overconvergent isocrystal on $\mathbb{U}$. Since $\mathcal{M}$ is overholonomic, so is $u^!\mathcal{M}$, by stability of overholonomicity under inverse image (no Frobenius structure needed), cf. [1, 1.3.14]. □

Since any overholonomic $\mathscr{D}^\dagger_{\mathcal{P}}$-module $\mathcal{M}$ is coherent [16, 3.1], we may view its support $\mathrm{Supp}(\mathcal{M})$ as a closed reduced subvariety of $\mathcal{P}_s$.

Proposition 2.3.2. *Let $\mathcal{M}$ be an irreducible object of Ovhol$(\mathbb{P}/L)$. There is an open dense smooth affine subscheme of an irreducible component of* $\mathrm{Supp}(\mathcal{M})$ *with the property: if $v : \mathbb{Y} \to \mathbb{P}$ denotes the corresponding immersion, then $\mathcal{E} := v^!\mathcal{M}$ is an irreducible overconvergent isocrystal on $\mathbb{Y}$. Moreover, the overconvergent isocrystal $\mathcal{E}$ lies in Ovhol$(\mathbb{Y}/L)$ and $v_{!+}(\mathcal{E}) = \mathcal{M}$.*

Proof. Take X_s an irreducible component of *Supp*$(\mathcal{M})$, then there exists some divisor $T'' \subset \mathcal{P}_s$ such that $\mathcal{U} = \mathcal{P}\backslash T''$ is affine, $Y'_s = X_s \bigcap (\mathcal{P}_s\backslash T'')$ is a smooth affine dense open subset of X_s, and T'' contains all the irreducible components of *Supp*$(\mathcal{M})$ which are not equal to X_s (6.0.2). Denote

$$\mathbb{U} := (\mathcal{U}_s, \mathcal{P}_s, \mathcal{P}_s) \quad \text{and} \quad \mathbb{Y}' := (Y'_s, X_s, \mathcal{P}_s).$$

Since the scheme Y'_s is smooth and affine, there exists a smooth affine formal scheme $\mathfrak{Y}'$ lifting Y'_s over Spf $\mathfrak{o}$, and since $\mathcal{U}$ is a formally smooth Spf $\mathfrak{o}$-formal scheme, the closed immersion $Y'_s \hookrightarrow \mathcal{U}_s$ can be lifted to a closed immersion $\tilde{k}$: $\mathfrak{Y}' \hookrightarrow \mathcal{U}$. Note that *Supp*$(\mathcal{M}_{|\mathcal{U}}) = Y'_s$ by our choice of Z. The overholonomic module $\mathcal{M}$ is a $\mathscr{D}^\dagger_{\mathcal{P}}$-module, so that $\mathcal{M}_{|\mathcal{U}}$ is a overholonomic $\mathscr{D}^\dagger_{\mathcal{U}}$-module with support in Y'_s, and $\mathcal{N} = \tilde{k}^!\mathcal{M}_{|\mathcal{U}}$ is an overholonomic $\mathscr{D}^\dagger_{\mathfrak{Y}'}$-module by Caro-Kashiwara theorem for overholonomic complexes [16, Théorème 2.11].

The module $\mathcal{N}$ is in particular overcoherent after any base change with finite extraordinary fibers and using [17, Théorème 3.7] we see that there exists some divisor T' of $\mathfrak{Y}'$ such that $({}^{\dagger}T')\mathcal{N}$ is an overconvergent isocrystal. This isocrystal is overholonomic after any base change as well, by stability of overholonomicity by localization [16, Proposition 2.4, (5)]. It can not be zero, otherwise $Supp(\mathcal{N})$ would be contained in T'. Let $\overline{T}'$ be the closure of T' in X_s.

Then by 6.0.3 $\overline{T}' \bigcap Y'_s = T'$ and by 6.0.1 there exists a divisor T of P_s such that $\mathcal{V} = \mathcal{P}\backslash T$ is affine, $\overline{T}' \bigcup T'' \subset T$ and

$$\mathfrak{Y} = \mathcal{V} \bigcap \mathfrak{Y}' \subset \mathfrak{Y}'\backslash T'$$

is open, hence dense, in Y'_s (note that Y'_s is irreducible, since its closure is). Note that $Y_s = \mathcal{V} \bigcap X_s$, so that the frame $\mathbb{Y} = (Y_s, X_s, \mathcal{P}_s)$ is smooth outside the divisor T. Denote by v the c-affine immersion $\mathbb{Y} \to \mathbb{P}$ and by k the closed immersion $\mathfrak{Y} \hookrightarrow \mathcal{V}$. As $\mathcal{N}_{|\mathfrak{Y}'\backslash T'}$ is a locally free $\mathcal{O}_{\mathfrak{Y}'\backslash T'}$-module of finite type, the module $\mathcal{N}_{|\mathfrak{Y}} = k^!\mathcal{M}_{|\mathcal{V}}$ is a locally free $\mathcal{O}_{\mathfrak{Y}}$-module of finite type and we can apply 2.1.4 and 2.1.5 to see that $\mathcal{E} = ({}^{\dagger}T)\mathcal{M} \in \mathrm{Isoc}^{\dagger\dagger}(\mathbb{Y}/L)$. It is non zero by construction.
Then by 2.1.2, we see that $\mathcal{E} = ({}^{\dagger}T)\mathcal{M} = v^!\mathcal{M}$. In particular, $\mathcal{E}$ is overholonomic after base change, by stability of overholonomicity under inverse image [1, 1.3.14]. Denote $\mathbb{V} = (\mathcal{U}_s, \mathcal{P}_s, \mathcal{P})$, a the c-open affine immersion $\mathbb{V} \to \mathbb{P}$. As a is the inclusion of the complement of a divisor T of $\mathcal{P}_s$, $a^{!0}\mathcal{M} = a^!\mathcal{M} = ({}^{\dagger}T)\mathcal{M} \in \mathrm{Ovhol}(\mathbb{V}/L)$ and is irreducible by [1, Lemma 1.4.6], since it is non zero. Since the module $({}^{\dagger}T)\mathcal{M}$ has support in X again by 2.1.2, we can apply Abe-Caro's version of Kashiwara's theorem [1, 1.3.2(iii)] for the c-closed immersion $\mathbb{Y} \hookrightarrow \mathbb{V}$ which implies that $\mathcal{E} = v^!\mathcal{M} \in \mathrm{Ovhol}(\mathbb{Y}/L)$ is irreducible. Moreover, by adjointness [1, 1.1.10]

$$\mathrm{Hom}(v_!\mathcal{E}, \mathcal{M}) = \mathrm{Hom}(\mathcal{E}, v^!\mathcal{M}) \neq 0$$

and there is therefore a non-zero morphism $v_!\mathcal{E} \to \mathcal{M}$. In other words, $\mathcal{M}$ is a quotient of $v_!\mathcal{E}$. But $v_!\mathcal{E} = v_!^0\mathcal{E}$, since Y_s is affine, and $v_{!+}\mathcal{E}$ is the unique irreducible quotient of $v_!^0\mathcal{E}$ [1, 1.4.7(ii)]. We therefore must have $v_{!+}\mathcal{E} = \mathcal{M}$. □

Consider now a pair $(Y, \mathcal{E})$ where $Y \subseteq \mathcal{P}_s$ is a connected smooth locally closed subvariety and $\mathcal{E}$ is an irreducible overconvergent isocrystal on $\mathbb{Y} = (Y, X)$, which belongs to $\mathrm{Ovhol}(\mathbb{Y}/L)$, the category of overholonomic modules on $\mathbb{Y}$ that is stable after base change. We write

$$\mathcal{L}(Y, \mathcal{E}) := v_{!+}(\mathcal{E}) \in \mathrm{Ovhol}(\mathbb{P}/L).$$

Remark: We recall that *any* overconvergent F-isocrystal on $\mathbb{Y} = (Y, X)$ is automatically overholonomic [1, 1.2.14].

Proposition 2.3.3. *The $\mathscr{D}^{\dagger}_{\mathcal{P}}$-module $\mathcal{L}(Y, \mathcal{E})$ is an irreducible object of Ovhol$(\mathbb{P}/L)$, with support $\overline{Y}$ and satisfies $v^!\mathcal{L}(Y, \mathcal{E}) = \mathcal{E}$.*

Proof. The irreducibility statement and the fact that $0 \neq v^!\mathcal{L}(Y, \mathcal{E}) \subset v^!v_+^0\mathcal{E} = \mathcal{E}$ follow from [1, 1.4.7(i)] and its proof. As it is irreducible as an overholonomic $\mathscr{D}^{\dagger}_{\mathcal{P}}$-module, it is a fortiori irreducible as an element of $\mathrm{Ovhol}(\mathbb{P}/L)$ (with base change condition). Since $\mathcal{E}$ is irreducible, $v^!\mathcal{L}(Y,\mathcal{E}) = \mathcal{E}$ as claimed or by [1, Lemma 1.4.5(ii)]. Finally, if $k : \mathbb{Y} \to \mathbb{U}$ is a c-closed immersion and $u : \mathbb{U} \to \mathbb{P}$ a c-open immersion such that $v = u \circ k$, then $v_{!+} = u_{!+} \circ k_{!+}$ [1, 1.4.5(i)]. The support of $k_{!+}\mathcal{E} = k_+\mathcal{E}$ equals Y and the support of $\mathcal{L}(Y,\mathcal{E}) = u_{!+}k_{!+}\mathcal{E}$ equals $\overline{Y}$. □

Two pairs are said to be *equivalent* $(Y,\mathcal{E}) \sim (Y',\mathcal{E}')$ if $\overline{Y} = \overline{Y'}$ and there is an open dense $U \subset \overline{Y}$ contained in the intersection $Y \cap Y'$ such that $u^!\mathcal{E} \simeq u'^!\mathcal{E}'$. Here u denotes the c-open immersion $\mathbb{U} = (U, \overline{Y}, \mathcal{P}) \to \mathbb{Y}$ and similarly for u'. This defines an equivalence relation $\sim$ on the set of pairs.

Theorem 2.3.4. *The correspondence $(Y,\mathcal{E}) \mapsto \mathcal{L}(Y,\mathcal{E})$ induces a bijection*

$$\{\textit{pairs } (Y,\mathcal{E})\}/\sim \xrightarrow{\simeq} \{\textit{irreducible objects of Ovhol}(\mathbb{Y}/L)\}/\simeq$$

Proof. Let us show that the map in question is well-defined. Let $(Y,\mathcal{E})$ and $(Y',\mathcal{E}')$ be two equivalent couples. Choose an open dense $U \subset \overline{Y}$ contained in the intersection $Y \cap Y'$ such that $u^!\mathcal{E} \simeq u'^!\mathcal{E}'$. Note that $v \circ u = v' \circ u'$. Define $\mathcal{F} = (v \circ u)^!\mathcal{L}(Y,\mathcal{E})$ and similarly for $\mathcal{L}(Y',\mathcal{E}')$. Then $(v \circ u)_{!+}\mathcal{F} = \mathcal{L}(Y,\mathcal{E})$ according to 2.3.2 and $\mathcal{F} = u^!v^!\mathcal{L}(Y,\mathcal{E}) = u^!\mathcal{E}$ according to 2.3.3. Hence, $\mathcal{F} \simeq \mathcal{F}'$ and we obtain $\mathcal{L}(Y,\mathcal{E}) \simeq \mathcal{L}(Y',\mathcal{E}')$.
Let us next show that the map is injective. So suppose that $\mathcal{L}(Y,\mathcal{E}) \simeq \mathcal{L}(Y',\mathcal{E}')$ for two couples $(Y,\mathcal{E})$ and $(Y',\mathcal{E}')$. Then 2.3.3 implies $\overline{Y} = \overline{Y'}$ and moreover, if $U \subset \overline{Y}$ is open dense and contained in the intersection $Y \cap Y'$, then $(v \circ u)^!\mathcal{L}(Y,\mathcal{E}) = u^!\mathcal{E}$. Since $v \circ u = v' \circ u'$, we obtain $u^!\mathcal{E} \simeq u'^!\mathcal{E}'$ as desired. This proves the injectivity. The surjectivity of the map is a direct consequence of 2.3.2. □

Let $Y \subseteq \mathcal{P}_s$ be a smooth locally closed subvariety and $\mathbb{Y} = (Y, X)$.

Definition 2.3.5. Let $d := \dim(\mathcal{P}_s) - \dim(Y)$. We define the *constant overholonomic module* on the frame $\mathbb{Y}$ to be

$$\mathcal{O}_{\mathbb{Y}} = \mathbb{R}\Gamma_{\mathbb{Y}}(\mathcal{O}_{\mathcal{P},\mathbb{Q}})[d].$$

Proposition 2.3.6. *Suppose that Y is connected and there exists a smooth formal scheme $\mathfrak{Y}$ over $\mathfrak{o}$, so that the immersion $Y \to \mathcal{P}$ lifts to some morphism of formal schemes $\mathcal{Y} \hookrightarrow \mathcal{P}$. The module $\mathcal{O}_{\mathbb{Y}}$ lies in F-$\mathrm{Isoc}^{\dagger\dagger}(\mathbb{Y}/L)$. If the rigid-analytic generic fiber $\mathcal{Y}_L$ is connected, then $\mathcal{O}_{\mathbb{Y}}$ is an irreducible object in the category* $\mathrm{Ovhol}(\mathbb{Y}/L)$.

Proof. Denote $Z = X\backslash Y$ and $\mathcal{U} = \mathcal{P}\backslash Z$. We have the closed immersion of smooth formal schemes $v : \mathcal{Y} \hookrightarrow \mathcal{U}$. Then, by [7, Proposition 1.4], we see that

$$\mathcal{O}_{\mathbb{Y}|\mathcal{U}} = \mathbb{R}\Gamma_{\mathcal{Y}}(\mathcal{O}_{\mathcal{U},\mathbb{Q}})[d] \simeq v_+v^!\mathcal{O}_{\mathcal{U},\mathbb{Q}}[d] = v_+\mathcal{O}_{\mathcal{Y}}.$$

This coincides with $\mathrm{sp}_+\mathcal{O}_{\mathbb{Y}}$ and hence lies in the category F-$\mathrm{Isoc}^{\dagger\dagger}(Y, \mathcal{U}/L)$, in the notation of [1, 1.2.14]. This shows $\mathcal{O}_{\mathbb{Y}} \in F$-$\mathrm{Isoc}^{\dagger\dagger}(\mathbb{Y}/L)$.

The irreducibility statement is based on the following lemma.

Auxiliary lemma. Let $\mathcal{Q}$ be a connected smooth formal scheme over $\mathfrak{o}$ and $\mathcal{Q}_L$ its generic fiber (as rigid analytic space). Assume furthermore that $\mathcal{Q}_L$ is connected.

(i) The constant isocrystal $\mathcal{O}_{Q_L}$ is irreducible in the category of convergent isocrystals.

(ii) The coherent $\mathscr{D}^\dagger_{\mathcal{Q}}$-module $\mathcal{O}_{\mathcal{Q},\mathbb{Q}}$ is irreducible in the category of $\mathscr{D}^\dagger_{\mathcal{Q}}$-modules.

Proof of the auxiliary lemma. We begin by (i). Let E be a subobject of $\mathcal{O}_{Q_L}$ in the abelian category of convergent isocrystals over $\mathcal{Q}_L$, and $E' = \mathcal{O}_{\mathcal{Q}_L}/E$ be the quotient. As convergent isocrystals over $\mathcal{Q}_L$, E and E' are locally free $\mathcal{O}_{\mathcal{Q}_L}$-modules so that there exists an admissible cover by affinoids $\mathcal{U}_i$ $(i \in I)$ such that $E_{|\mathcal{U}_i}$ and $E'_{|\mathcal{U}_i}$ are free $\mathcal{O}_{\mathcal{U}_i}$-modules for each i. Fix i_0 and denote by $A = \Gamma(\mathcal{U}_{i_0}, \mathcal{O}_{\mathcal{U}_{i_0}})$. Since $\mathcal{U}_{i_0}$ is affinoid, we have an exact sequence of free A-modules

$$0 \to \Gamma(\mathcal{U}_{i_0}, E) \to A \to \Gamma(\mathcal{U}_{i_0}, E') \to 0.$$

Take x a point of $\mathcal{U}_{i_0}$, and $L(x)$ its residue field, then the previous exact sequence remains exact after tensoring by $L(x)$, meaning that $\Gamma(\mathcal{U}_{i_0}, E)$ is either equal to 0 or to A. Assume for example that this is equal to 0, so that $E_{|\mathcal{U}_{i_0}} = 0$ by Tate's acyclicity theorem. By Zorn's lemma there is a maximal subset $J \subset I$ such that $E_{|U_i} = 0$ for each $i \in J$. Assume that $J \neq I$ then $J' = I \backslash J$ is not empty. By connectedness, the union $\bigcup_{i \in J'} U_i$ intersects the union $\bigcup_{i \in J} U_i$, thus there exist $l \in J'$, $i \in J$ such that $U_l \bigcap U_i \neq \varnothing$. Since $E_{|U_l}$ is either equal to 0 or to $\mathcal{O}_{U_l}$, we see that it is zero by restricting to $U_l \bigcap U_i$, which contradicts the fact that $J \neq I$. This proves (i).
For (ii) we use then that the abelian category of convergent isocrystals over the generic fiber $\mathcal{Q}_L$ of the formal scheme $\mathcal{Q}$ is equivalent to the category of coherent $\mathscr{D}^\dagger_{\mathcal{Q}}$-modules, that are coherent $\mathcal{O}_{\mathcal{Q},\mathbb{Q}}$-modules ([6, 4.1.4]). The functors sp_* and sp^* realize this equivalence of categories. Let $\mathcal{E}$ be a non-zero coherent $\mathscr{D}^\dagger_{\mathcal{Q}}$-submodule of $\mathcal{O}_{\mathcal{Q},\mathbb{Q}}$, then $E = sp^*\mathcal{E}$ is a convergent isocrystal, that is a subobject of $\mathcal{O}_{Q_L}$. By (i), it is either 0 or equal to the constant convergent isocrystal $\mathcal{O}_{\mathcal{Q}_L}$. Thus $\mathcal{E}$ is either 0 or $\mathcal{O}_{\mathcal{Q},\mathbb{Q}}$ and this proves (ii). Thus the auxiliary lemma is proved.

Let us come back to the proof of the proposition. Let $\alpha : \mathcal{E} \hookrightarrow \mathcal{O}_{\mathbb{Y}}$ be an injective morphism in the category $\mathrm{Ovhol}(\mathbb{Y}/L)$. As remarked in the beginning of the proof,

$$\mathcal{O}_{\mathbb{Y}|\mathcal{U}} = \mathbb{R}\Gamma_{\mathcal{Y}}(\mathcal{O}_{\mathcal{U},\mathbb{Q}})[d] \simeq v_+ v^! \mathcal{O}_{\mathcal{U},\mathbb{Q}}[d] = v_+\mathcal{O}_{\mathcal{Y}}.$$

By Kashiwara's theorem for the closed immersion $v : \mathcal{Y} \hookrightarrow \mathcal{U}$ [9] and the previous lemma, $v_+\mathcal{O}_{\mathcal{Y}}$ is irreducible in the category of coherent $\mathscr{D}^\dagger_{\mathcal{U}}$-modules with support in $\mathcal{Y}$, so that $\mathcal{E}_{|\mathcal{U}}$ is either 0 or equal to $v_+\mathcal{O}_{\mathcal{Y}}$. Using 2.1.1, we conclude that $\mathcal{E}$ is either 0 or equal to $\mathcal{O}_{\mathbb{Y}}$. □

Example: If T is a divisor in $\mathcal{P}_s$, $\mathcal{U} := \mathcal{P}\backslash T$ and $\mathbb{Y} = (\mathcal{U}_s, \mathcal{P}_s, \mathcal{P})$, then $\mathrm{Ovhol}(\mathbb{Y}/L)$ is the usual category of overholonomic $\mathscr{D}^\dagger_{\mathcal{P}}({}^\dagger T)$-modules. In this case, if $\mathcal{U}$ and its generic fiber $\mathcal{U}_L$ are connected, then the constant overholonomic module $\mathcal{O}_{\mathbb{Y}} = \mathcal{O}_{\mathcal{P},\mathbb{Q}}({}^\dagger T)$ is an irreducible $\mathscr{D}^\dagger_{\mathcal{P}}({}^\dagger T)$-module by the previous proposition (applied to $Y = \mathcal{U}_s$).

Proposition 2.3.7. *The module $v_{!+}(\mathcal{O}_{\mathbb{Y}})$ is an overholonomic F-$\mathscr{D}^{\dagger}_{\mathcal{P}}$-module, which is irreducible as $\mathscr{D}^{\dagger}_{\mathcal{P}}$-module.*

Proof. This follows from the theorem 2.3.4 and the above proposition. □

3. Some compatibility results between generic and special fibre

We keep the notations introduced in the preceding section. In this section, we place ourselves into certain integral situations involving schemes over $\mathfrak{o}$ and establish various compatibilities between the classical intermediate extensions on generic fibres and Abe-Caro intermediate extensions arising after reduction on the special fibre. We will focus in particular on the cases of open immersions and proper morphisms.

The results of this section are then applied in the final section 5 in the case of highest weight representations, in order to compare intermediate extensions over the Bruhat cells in generic and special fibre, cf. prop. 5.1.8 and thm. 5.1.9.

3.1. **Notations.** For a $\mathfrak{o}$-scheme X, we write X_s and $X_{\mathbb{Q}}$ for its special and generic fiber respectively. We denote

$$X_i = X \times \operatorname{Spec} \mathfrak{o}/\varpi^{i+1}$$

and write $\mathfrak{X}$ for the associated formal scheme obtained by ϖ-adic completion. We also have the frame $\mathbb{X} = (X_s, X_s, \mathfrak{X})$.
If $f : X \to Y$ is a morphism of $\mathfrak{o}$-schemes, then $f_s, f_{\mathbb{Q}}, f_i$ and $\hat{f}$ will denote the induced morphisms $X_s \to Y_s, X_{\mathbb{Q}} \to Y_{\mathbb{Q}}, X_i \to Y_i$ and $\mathfrak{X} \to \mathfrak{Y}$ respectively. Moreover, $F = (f_s, f_s, \hat{f})$ will denote the induced morphism of frames $\mathbb{X} \to \mathbb{Y}$.

3.2. **Open immersions.** Let P be a smooth scheme over $\mathfrak{o}$. The closed immersions $P_i \hookrightarrow \mathcal{P}$ for any i give rise to a canonical ringed space morphism

$$\alpha : \mathcal{P} = \varinjlim_i P_i \to P.$$

This morphism α comes with the diagram

$$\mathcal{P} \xrightarrow{\alpha} P \xleftarrow{j} P_{\mathbb{Q}}$$

which will be our basic underlying structure in the following.

We record a first simple property.

Lemma 3.2.1. *(i) There is a canonical isomorphism*

$$\mathcal{O}_{P,\mathbb{Q}} \simeq j_* \mathcal{O}_{P_{\mathbb{Q}}}.$$

(ii) There is a canonical isomorphism

$$\mathscr{D}^{(m)}_{P,\mathbb{Q}} \simeq j_* \mathscr{D}_{P_{\mathbb{Q}}}$$

for any m.

Proof. For the first isomorphism, we see that there is a canonical morphism of quasi-coherent $\mathcal{O}_P$-sheaves, $\mathcal{O}_P \to j_*\mathcal{O}_{P_{\mathbb{Q}}}$, sending a local section f to f. After tensoring with $\mathbb{Q}$, we get a map $\mathcal{O}_{P,\mathbb{Q}} \to j_*\mathcal{O}_{P_{\mathbb{Q}}}$ and if $P = \operatorname{Spec} A$, this morphism is the Identity of $A_{\mathbb{Q}} = \Gamma(P, \mathcal{O}_{P,\mathbb{Q}}) = \Gamma(P, j_*\mathcal{O}_{P_{\mathbb{Q}}})$. This proves (i). For (ii), we start with the canonical morphism

$$\mathscr{D}_P^{(m)} \to j_*\mathscr{D}_{P_{\mathbb{Q}}} \simeq j_*\mathscr{D}_{P,\mathbb{Q}}^{(m)}.$$

We deduce from this a morphism $\mathscr{D}_{P,\mathbb{Q}}^{(m)} \to j_*\mathscr{D}_{P_{\mathbb{Q}}}$. To check that it is an isomorphism, it is enough to consider the case where P is affine with local coordinates $x_1, \ldots, x_M$. Then both sheaves are free $\mathcal{O}_{P,\mathbb{Q}}$-modules with basis $\underline{\partial}^{\underline{k}}$ and we conclude using (i). □

If $\mathcal{E}$ is a quasi-coherent $\mathcal{O}_{P_{\mathbb{Q}}}$-module, one defines

$$\overline{\mathcal{E}} := \alpha^{-1} j_*\mathcal{E}.$$

Lemma 3.2.2. *The formation $\mathcal{E} \mapsto \overline{\mathcal{E}}$ is an exact functor from quasi-coherent $\mathcal{O}_{P_{\mathbb{Q}}}$-modules to $\overline{\mathcal{O}}_{P_{\mathbb{Q}}}$-modules. It extends to a derived functor $D^b_{qcoh}(\mathcal{O}_{P_{\mathbb{Q}}}) \to D^b(\overline{\mathcal{O}}_{P_{\mathbb{Q}}})$.*

Proof. This statement comes from the fact that the functor j_* is exact on quasi-coherent $\mathcal{O}_{P_{\mathbb{Q}}}$-sheaves, since j is affine, as well as α^{-1}. The functor $\mathcal{E} \mapsto \overline{\mathcal{E}}$ is thus exact as the composition of two exact funtors. □

In particular, one can consider the sheaf $\overline{\mathscr{D}}_{P_{\mathbb{Q}}}$ over the formal scheme $\mathcal{P}$.

Lemma 3.2.3. *There is an injective flat morphism of sheaves of rings*

$$\overline{\mathscr{D}}_{P_{\mathbb{Q}}} \hookrightarrow \mathscr{D}^{\dagger}_{\mathcal{P}}.$$

Proof. If $U \subset P$ is an open affine of P with local coordinates $x_1, ..., x_M$, then we have the following description

$$\Gamma(U, \alpha_*(\mathscr{D}^{\dagger}_{\mathcal{P}})) = \left\{ \sum_{\underline{\nu}} a_{\underline{\nu}} \underline{\partial}^{[\underline{\nu}]} \mid a_{\underline{\nu}} \in \mathcal{O}_{\mathfrak{U}} \otimes \mathbb{Q} \mid \exists c > 0, \eta < 1, \, ||a_{\underline{\nu}}|| \leqslant c\eta^{|\underline{\nu}|} \right\}$$

and

$$\Gamma(U, \mathscr{D}_{P,\mathbb{Q}}) = \left\{ \sum_{\underline{\nu}, finite} a_{\underline{\nu}} \underline{\partial}^{[\underline{\nu}]} \mid a_{\underline{\nu}} \in \mathcal{O}_U \otimes \mathbb{Q} \right\}.$$

This gives the inclusion. For the flatness, we know that $\mathscr{D}^{\dagger}_{\mathcal{P}}$ is flat over $\widehat{\mathscr{D}}^{(0)}_{\mathcal{P},\mathbb{Q}}$ [6, Cor. 3.5.4]. Moreover the sheaf $\widehat{\mathscr{D}}^{(0)}_{\mathcal{P}}$ is flat over $\mathscr{D}^{(0)}_P$, by completion, so that $\mathscr{D}^{\dagger}_{\mathcal{P}}$ is flat over $\overline{\mathscr{D}}_{P_{\mathbb{Q}}}$. □

The proof of the following lemma is easy and left to the reader.

Lemma 3.2.4. *(i) Let $\mathcal{E}$ be a coherent $\mathscr{D}_{P_{\mathbb{Q}}}$-module, then $\mathscr{D}^{\dagger}_{\mathcal{P}} \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} \overline{\mathcal{E}}$ is a coherent $\mathscr{D}^{\dagger}_{\mathcal{P}}$-module.*

(ii) Let $\mathcal{E} \in D^b_{coh}(\mathscr{D}_{P_{\mathbb{Q}}})$, then $\mathscr{D}^{\dagger}_{\mathcal{P}} \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} \overline{\mathcal{E}} \in D^b_{coh}(\mathscr{D}^{\dagger}_{\mathcal{P}})$.

The following proposition of commutation of duality with scalar extension is due to A. Virrion. Her statement involves perfect complexes, but as $P_{\mathbb{Q}}$ is smooth, the sheaf $\overline{\mathscr{D}}_{P_{\mathbb{Q}}}$ has finite cohomological dimension and the category $D^b_{coh}(\overline{\mathscr{D}}_{P_{\mathbb{Q}}})$ coincides with the category of perfect complexes of $\overline{\mathscr{D}}_{P_{\mathbb{Q}}}$-modules.

Proposition 3.2.5. *Let $\mathcal{E} \in D^b_{coh}(\mathscr{D}_{P_{\mathbb{Q}}})$, then*

$$\mathscr{D}^\dagger_{\mathcal{P}} \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} \mathbb{D}_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}}(\overline{\mathcal{E}}) \simeq \mathbb{D}_{\mathscr{D}^\dagger_{\mathcal{P}}}(\mathscr{D}^\dagger_{\mathcal{P}} \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} \overline{\mathcal{E}}).$$

Proof. This is [55, 1.4,4.4]. □

Let us recall that, if $d = \dim(P_{\mathbb{Q}})$,

$$\mathbb{D}_{\mathscr{D}_{P_{\mathbb{Q}}}}(\mathcal{E}) := \mathbb{R}\mathcal{H}om_{\mathscr{D}_{P_{\mathbb{Q}}}}(\mathcal{E}, \mathscr{D}_{P_{\mathbb{Q}}}[d]) \otimes_{\mathcal{O}_{P_{\mathbb{Q}}}} \omega_{P_{\mathbb{Q}}}.$$

Definition 3.2.6. Let P be a smooth $\mathfrak{o}$-scheme, $Z \subset P$ a divisor, we say that Z is a *transversal divisor* if Z_s and $Z_{\mathbb{Q}}$ are divisors respectively of P_s and $P_{\mathbb{Q}}$.

In the following, let Z be a transversal divisor in P.

We write j for the open immersion $P\backslash Z \subset P$. We can define, for $\mathcal{E}$ a coherent $\mathscr{D}_{P_{\mathbb{Q}}}$-module,

$$(*Z_{\mathbb{Q}})\mathcal{E} = \mathscr{D}_{P_{\mathbb{Q}}}(*Z_{\mathbb{Q}}) \otimes_{\mathscr{D}_{P_{\mathbb{Q}}}} \mathcal{E}.$$

Note that $(*Z_{\mathbb{Q}})\mathcal{E} = j_{\mathbb{Q}+}j_{\mathbb{Q}!}\mathcal{E}$. In the same way, we define for $\mathcal{E}$ a coherent $\mathscr{D}^\dagger_{\mathcal{P}}$-module,

$$({}^\dagger Z_s)\mathcal{E} = \mathscr{D}^\dagger_{\mathcal{P}}({}^\dagger Z_s) \otimes_{\mathscr{D}^\dagger_{\mathcal{P}}} \mathcal{E}.$$

Let $Y = P\backslash Z$ with immersion $j : Y \hookrightarrow P$ and let J be the frame morphism

$$J : \mathbb{Y} := (Y_s, P_s, \mathcal{P}) \to \mathbb{P} := (P_s, P_s, \mathcal{P}).$$

Then $({}^\dagger Z_s)\mathcal{E} = J_+ J^! \mathcal{E}$. By definition, in this situation, J_+ is the forget functor from the category $\mathrm{Ovhol}(\mathbb{Y}/L)$ to the category $\mathrm{Ovhol}(\mathbb{P}/L)$. Moreover the functor $j_{\mathbb{Q}+}$ is exact since $Z_{\mathbb{Q}}$ is a divisor of $P_{\mathbb{Q}}$ and induces an equivalence of categories between coherent $\mathscr{D}_{P_{\mathbb{Q}}}(*Z_{\mathbb{Q}})$-modules and coherent $\mathscr{D}_{Y_{\mathbb{Q}}}$-modules. At the level of sheaves of $\mathcal{O}_{Y_{\mathbb{Q}}}$-modules, $j_{\mathbb{Q}+} = j_{\mathbb{Q}*}$. Recall also that in this situation objects of $\mathrm{Ovhol}(\mathbb{Y}/L)$ consist of degree zero complexes of $\mathscr{D}^\dagger_{\mathcal{P}}({}^\dagger Z_s)$-modules by [1, Remark 1.2.7 (iii)].

Proposition 3.2.7. *Let $\mathcal{E} \in D^b_{hol}(Y_{\mathbb{Q}})$, such that $\mathcal{F} = \mathscr{D}^\dagger_{\mathcal{P}} \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} \overline{j_{\mathbb{Q}+}\mathcal{E}} \in D^b_{ovhol}(\mathbb{Y})$. Let $c_{\mathbb{Q}}$ resp. C be the canonical isomorphism $\mathcal{E} \simeq \mathbb{D}_{Y_{\mathbb{Q}}} \circ \mathbb{D}_{Y_{\mathbb{Q}}}(\mathcal{E})$ resp. $\mathcal{F} \simeq \mathbb{D}_{\mathbb{Y}} \circ \mathbb{D}_{\mathbb{Y}}(\mathcal{F})$. Then we have a commutative diagram*

$$\begin{tikzcd}
\overline{j_{\mathbb{Q}+}\mathcal{E}} \arrow[rr, "\overline{j_{\mathbb{Q}+}c_{\mathbb{Q}}}"] \arrow[d, "1\otimes id_{j_{\mathbb{Q}+}\mathcal{E}}"] & & \overline{j_{\mathbb{Q}+}\mathbb{D}_{Y_{\mathbb{Q}}} \circ \mathbb{D}_{Y_{\mathbb{Q}}}(\mathcal{E})} \arrow[d] \\
J_+(\mathscr{D}^\dagger_{\mathcal{P}} \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} \overline{j_{\mathbb{Q}+}\mathcal{E}}) \arrow[r, "J_+C"] & J_+\mathbb{D}_{\mathbb{Y}} \circ \mathbb{D}_{\mathbb{Y}}\left(\mathscr{D}^\dagger_{\mathcal{P}} \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} \overline{j_{\mathbb{Q}+}\mathcal{E}}\right). &
\end{tikzcd}$$

Proof. Let us first remark that the sheaf $\mathscr{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}Z_s)$ is flat over $\mathscr{D}^{\dagger}_{\mathcal{P}}$ and thus flat over $\overline{\mathscr{D}}_{P_{\mathbb{Q}}}$. This is why no derived tensor product appears in the previous diagram. Moreover, it is enough to prove the statement for a single $\mathscr{D}_{Y_{\mathbb{Q}}}$-holonomic module $\mathcal{E}$ such that $\mathcal{F} = \mathscr{D}^{\dagger}_{\mathcal{P}} \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} \overline{j_{\mathbb{Q}*}\mathcal{E}}$ is an overholonomic module over $\mathbb{Y}$. In this case, all complexes are single modules in degree 0. The top horizontal arrow of the diagram is induced by the following map :

$$\begin{array}{l} \mathcal{E} \longrightarrow \mathcal{H}om_{\mathscr{D}_{Y_{\mathbb{Q}}}}(\mathcal{H}om_{\mathscr{D}_{Y_{\mathbb{Q}}}}(\mathcal{E}, \mathscr{D}_{Y_{\mathbb{Q}}})) \longrightarrow \mathbb{D}_{Y_{\mathbb{Q}}} \circ \mathbb{D}_{Y_{\mathbb{Q}}}(\mathcal{E}) \\ x \longmapsto ev_x(\varphi) = \varphi(x). \end{array}$$

Recall that $C : \mathcal{F} \to \mathbb{D}_{\mathbb{Y}}\mathbb{D}_{\mathbb{Y}}\mathcal{F}$ is defined in our case as follows: as $\mathcal{F}$ is overholonomic over $\mathbb{Y}$, one has

$$\mathcal{F} \simeq ({}^{\dagger}Z_s)\mathcal{F} = \mathscr{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}Z_s) \otimes_{\mathscr{D}^{\dagger}_{\mathcal{P}}} \mathcal{F}.$$

We therefore see using the base change result [55, 1.4,4.4] that

$$\begin{aligned} \mathbb{D}_{\mathbb{Y}}(\mathcal{F}) &= \mathscr{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}Z_s) \otimes_{\mathscr{D}^{\dagger}_{\mathcal{P}}} \mathbb{D}_{\mathbb{P}}(\mathcal{F}) \\ &= \mathbb{R}\mathcal{H}om_{\mathscr{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}Z_s)}(\mathcal{F}, \mathscr{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}Z_s)[d]) \otimes_{\mathcal{O}_{\mathcal{P}}} \omega_{\mathcal{P}}, \\ &= \mathbb{R}\mathcal{H}om_{\mathscr{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}Z_s)}(\mathcal{F}, \mathscr{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}Z_s)[d]) \otimes_{\mathcal{O}_{\mathcal{P}}({}^{\dagger}Z_s)} \omega_{\mathcal{P}}({}^{\dagger}Z_s), \end{aligned}$$

and the canonical map C is then given by the following composition

$$\mathcal{F} \longrightarrow \mathcal{H}om_{\mathscr{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}Z_s)}(\mathcal{H}om_{\mathscr{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}Z_s)}(\mathcal{F}, \mathscr{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}Z_s))) \longrightarrow \mathbb{D}_{\mathbb{Y}} \circ \mathbb{D}_{\mathbb{Y}}(\mathcal{F}) .$$

Note that C is an isomorphism, since it is an isomorphism when restricted to $\mathcal{P}\backslash Z$ by [6, 4.3.10]. Moreover, one has a canonical isomorphism

$$\mathcal{F} \simeq \mathscr{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}Z_s) \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}(*Z_{\mathbb{Q}})} \overline{j_{\mathbb{Q}+}\mathcal{E}},$$

so that we can use again [55, 1.4,4.4], to obtain the following isomorphisms

$$\mathbb{D}_{\mathbb{Y}}(\mathcal{F}) \simeq \mathscr{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}Z_s) \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}(*Z_{\mathbb{Q}})} \mathbb{R}\mathcal{H}om_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}(*Z_{\mathbb{Q}})}(\overline{j_{\mathbb{Q}+}\mathcal{E}}, \overline{\mathscr{D}}_{P_{\mathbb{Q}}}(*Z_{\mathbb{Q}})[d]) \otimes_{\overline{\mathcal{O}}_P} \overline{\omega}_{P_{\mathbb{Q}}}(*Z_{\mathbb{Q}}),$$

$$\mathbb{D}_{\mathbb{Y}}(\mathcal{F}) \simeq \mathscr{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}Z_s) \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}(*Z_{\mathbb{Q}})} \overline{j_{\mathbb{Q}+}\mathbb{D}_{Y_{\mathbb{Q}}}\mathcal{E}},$$

where we identify

$$j_{\mathbb{Q}+}\mathbb{D}_{Y_{\mathbb{Q}}}\mathcal{E} = \mathbb{R}\mathcal{H}om_{\mathscr{D}_{P_{\mathbb{Q}}}(*Z_{\mathbb{Q}})}(j_{\mathbb{Q}+}\mathcal{E}, \mathscr{D}_{P_{\mathbb{Q}}}(*Z_{\mathbb{Q}})[d]) \otimes_{\mathcal{O}_P} \omega_{P_{\mathbb{Q}}}(*Z_{\mathbb{Q}}).$$

Using again [55, 1.4,4.4] between the sheaves $\overline{\mathscr{D}}_{P_{\mathbb{Q}}}(*Z_{\mathbb{Q}})$ and $\mathscr{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}Z_s)$, we find a canonical isomorphism

$$\mathbb{D}_{\mathbb{Y}}\mathbb{D}_{\mathbb{Y}}(\mathcal{F}) \simeq \mathscr{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}Z_s) \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}(*Z_{\mathbb{Q}})} \overline{j_{\mathbb{Q}+}\mathbb{D}_{Y_{\mathbb{Q}}}\mathbb{D}_{Y_{\mathbb{Q}}}\mathcal{E}},$$

which allows us to write the diagram of the statement in the following way

$$\begin{tikzcd}
\overline{j_{\mathbb{Q}+}\mathcal{E}} \arrow[r, "\overline{j_{\mathbb{Q}+}c_{\mathbb{Q}}}"] \arrow[d, "1\otimes id_{j_{\mathbb{Q}+}\mathcal{E}}"] & \overline{j_{\mathbb{Q}+}\mathbb{D}_{Y_{\mathbb{Q}}}\circ\mathbb{D}_{Y_{\mathbb{Q}}}(\mathcal{E})} \arrow[d, "1\otimes id"] \\
J_+(\mathscr{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}Z_s)\otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}(*Z_s)}\overline{j_{\mathbb{Q}+}\mathcal{E}}) \arrow[r, "J_+C"] & \mathscr{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}Z_s)\otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}(*Z_{\mathbb{Q}})}\overline{j_{\mathbb{Q}+}\mathbb{D}_{Y_{\mathbb{Q}}}\mathbb{D}_{Y_{\mathbb{Q}}}\mathcal{E}}.
\end{tikzcd}$$

The commutativity of this diagram follows then from the identity $(1\otimes id)(ev_x) = ev(1\otimes x)$ for a local section $x \in \overline{j_{\mathbb{Q}+}\mathcal{E}}$. □

Corollary 3.2.8. *In the situation of the proposition, let $\mathcal{E} \in D^b_{hol}(Y_{\mathbb{Q}})$, such that $\mathcal{F} = \mathscr{D}^{\dagger}_{\mathcal{P}}\otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}}\overline{j_{\mathbb{Q}+}\mathcal{E}} \in D^b_{ovhol}(\mathbb{Y})$. We then have a commutative diagram*

$$\begin{tikzcd}
\overline{j_{\mathbb{Q}+}\mathcal{E}} \arrow[r, "\sim"] \arrow[d] & \overline{j_{\mathbb{Q}+}j^{!}_{\mathbb{Q}}j_{\mathbb{Q}!}\mathcal{E}} \arrow[d] \\
\mathscr{D}^{\dagger}_{\mathcal{P}}\otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}}\overline{j_{\mathbb{Q}+}\mathcal{E}} \arrow[r, "\sim"] & J_+J^!J_!(\mathscr{D}^{\dagger}_{\mathcal{P}}\otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}}\overline{j_{\mathbb{Q}+}\mathcal{E}}).
\end{tikzcd}$$

Proof. We have the following equality as functors on $D^b_{hol}(Y_{\mathbb{Q}})$

$$\begin{aligned}
j^{!}_{\mathbb{Q}}j_{\mathbb{Q}!} &= j^{!}_{\mathbb{Q}}\mathbb{D}_{P_{\mathbb{Q}}}j_{\mathbb{Q}+}\mathbb{D}_{Y_{\mathbb{Q}}} \\
&= \mathbb{D}_{Y_{\mathbb{Q}}}j^{!}_{\mathbb{Q}}j_{\mathbb{Q}+}\mathbb{D}_{Y_{\mathbb{Q}}} \\
&= \mathbb{D}_{Y_{\mathbb{Q}}}\mathbb{D}_{Y_{\mathbb{Q}}} \simeq \mathrm{id}.
\end{aligned}$$

On the other hand let us notice that $J^! = \mathscr{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}Z_s)\otimes_{\mathscr{D}^{\dagger}_{\mathcal{P}}}\cdot$ is a scalar extension so that again by [55, 1.4,4.4], $J^!\mathbb{D}_{\mathbb{P}} = \mathbb{D}_{\mathbb{Y}}J^!$. Moreover if $\mathcal{F} \in D^b_{\mathrm{ovhol}}(\mathbb{Y})$, $J^!J_+\mathcal{F} = \mathscr{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}Z_s)\otimes_{\mathscr{D}^{\dagger}_{\mathcal{P}}}\mathcal{F} \simeq \mathcal{F}$ by definition of elements of $D^b_{\mathrm{ovhol}}(\mathbb{Y})$. Using these remarks, we compute

$$\begin{aligned}
J^!J_! &= J^!\mathbb{D}_{\mathbb{P}}J_+\mathbb{D}_{\mathbb{Y}} \\
&= \mathbb{D}_{\mathbb{Y}}J^!J_+\mathbb{D}_{\mathbb{Y}} \\
&= \mathbb{D}_{\mathbb{Y}}\mathbb{D}_{\mathbb{Y}} \simeq \mathrm{id},
\end{aligned}$$

so that the diagram of the corollary is the same as the diagram of the previous proposition 3.2.7. □

We next give another compatibility statement.

Proposition 3.2.9. *Let $\mathcal{E} \in D^b_{hol}(P_{\mathbb{Q}})$, such that $\mathcal{F} = \mathscr{D}^{\dagger}_{\mathcal{P}}\otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}}\overline{j_{\mathbb{Q}+}\mathcal{E}} \in D^b_{ovhol}(\mathbb{P})$. Let $can : \mathcal{E} \to j_{\mathbb{Q}+}j^{!}_{\mathbb{Q}}\mathcal{E}$ and $CAN : \mathcal{F} \to J_+J^!\mathcal{F}$ be the canonical morphisms. Then the* *following diagram is commutative*

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "can"] \arrow[d] & j_{\mathbb{Q}+}j_{\mathbb{Q}}^{!}\mathcal{E} \arrow[d] \\
\mathscr{D}^{\dagger}_{\mathcal{P}} \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} \mathcal{E} \arrow[r, "CAN"] & J_{+}J^{!}(\mathscr{D}^{\dagger}_{\mathcal{P}} \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} \mathcal{E}).
\end{tikzcd}$$

Proof. If we explicit all functors in our situation, we find the following diagram that is clearly commutative

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "can"] \arrow[d] & \overline{\mathscr{D}}_{P_{\mathbb{Q}}}(*Z_{\mathbb{Q}}) \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} \mathcal{E} \arrow[d] \\
\mathscr{D}^{\dagger}_{\mathcal{P}} \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} \mathcal{E} \arrow[r, "CAN"] & \mathscr{D}^{\dagger}_{\mathcal{P}}(^{\dagger}Z_s) \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} \mathcal{E}.
\end{tikzcd}$$

□

Recall that a relative normal crossing divisor is transversal.

Proposition 3.2.10. *Let $Z \subset P$ be a relative normal crossing divisor. Then one has*

$$\mathcal{O}_{\mathcal{P},\mathbb{Q}}(^{\dagger}Z_s) \simeq \mathscr{D}^{\dagger}_{\mathcal{P}} \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} \overline{j_{\mathbb{Q}*}\mathcal{O}_{Y_{\mathbb{Q}}}}.$$

Proof. This is a result of Berthelot, cf. [5, 4.3.2]. □

Note that the sheaf $\overline{j_*\mathcal{O}_{Y_{\mathbb{Q}}}}$ is equal to $\overline{\mathcal{O}_{P_{\mathbb{Q}}}(*Z_{\mathbb{Q}})}$ and the isomorphism is given by the canonical inclusion of sheaves of rings $\overline{\mathcal{O}_{P_{\mathbb{Q}}}(*Z_{\mathbb{Q}})} \hookrightarrow \mathcal{O}_{\mathcal{P},\mathbb{Q}}(^{\dagger}Z_s)$, sending 1 to 1. This allows us to identify $\mathcal{O}_{\mathcal{P},\mathbb{Q}}(^{\dagger}Z_s)$ with $\mathscr{D}^{\dagger}_{\mathcal{P}} \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} \overline{j_*\mathcal{O}_{Y_{\mathbb{Q}}}} = \mathcal{O}_{\mathbb{Y}}$. In the same situation as in 3.2.7 we have

Proposition 3.2.11. *Let $Z \subset P$ be a relative normal crossing divisor.*

(i) $\mathbb{D}_{Y_{\mathbb{Q}}}\mathcal{O}_{Y_{\mathbb{Q}}} = \mathcal{O}_{Y_{\mathbb{Q}}}$, $\mathbb{D}_{\mathbb{Y}}\mathcal{O}_{\mathbb{Y}} = \mathcal{O}_{\mathbb{Y}}$,

(ii) there is a canonical isomorphism $J_!\mathcal{O}_{\mathbb{Y}} \simeq \mathscr{D}^{\dagger}_{\mathcal{P}} \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} j_{\mathbb{Q}!}\mathcal{O}_{Y_{\mathbb{Q}}}$.

Proof. The fact that $\mathbb{D}_{Y_{\mathbb{Q}}}\mathcal{O}_{Y_{\mathbb{Q}}} = \mathcal{O}_{Y_{\mathbb{Q}}}$ is classical and comes from the fact that on the smooth scheme $Y_{\mathbb{Q}}$, the $\mathscr{D}_{Y_{\mathbb{Q}}}$-module $\mathcal{O}_{Y_{\mathbb{Q}}}$ admits a resolution by the Spencer complex, and that this complex is auto-dual. To see the second statement, we use the fact proved in [44, Lemme 4.2.1] that $\mathcal{O}_{\mathbb{Y}} = \mathcal{O}_{\mathcal{P},\mathbb{Q}}(^{\dagger}Z_s)$ also admits a resolution by a Spencer complex (with $d = \dim Y_{\mathbb{Q}}$)

$$0 \to \mathscr{D}^{\dagger}_{\mathcal{P}}(^{\dagger}Z_s) \otimes_{\mathcal{O}_{\mathcal{P}}} \Lambda^d \mathcal{T}_{\mathcal{P}} \to \ldots \to \mathscr{D}^{\dagger}_{\mathcal{P}}(^{\dagger}Z_s) \otimes_{\mathcal{O}_{\mathcal{P}}} \Lambda^1 \mathcal{T}_{\mathcal{P}} \to \mathscr{D}^{\dagger}_{\mathcal{P}}(^{\dagger}Z_s)$$

that is auto-dual for the functor $\mathbb{D}_{\mathbb{Y}} = R\mathcal{H}om_{\mathscr{D}^\dagger_{\mathcal{P}}({}^\dagger Z_s)}(\,\cdot\,, \mathscr{D}^\dagger_{\mathcal{P}}({}^\dagger Z_s)[d]) \otimes_{\mathcal{O}_{\mathcal{P}}} \omega_{\mathcal{P}}$. This proves (i) and (ii) follows from the computation

$$\begin{aligned} J_!\mathcal{O}_{\mathbb{Y}} &= \mathbb{D}_{\mathbb{P}} J_+ \mathbb{D}_{\mathbb{Y}} \mathcal{O}_{\mathbb{Y}} \\ &= \mathbb{D}_{\mathbb{P}} J_+ \mathcal{O}_{\mathbb{Y}} \\ &= \mathbb{D}_{\mathbb{P}}(\mathscr{D}^\dagger_{\mathcal{P}} \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} \overline{j_+\mathcal{O}_{Y_{\mathbb{Q}}}}) \text{ by 3.2.10,} \\ &\simeq \mathscr{D}^\dagger_{\mathcal{P}} \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} \overline{\mathbb{D}_{P_{\mathbb{Q}}}(j_+\mathcal{O}_{Y_{\mathbb{Q}}})} \text{ by 3.2.5.} \end{aligned}$$

□

3.3. **Proper morphisms.** Before giving compatibility results for direct images relative to proper morphisms, we establish two auxiliary lemmas.

Lemma 3.3.1. *Let $f : P \to Q$ be a morphism of smooth $\mathfrak{o}$-schemes and $\mathcal{F} \in D^+_{qcoh}(\mathcal{O}_{P_{\mathbb{Q}}})$ with $\overline{\mathcal{F}} \in D^+(\overline{\mathcal{O}}_{P_{\mathbb{Q}}})$. There is a natural map $\overline{Rf_{\mathbb{Q}*}(\mathcal{F})} \to R\hat{f}_*(\overline{\mathcal{F}})$.*

Proof. We have the following diagram

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\alpha"] \arrow[d, "\hat{f}"] & P \arrow[d, "f"] & P_{\mathbb{Q}} \arrow[l, "j"'] \arrow[d, "f_{\mathbb{Q}}"] \\ \mathcal{Q} \arrow[r, "\alpha"] & Q & Q_{\mathbb{Q}}, \arrow[l, "j"'] \end{tikzcd}$$

in which both squares are commutative diagrams (the left one is commutative as it is commutative when $\mathcal{P}$ and $\mathcal{Q}$ are replaced by P_i and Q_i). Let $\mathcal{E}$ be a quasi-coherent sheaf on $P_{\mathbb{Q}}$, we have a canonical map $j_*\mathcal{E} \to \alpha_*\alpha^{-1}j_*\mathcal{E}$. If we compose this map by f_*, we get by adjunction by α a map $\overline{f_{\mathbb{Q}*}\mathcal{E}} \to \hat{f}_*\overline{\mathcal{E}}$. Let $\mathcal{F} \xrightarrow{\simeq} \mathcal{I}_{\bullet,\bullet}$ be an injective resolution of $\mathcal{F}$ by a double complex of quasi-coherent $\mathcal{O}_{P_{\mathbb{Q}}}$-modules. As the functor $\mathcal{E} \mapsto \overline{\mathcal{E}}$ is exact on quasi-coherent $\mathcal{O}_{P_{\mathbb{Q}}}$-modules, we have a quasi-isomorphism $\mathbb{R}\hat{f}_*(\overline{\mathcal{F}}) \simeq \mathbb{R}\hat{f}_*(\overline{\mathcal{I}}_{\bullet,\bullet})$. We finally obtain the map of the lemma by the following composition

$$\overline{\mathbb{R}f_{\mathbb{Q}*}(\mathcal{F})} \simeq \overline{f_{\mathbb{Q}*}(\mathcal{I}_{\bullet,\bullet})} \to \hat{f}_*(\overline{\mathcal{I}}_{\bullet,\bullet}) \to \mathbb{R}\hat{f}_*(\overline{\mathcal{I}}_{\bullet,\bullet}) \simeq \mathbb{R}\hat{f}_*(\overline{\mathcal{F}}).$$

□

Lemma 3.3.2. *Let $f : P \to Q$ be a morphism of smooth $\mathfrak{o}$-schemes, $\mathcal{E} \in D^b_{coh}(\mathscr{D}_{P_{\mathbb{Q}}})$. There is a canonical morphism in $D^+(\overline{\mathscr{D}}_{Q_{\mathbb{Q}}})$*

$$\mathscr{D}^\dagger_{\mathcal{Q}} \otimes_{\overline{\mathscr{D}}_{Q_{\mathbb{Q}}}} \overline{f_{\mathbb{Q}+}(\mathcal{E})} \to \hat{f}_+\left(\mathscr{D}^\dagger_{\mathcal{P}} \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} \overline{\mathcal{E}}\right).$$

Proof. It is enough to prove that there is a morphism

$$\overline{f_{\mathbb{Q}+}(\mathcal{E})} \to \hat{f}_+\left(\mathscr{D}^\dagger_{\mathcal{P}} \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} \overline{\mathcal{E}}\right).$$

Denote the transfer sheaves $\mathscr{D}_{Q_{\mathbb{Q}}\leftarrow P_{\mathbb{Q}}} = \omega_{P/Q} \otimes_{\mathcal{O}_{P_{\mathbb{Q}}}} f_{\mathbb{Q}}^{*}\mathscr{D}_{Q_{\mathbb{Q}}}$, and

$$\widehat{\mathscr{D}}^{(m)}_{\mathcal{P}\to\mathcal{Q}} = \varprojlim_i f_i^{*}\mathscr{D}^{(m)}_{Q_i}, \widehat{\mathscr{D}}^{(m)}_{\mathcal{Q}\leftarrow\mathcal{P}} = \omega_{\mathcal{P}/\mathcal{Q}} \otimes_{\mathcal{O}_{\mathcal{P}}} \widehat{\mathscr{D}}^{(m)}_{\mathcal{P}\to\mathcal{Q}},\ \mathscr{D}^{\dagger}_{\mathcal{Q}\leftarrow\mathcal{P}} = \varinjlim_m \widehat{\mathscr{D}}^{(m)}_{\mathcal{Q}\leftarrow\mathcal{P},\mathbb{Q}}.$$

Recall that

$$f_{\mathbb{Q}+}(\mathcal{E}) = Rf_{\mathbb{Q}*}\left(\mathscr{D}_{Q_{\mathbb{Q}}\leftarrow P_{\mathbb{Q}}} \otimes^{\mathbf{L}}_{\mathscr{D}_{P_{\mathbb{Q}}}} \mathcal{E}\right), \hat{f}_{+}\left(\mathscr{D}^{\dagger}_{\mathcal{P}} \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} \overline{\mathcal{E}}\right) = R\hat{f}_{*}\left(\mathscr{D}^{\dagger}_{\mathcal{Q}\leftarrow\mathcal{P}} \otimes^{\mathbf{L}}_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} \overline{\mathcal{E}}\right).$$

Note that we have

$$\overline{f_{\mathbb{Q}}^{-1}\mathscr{D}_{Q_{\mathbb{Q}}}} \simeq \alpha^{-1}f^{-1}(\mathscr{D}_Q \otimes \mathbb{Q}) = \hat{f}^{-1}\alpha^{-1}(\mathscr{D}_Q \otimes \mathbb{Q}),$$

and for all $m \geqslant 0$, we have maps: $\alpha^{-1}(\mathscr{D}_Q) \to \widehat{\mathscr{D}}^{(m)}_{Q}$. The latter induce maps $\hat{f}^{*}(\alpha^{-1}(\mathscr{D}_Q)) \to \widehat{\mathscr{D}}^{(m)}_{\mathcal{P}\to\mathcal{Q}}$, which in turn give rise to maps of transfer sheaves $\overline{\mathscr{D}}_{Q_{\mathbb{Q}}\leftarrow P_{\mathbb{Q}}} \to \mathscr{D}^{\dagger}_{\mathcal{Q}\leftarrow\mathcal{P}}$.
Take $\mathcal{E} \in D^b_{coh}(\mathscr{D}_{P_{\mathbb{Q}}})$. Since $\mathscr{D}_{Q_{\mathbb{Q}}\leftarrow P_{\mathbb{Q}}}$ is a quasi-coherent $\mathcal{O}_{P_{\mathbb{Q}}}$-module, we see that

$$\mathscr{D}_{Q_{\mathbb{Q}}\leftarrow P_{\mathbb{Q}}} \otimes^{\mathbf{L}}_{\mathscr{D}_{P_{\mathbb{Q}}}} \mathcal{E} \in D^b_{qcoh}(\mathcal{O}_{P_{\mathbb{Q}}}),$$

so that we can apply 3.3.1 to this complex of sheaves. The map of the statement arises then from the composition

$$\overline{Rf_{\mathbb{Q}*}\left(\mathscr{D}_{Q_{\mathbb{Q}}\leftarrow P_{\mathbb{Q}}} \otimes^{\mathbf{L}}_{\mathscr{D}_{P_{\mathbb{Q}}}} \mathcal{E}\right)} \to R\hat{f}_{*}\left(\overline{\mathscr{D}_{Q_{\mathbb{Q}}\leftarrow P_{\mathbb{Q}}} \otimes^{\mathbf{L}}_{\mathscr{D}_{P_{\mathbb{Q}}}} \mathcal{E}}\right) \to R\hat{f}_{*}\left(\mathscr{D}^{\dagger}_{\mathcal{Q}\leftarrow\mathcal{P}} \otimes^{\mathbf{L}}_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} \overline{\mathcal{E}}\right).$$

□

We assume from now on for the rest for this subsection that

$$f : P \longrightarrow Q$$

is a proper morphism between smooth $\mathfrak{o}$-schemes.

Both sheaves $\mathscr{D}_{P_{\mathbb{Q}}}$ and $\mathscr{D}^{\dagger}_{\mathcal{P}}$ have finite cohomological dimension [8, 4.4.8], as well as Rf_* since f is proper. Take $* \in \{-, b\}$ and $\mathcal{E} \in D^{*}_{coh}(\mathscr{D}_{P_{\mathbb{Q}}})$, then $f_{\mathbb{Q}+}(\mathcal{E})$ (resp. $\hat{f}_{+}(\mathscr{D}^{\dagger}_{\mathcal{P}} \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} \overline{\mathcal{E}})$ are objects of $D^{*}_{coh}(\overline{\mathscr{D}}_{Q_{\mathbb{Q}}})$, (resp. $D^{*}_{coh}(\mathscr{D}^{\dagger}_{\mathcal{Q}})$), and thanks to the lemma, there is a map

$$\mathscr{D}^{\dagger}_{\mathcal{Q}} \otimes_{\overline{\mathscr{D}}_{Q_{\mathbb{Q}}}} \overline{f_{\mathbb{Q}+}(\mathcal{E})} \to \hat{f}_{+}(\mathscr{D}^{\dagger}_{\mathcal{P}} \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} \overline{\mathcal{E}}).$$

Our goal (see 3.3.6) is to prove that this map is an isomorphism, provided that P and Q are *projective* $\mathfrak{o}$-schemes.

As usual, we will factorize f into a closed immersion followed by a projection. We first deal with the case of a closed immersion. So let $i : P \hookrightarrow Q$ be a closed immersion of smooth $\mathfrak{o}$-schemes, defined by some sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Q$. We then have the following compatibility result for closed immersions

Proposition 3.3.3. *Let $* \in \{-, b\}$ Let $\mathcal{E} \in D^{*}_{coh}(\mathscr{D}_{P_{\mathbb{Q}}})$, then there is a canonical isomorphism in $D^{*}_{coh}(\mathscr{D}^{\dagger}_{Q})$*

$$\mathscr{D}^{\dagger}_{\mathcal{Q}} \otimes_{\overline{\mathscr{D}}_{Q_{\mathbb{Q}}}} \overline{i_{\mathbb{Q}+}(\mathcal{E})} \simeq \hat{\imath}_{+}\left(\mathscr{D}^{\dagger}_{\mathcal{P}} \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} \overline{\mathcal{E}}\right).$$

Proof. It is well known that $i_{+\mathbb{Q}}$ sends $D^*_{coh}(\mathscr{D}_{P_\mathbb{Q}})$ to $D^*_{coh}(\mathscr{D}_{Q_\mathbb{Q}})$, resp. $\hat{\imath}_+$ sends $D^*_{coh}(\mathscr{D}^\dagger_P)$ to $D^*_{coh}(\mathscr{D}^\dagger_Q)$. Finally both functors send $D^*_{coh}(\mathscr{D}_{P_\mathbb{Q}})$ to $D^*_{coh}(\mathscr{D}^\dagger_Q)$. The map from the LHS to the RHS is the one given in the previous lemma 3.3.2. Since i is a closed immersion, i is affine, it has finite cohomological dimension and both functors are way out left in the sense of [34, I,7]. Proving that the map is an isomorphism is a local question on Q, so that we can assume that Q is affine and P as well. In this case any coherent $\mathscr{D}_{P_\mathbb{Q}}$-module is a quotient of a finite free $\mathscr{D}_{P_\mathbb{Q}}$-module, and using a standard dévissage argument for way out left functors [34, I,7, (iv)] we are reduced to prove the lemma in the case where $\mathcal{E} = \mathscr{D}_{P_\mathbb{Q}}$. In this case, we have the following formulas

$$i_{\mathbb{Q}+}(\mathscr{D}_{P_\mathbb{Q}}) = i_*\left(\mathscr{D}_{Q_\mathbb{Q}\leftarrow P_\mathbb{Q}}\right),\ \hat{\imath}_+\left(\mathscr{D}^\dagger_{\mathcal{P}}\right) = \hat{\imath}_*\left(\mathscr{D}^\dagger_{\mathcal{Q}\leftarrow\mathcal{P}}\right)$$

As $\hat{\imath}$ is a quasi-compact morphism, $\mathbb{R}\hat{\imath}_* = \hat{\imath}_*$ commutes with inductive limits so that

$$\hat{\imath}_*\left(\mathscr{D}^\dagger_{\mathcal{Q}\hookleftarrow\mathcal{P}}\right) = \varinjlim_m \hat{\imath}_*\left(\widehat{\mathscr{D}}^{(m)}_{\mathcal{Q}\hookleftarrow\mathcal{P},\mathbb{Q}}\right).$$

Let us fix an integer m, we have to show that

$$\hat{\imath}_*\left(\widehat{\mathscr{D}}^{(m)}_{\mathcal{Q}\hookleftarrow\mathcal{P},\mathbb{Q}}\right) \simeq \widehat{\mathscr{D}}^{(m)}_{\mathcal{Q},\mathbb{Q}} \otimes_{\overline{\mathscr{D}}_{P_\mathbb{Q}}} \overline{i_*\left(\mathscr{D}_{Q_\mathbb{Q}\leftarrow P_\mathbb{Q}}\right)}. \tag{3.3.4}$$

We first compute the left hand side of this formula. By [9, Thm.3.5.3], we know that $\hat{\imath}_+(\widehat{\mathscr{D}}^{(m)}_{\mathcal{P}})$ is a coherent $\widehat{\mathscr{D}}^{(m)}_{\mathcal{Q}}$-module, and as $\hat{\imath}$ is affine, we have by [30, prop. 13.2.3]

$$\hat{\imath}_+(\widehat{\mathscr{D}}^{(m)}_{\mathcal{P}}) \simeq \varprojlim_i i_*(\mathscr{D}^{(m)}_{Q_i\hookleftarrow P_i}). \tag{3.3.5}$$

We now come to the right hand side of the formula 3.3.4. We need the

Auxiliary lemma. The sheaf $i_+(\mathscr{D}^{(m)}_P)$ is a coherent $\mathscr{D}^{(m)}_Q$-module.

Proof. We have

$$\begin{aligned} i_+\mathscr{D}^{(m)}_P &= i_*\left(i^*\omega_Q^{-1}\otimes_{\mathcal{O}_Q} i^*\mathscr{D}^{(m)}_Q \otimes_{\mathcal{O}_P}\omega_P\right) \\ &\simeq \omega_Q^{-1}\otimes_{\mathcal{O}_Q}\mathscr{D}^{(m)}_Q\otimes_{\mathcal{O}_Q} i_*\omega_P \text{ by the projection formula,}\end{aligned}$$

the left $\mathscr{D}^{(m)}_Q$-module structure being given by the one of $\omega_Q^{-1}\otimes_{\mathcal{O}_Q}\mathscr{D}^{(m)}_Q$, that is by the right structure on $\mathscr{D}^{(m)}_Q$ twisted on the left, which makes this left $\mathscr{D}^{(m)}_Q$-module a coherent module. This proves the auxiliary lemma.

Returning back to the proof of the proposition, consider the $\widehat{\mathscr{D}}^{(m)}_{\mathcal{Q}}$-module

$$\mathcal{M} = \widehat{\mathscr{D}}^{(m)}_{\mathcal{Q}} \otimes_{\alpha^{-1}\mathscr{D}^{(m)}_Q} \alpha^{-1} i_+\mathscr{D}^{(m)}_P.$$

The auxiliary lemma implies that $\mathcal{M}$ is coherent, and so [6, 3.2.4] implies

$$\begin{aligned}\mathcal{M} &\simeq \varprojlim_i \mathscr{D}^{(m)}_{Q_i} \otimes_{\mathscr{D}^{(m)}_{Q_i}} i_* \mathscr{D}^{(m)}_{Q_i \leftarrow P_i} \\ &\simeq \varprojlim_i i_* \mathscr{D}^{(m)}_{Q_i \leftarrow P_i}\end{aligned}$$

As $\mathcal{M}_{\mathbb{Q}}$ coincides with $\widehat{\mathscr{D}}^{(m)}_{\mathcal{Q}} \otimes_{\overline{\mathscr{D}}_{Q_{\mathbb{Q}}}} \overline{i_{\mathbb{Q}+}(\mathscr{D}_{P_{\mathbb{Q}}})}$, this module is isomorphic with the right-hand side of 3.3.4. Comparing with the left-hand side 3.3.5 proves the proposition. □

As before, let $* \in \{b, -\}$.

Proposition 3.3.6. *Let P, Q be smooth and projective $\mathfrak{o}$-schemes, let $f : P \to Q$ be a proper morphism, and $\hat{f} : \mathcal{P} \to \mathcal{Q}$ be the formal completion of f. Let $\mathcal{E} \in D^*_{coh}(\mathscr{D}_{P_{\mathbb{Q}}})$, then there is a functorial isomorphism in $D^*_{coh}(\mathscr{D}^\dagger_{\mathcal{Q}})$*

$$\mathscr{D}^\dagger_{\mathcal{Q}} \otimes_{\overline{\mathscr{D}}_{Q_{\mathbb{Q}}}} \overline{f_{\mathbb{Q}+}(\mathcal{E})} \simeq \hat{f}_+ \left(\mathscr{D}^\dagger_{\mathcal{P}} \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} \overline{\mathcal{E}} \right).$$

Proof. We already noticed that both functors send objects of $D^*_{coh}(\mathscr{D}_{P_{\mathbb{Q}}})$ to objects of $D^*_{coh}(\mathscr{D}^\dagger_{\mathcal{Q}})$, as f has finite cohomological dimension. Moreover both functors are way out left in the sense of [34, I,7]. The map from LHS to RHS was defined in 3.3.2. Using [53, Tag 0C4Q], we know that f is projective. Then, using the previous compatibility result 3.3.3 for closed immersions, it is enough to prove the statement when P is a relative projective space over Q, say $P = \mathbb{P}^M_Q$ and $f : \mathbb{P}^M_Q \to Q$ is the canonical map. Since the question is local on Q, we can (and we do assume) that Q is affine, smooth with coordinates $t_1, \ldots, t_s$. Let $\mathcal{E}$ be a coherent $\mathscr{D}_{P,\mathbb{Q}}$-module. As $P_{\mathbb{Q}}$ is a noetherian space, $\mathcal{E}$ is an inductive limit of its sub $\mathcal{O}_{P_{\mathbb{Q}}}$-coherent sheaves, so that there is a $\mathcal{O}_{P_{\mathbb{Q}}}$-coherent sheaf $\mathcal{E}'$ and a surjection of $\mathscr{D}_{P_{\mathbb{Q}}}$-modules $\mathscr{D}_{P_{\mathbb{Q}}} \otimes_{\mathcal{O}_{P_{\mathbb{Q}}}} \mathcal{E}' \twoheadrightarrow \mathcal{E}$, where the $\mathscr{D}_{P_{\mathbb{Q}}}$-module structure on the left hand side is given by the one of $\mathscr{D}_{P_{\mathbb{Q}}}$. By Serre's theorem, for some $a, r \in \mathbb{N}$, there is a surjection of coherent $\mathcal{O}_{P_{\mathbb{Q}}}$-modules $\mathcal{O}_{P_{\mathbb{Q}}}(-a)^r \twoheadrightarrow \mathcal{E}'$, and we see that there is a surjection of coherent $\mathscr{D}_{P_{\mathbb{Q}}}$-modules $\mathscr{D}_{P_{\mathbb{Q}}}(-a)^r \twoheadrightarrow \mathcal{E}$. Iterating this process, we see that each coherent $\mathscr{D}_{P_{\mathbb{Q}}}$-module has some resolution by $\mathscr{D}_{P_{\mathbb{Q}}}$-modules of the type $\mathscr{D}_{P_{\mathbb{Q}}}(-a)^r$. Finally using again the dévissage argument for way out left functors of [34, I,7, (iv)] we are reduced to prove the proposition for a projective morphism $f : P = \mathbb{P}^M_Q \to Q$, with Q affine, endowed with coordinates, and $\mathcal{E} = \mathscr{D}_{P_{\mathbb{Q}}}(-a)$, with $a \in \mathbb{N}$. Let us assume this from now on.
Since $\mathbb{R}f_*$ commutes with inductive limits, because $\mathbb{P}^M_Q$ and Q are quasi-compact, it is also enough to prove that, for all m, we have

$$\widehat{\mathscr{D}}^{(m)}_{\mathcal{Q}} \otimes_{\overline{\mathscr{D}}_{Q_{\mathbb{Q}}}} \overline{f_+(\mathscr{D}_{P_{\mathbb{Q}}}(-a))} \simeq \hat{f}_+ \left(\widehat{\mathscr{D}}^{(m)}_{\mathcal{P},\mathbb{Q}} \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} \overline{\mathscr{D}}_{P_{\mathbb{Q}}}(-a) \right). \tag{3.3.7}$$

The following lemma therefore completes the proof of the proposition. □

Lemma 3.3.8. *Assertion 3.3.7 is true for any m.*

Proof. Let $\mathcal{F} = \mathscr{D}_P^{(m)}(-a)$, we have

$$\begin{aligned} f_+(\mathcal{F}) &= \mathbb{R}f_*\left(\mathscr{D}^{(m)}_{Q\leftarrow P}\otimes_{\mathscr{D}_P^{(m)}}\mathscr{D}_P^{(m)}(-a)\right) \\ &= \mathbb{R}f_*\left(f^*\mathscr{D}_Q^{(m)}\otimes_{\mathcal{O}_P}\omega_{P/Q}(-a)\right), \\ &= (\mathscr{D}_Q^{(m)}\otimes_{\mathcal{O}_Q}\omega_Q^{-1})\otimes_{\mathcal{O}_Q}\mathbb{R}f_*(\omega_P(-a)) \text{ (by the projection formula)}, \end{aligned}$$

where the left $\mathscr{D}_Q^{(m)}$-module structure is given by the left structure of $\mathscr{D}_Q^{(m)}\otimes_{\mathcal{O}_Q}\omega_Q^{-1}$, obtained by twisting the right structure of $\mathscr{D}_Q^{(m)}$. As ω_Q is free of rank 1,

$$\omega_P \simeq \omega_P\otimes_{\mathcal{O}_P} f^*\omega_Q^{-1}\simeq\omega_{P/Q}\simeq\mathcal{O}_P(-M-1),$$

where $M := \dim P_{\mathbb{Q}} - \dim Q_{\mathbb{Q}}$. We refer for example to [33, III,thm. 5.1] for the computation of $\mathbb{R}f_*(\mathcal{O}_P(-M-1))$ over any affine base Q, which is a complex of finite free $\mathcal{O}_Q$-modules. More precisely, denote

$$d = \max\{rank(H^0(P,\mathcal{O}_P(-a-M-1))), rank(H^M(P,\mathcal{O}_P(-a-M-1)))\}.$$

There are several cases:

(i) If $a\leqslant -M-1$, then $f_+(\mathcal{F})\simeq\mathscr{D}_Q^{(m)}\otimes_{\mathcal{O}_Q}\mathcal{O}_Q^d$ is concentrated in degree 0,
(ii) if $a\geqslant 0$, then $f_+(\mathcal{F})\simeq\mathscr{D}_Q^{(m)}\otimes_{\mathcal{O}_Q}\mathcal{O}_Q^d[-M]$ is concentrated in degree M,
(iii) if $-M\leqslant a\leqslant -1$, then $f_+(\mathcal{F})$=0.

Note also that we have the following isomorphism of (twisted) left $\widehat{\mathscr{D}}^{(m)}_{\mathcal{Q},\mathbb{Q}}$-modules

$$\widehat{\mathscr{D}}^{(m)}_{\mathcal{Q},\mathbb{Q}}\otimes_{\overline{\mathscr{D}}_Q^{(m)}}(\overline{\mathscr{D}}_Q^{(m)}\otimes_{\overline{\mathcal{O}}_{Q_{\mathbb{Q}}}}\overline{\omega_Q}^{-1})\simeq\widehat{\mathscr{D}}^{(m)}_{\mathcal{Q},\mathbb{Q}}\otimes_{\mathcal{O}_{\mathcal{Q}}}\omega_{\mathcal{Q}}^{-1}.$$

We will first compute the left-hand side of 3.3.7. Let us denote

$$\begin{aligned} \mathcal{A} &:= \widehat{\mathscr{D}}^{(m)}_{\mathcal{Q}}\otimes_{\alpha^{-1}\mathscr{D}_Q^{(m)}}\alpha^{-1}f_+(\mathcal{F})\in D^b_{coh}(\widehat{\mathscr{D}}^{(m)}_{\mathcal{Q}}) \\ &= (\widehat{\mathscr{D}}^{(m)}_{\mathcal{Q}}\otimes_{\mathcal{O}_{\mathcal{Q}}}\omega_{\mathcal{Q}}^{-1})\otimes_{\alpha^{-1}\mathcal{O}_Q}\alpha^{-1}\mathbb{R}f_*(\omega_P(-a)), \end{aligned}$$

that consists of a complex concentrated in at most one degree where it is isomorphic to a direct sum of d copies of $\widehat{\mathscr{D}}^{(m)}_{\mathcal{Q}}$. In particular, by 3.2.1 of [9], it satisfies $\mathcal{A}\simeq\mathbb{R}\varprojlim_i(\mathscr{D}^{(m)}_{Q_i}\otimes^L_{\widehat{\mathscr{D}}^{(m)}_{\mathcal{Q}}}\mathcal{A})$. Since $\mathcal{A}$ is a complex of finite free $\widehat{\mathscr{D}}^{(m)}_{\mathcal{Q}}$-modules, we have

$$\mathscr{D}^{(m)}_{Q_i}\otimes^L_{\widehat{\mathscr{D}}^{(m)}_{\mathcal{Q}}}\mathcal{A}=\mathscr{D}^{(m)}_{Q_i}\otimes_{\widehat{\mathscr{D}}^{(m)}_{\mathcal{Q}}}\mathcal{A}\simeq\mathscr{D}^{(m)}_{Q_i}\otimes_{\mathcal{O}_{Q_i}}\omega_{Q_i}^{-1}\otimes_{\alpha^{-1}\mathcal{O}_Q}\alpha^{-1}\mathbb{R}f_*(\omega_P(-a)),$$

is a complex either in degree M or 0, where it is isomorphic to a direct sum of d copies of $\mathscr{D}^{(m)}_{Q_i}$. Finally we have

$$\mathcal{A}\simeq\mathbb{R}\varprojlim_i\left(\mathscr{D}^{(m)}_{Q_i}\otimes_{\mathcal{O}_{Q_i}}\omega_{Q_i}^{-1}\otimes_{\alpha^{-1}\mathcal{O}_Q}\alpha^{-1}\mathbb{R}f_*(\omega_P(-a))\right).$$

To compute the right-hand side of 3.3.7, we introduce $\mathcal{B} = \hat{f}_+(\widehat{\mathscr{D}}^{(m)}_{\mathcal{P}}(-a))$, so that we have

$$\begin{aligned}
\mathcal{B} &\simeq \mathbb{R}\hat{f}_* \left(\hat{f}^*(\widehat{\mathscr{D}}^{(m)}_{\mathcal{Q}} \otimes_{\mathcal{O}_{\mathcal{Q}}} \omega^{-1}_{\mathcal{O}_{\mathcal{Q}}}) \otimes_{\mathcal{O}_{\mathcal{P}}} \omega_{\mathcal{P}}(-a) \right) \\
&\simeq \mathbb{R}\hat{f}_* \mathbb{R} \varprojlim_i \left(f_i^*(\mathscr{D}^{(m)}_{Q_i} \otimes_{\mathcal{O}_{Q_i}} \omega^{-1}_{Q_i}) \otimes_{\mathcal{O}_{P_i}} \omega_{P_i}(-a) \right) \\
&\simeq \mathbb{R} \varprojlim_i \mathbb{R} f_{i*} \left(f_i^*(\mathscr{D}^{(m)}_{Q_i} \otimes_{\mathcal{O}_{Q_i}} \omega^{-1}_{Q_i}) \otimes_{\mathcal{O}_{P_i}} \omega_{P_i}(-a) \right) \text{ by [53, Tag 0BKP]} \\
&\simeq \mathbb{R} \varprojlim_i \left(\mathscr{D}^{(m)}_{Q_i} \otimes_{\mathcal{O}_{Q_i}} \omega^{-1}_{Q_i} \otimes_{\mathcal{O}_{Q_i}} \mathbb{R} f_{i*} \omega_{P_i}(-a) \right).
\end{aligned}$$

Again, by using the computation of [33, III,thm. 5.1], we see that $\mathbb{R}f_{i*}\omega_{P_i}(-a) \simeq \mathbb{R}f_{i*}\mathcal{O}_{P_i}(-a-M-1)$, is a complex concentrated in only one degree and

$$\mathbb{R}f_{i*}\omega_{P_i}(-a) \simeq \mathcal{O}_{Q_i} \otimes_{\mathcal{O}_Q} \mathbb{R}f_*\omega_P(-a).$$

This finally shows that $\mathcal{B}$ is isomorphic to $\mathcal{A}$. This implies the lemma. □

3.4. **Compatibility for intermediate extensions of constant coefficients.** We now come to the main application of our previous compatibility results. For this we place ourselves in the following axiomatic situation (S):

(i) Y is an affine and smooth scheme over $\mathfrak{o}$.

(ii) there is an immersion $v : Y \hookrightarrow P$ into a smooth projective scheme P over $\mathfrak{o}$. Let $X := \overline{Y}$ be the Zariski closure of Y in P and $Z := X \backslash Y$.

(iii) There is a smooth and projective $\mathfrak{o}$-scheme X', a surjective morphism
$$b : X' \to X$$
inducing an isomorphism $Y' := b^{-1}Y \simeq Y$, such that $Z' = X' \backslash Y'$ is a transversal divisor as defined in 3.2.6 with normal crossings. We have the open immersion $j' : Y \simeq b^{-1}Y \hookrightarrow X'$.

As usual $\mathfrak{X}, \mathfrak{Y}$ etc. denote the formal schemes obtained from these schemes by p-adic completion, and X_s, Y_s etc. denote their special fiber. For simplicity, we also write v for the morphism of frames

$$v : \mathbb{Y} = (Y_s, X_s, \mathcal{P}) \longrightarrow (P_s, P_s, \mathcal{P}) = \mathbb{P}$$

induced by the immersion $v : Y \hookrightarrow P$. Let us introduce the composite morphism

$$g : X' \xrightarrow{b} X \hookrightarrow P.$$

By p-adic completion we obtain a morphism $\hat{g}$: $\mathfrak{X}' \to \mathcal{P}$, and a morphism of frames

$$u = (Id_{Y_s}, b_s, \hat{g}) : \mathbb{Y}' = (Y_s, X'_s, \mathfrak{X}') \to \mathbb{Y} = (Y_s, X_s, \mathcal{P}).$$

Denoting $G = (g_s, g_s, \hat{g})$ and $J' = (j'_s, \mathrm{id}_{X'_s}, \mathrm{id}_{\mathfrak{X}'})$, we then have the basic commutative diagram of frames:

$$\begin{tikzcd}
\mathbb{Y}' = (Y_s, X'_s, \mathfrak{X}') \arrow[r, "J'"] \arrow[d, "u"] & (X'_s, X'_s, \mathfrak{X}') = \mathbb{X}' \arrow[d, "G"] \\
\mathbb{Y} = (Y_s, X_s, \mathcal{P}) \arrow[r, "v"] & (P_s, P_s, \mathcal{P}) = \mathbb{P}.
\end{tikzcd}$$

The frame morphism u is c-affine, and the first morphism of this frame is equal to the identity, so that by [1, 1.2.8], we know that $u_!$ and u_+ are t-exact, and $u_! = u_+$, as functors of abelian categories $F\text{-Ovhol}(\mathbb{Y}'/L) \to F\text{-Ovhol}(\mathbb{Y}/L)$ and equal to $\mathcal{H}^0_t u_+ = \mathcal{H}^0_t u_!$. Let

$$Q := v \circ u = G \circ J',$$

which is a c-affine immersion, (in particular $Y_s \hookrightarrow P_s$ is an immersion). Note that we have

$$J'_+ \mathcal{O}_{\mathbb{Y}'} = \mathcal{O}_{\mathfrak{X}',\mathbb{Q}}({}^\dagger Z'_s),$$

and that in our case J'_+ is the forget functor $F\text{-Ovhol}(\mathbb{Y}'/L) \to F\text{-Ovhol}(\mathbb{X}'/L)$. Let $v_\mathbb{Q}$ be the immersion $Y_\mathbb{Q} \hookrightarrow P_\mathbb{Q}$. We now fix once and for all the following notations:

$$\begin{array}{rclcl}
\theta_{v_\mathbb{Q}} & = & \theta_{v_\mathbb{Q}, \mathcal{O}_{Y_\mathbb{Q}}} : v_{\mathbb{Q}!}\mathcal{O}_{Y_\mathbb{Q}} \to v_{\mathbb{Q}+}\mathcal{O}_{Y_\mathbb{Q}} & \text{resp.} & \theta_{j'_\mathbb{Q}} = \theta_{j'_\mathbb{Q}, \mathcal{O}_{Y_\mathbb{Q}}} \\
\theta_Q = \theta^0_{Q, \mathcal{O}_{\mathbb{Y}'}} & = & \theta_{Q, \mathcal{O}_{\mathbb{Y}'}} : Q_! \mathcal{O}_{\mathbb{Y}'} \to Q_+ \mathcal{O}_{\mathbb{Y}'} & \text{resp.} & \theta_{J'} = \theta_{J', \mathcal{O}_{\mathbb{Y}'}}.
\end{array}$$

We also need the two morphisms

$$\theta^{alg}_v = \overline{\theta}_{v_\mathbb{Q}} \qquad \text{resp.} \qquad \theta^{alg}_{j'} = \overline{\theta}_{j'_\mathbb{Q}}.$$

Our goal is to describe the relation between the classical intermediate extension $v_{\mathbb{Q}!+}\mathcal{O}_{Y_\mathbb{Q}}$ on the generic fibre and the Abe-Caro intermediate extension $v_{!+}\mathcal{O}_\mathbb{Y}$ on the special fibre. We start with the following lemma.

Lemma 3.4.1. *We have the following commutative diagram in F-Ovhol$(\mathbb{X}'/L)$, where all maps are canonical*

$$\begin{tikzcd}
\mathscr{D}^\dagger_{\mathfrak{X}'} \otimes_{\overline{\mathscr{D}}_{X'_\mathbb{Q}}} \overline{j'_{\mathbb{Q}!}\mathcal{O}_{Y_\mathbb{Q}}} \arrow[r] \arrow[d, "(1)"', "\simeq"] & \mathscr{D}^\dagger_{\mathfrak{X}'} \otimes_{\overline{\mathscr{D}}_{X'_\mathbb{Q}}} \overline{j'_{\mathbb{Q}+}\mathcal{O}_{Y_\mathbb{Q}}} \arrow[d, "(3)"', "\simeq"] \\
J'_! \mathcal{O}_{\mathbb{Y}'} \arrow[r, "\theta_{J'}"] & J'_+ \mathcal{O}_{\mathbb{Y}'}
\end{tikzcd}$$

and where the upper horizontal arrow equals $\mathscr{D}^\dagger_{\mathfrak{X}'} \otimes \theta^{alg}_{j'}$.

Proof. The diagram of the statement can be completed by the following diagram

$$\begin{tikzcd} \mathscr{D}^\dagger_{\mathfrak{X}'} \otimes_{\overline{\mathscr{D}}_{X'_{\mathbb{Q}}}} \overline{j'_{\mathbb{Q}!}\mathcal{O}_{Y_{\mathbb{Q}}}} \arrow[r, "c_{\mathbb{Q}}"] \arrow[d, "(1)"'] \arrow[rr, bend left, "\mathscr{D}^\dagger_{\mathfrak{X}'}\otimes\theta^{alg}_{j'}"] & \mathscr{D}^\dagger_{\mathfrak{X}'} \otimes_{\overline{\mathscr{D}}_{X'_{\mathbb{Q}}}} \overline{j'_{\mathbb{Q}+}j'^!_{\mathbb{Q}}j'_{\mathbb{Q}!}\mathcal{O}_{Y_{\mathbb{Q}}}} \arrow[r, "\simeq"] \arrow[d, "(2)"'] & \mathscr{D}^\dagger_{\mathfrak{X}'} \otimes_{\overline{\mathscr{D}}_{X'_{\mathbb{Q}}}} \overline{j'_{\mathbb{Q}+}\mathcal{O}_{Y_{\mathbb{Q}}}} \arrow[d, "(3)"', "\simeq"] \\ J'_!\mathcal{O}_{\mathbb{Y}'} \arrow[r, "C"] \arrow[rr, bend right, "\theta_{J'}"'] & J'_+J'^!J'_!\mathcal{O}_{\mathbb{Y}'} \arrow[r, "\simeq"] & J'_+\mathcal{O}_{\mathbb{Y}'}. \end{tikzcd}$$

Let us prove that both squares of this diagram are commutative. The isomorphism (3) is given by Berthelot's result 3.2.10. The right square of this diagram is commutative by 3.2.8, horizontal maps of this square are isomorphisms, so that (2) is an isomorphism as well. The left square of this diagram is commutative by 3.2.9 applied to $\overline{j'_{\mathbb{Q}!}\mathcal{O}_{Y_{\mathbb{Q}}}}$ and 3.2.11. Moreover (ii) of 3.2.11 tells us that (1) is an isomorphism. We conclude that the external square is commutative with vertical arrows being isomorphisms. □

Recall that $Q = G \circ J' = v \circ u$, $v_{\mathbb{Q}} = g_{\mathbb{Q}} \circ j'_{\mathbb{Q}}$, $\theta^{alg}_v = \overline{\theta}_{v_{\mathbb{Q}},\mathcal{O}_{Y_{\mathbb{Q}}}}$ and $\theta_Q = \theta_{Q,\mathcal{O}_{\mathbb{Y}}}$.

Corollary 3.4.2. *There is a commutative diagram (with canonical vertical maps) in F-*Ovhol$(\mathbb{P})$

$$\begin{tikzcd} \mathscr{D}^\dagger_{\mathcal{P}} \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} \overline{v_{\mathbb{Q}!}\mathcal{O}_{Y_{\mathbb{Q}}}} \arrow[r] \arrow[d, "\simeq"] & \mathscr{D}^\dagger_{\mathcal{P}} \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} \overline{v_{\mathbb{Q}+}\mathcal{O}_{Y_{\mathbb{Q}}}} \arrow[d, "\simeq"] \\ Q_!\mathcal{O}_{\mathbb{Y}'} \arrow[r, "\theta_Q"] & Q_+\mathcal{O}_{\mathbb{Y}'} \end{tikzcd}$$

where the upper horizontal arrow equals the map $\mathscr{D}^\dagger_{\mathcal{P}} \otimes \theta^{alg}_v$.

Proof. As G is c-proper, $G_+ = G_!$, and using 2.2.1, we can see that $\theta_Q = G_+ \circ \theta_{J'}$. Similarly, we have the equality $v_{\mathbb{Q}} = g_{\mathbb{Q}} \circ j'_{\mathbb{Q}}$, and as $g_{\mathbb{Q}}$ is proper, $\theta_{v_{\mathbb{Q}}} = g_{\mathbb{Q}+} \circ \theta_{j'_{\mathbb{Q}}}$. We finally use the compatibility for projective morphisms 3.3.6, and we observe that we obtain the diagram of the corollary after applying $\hat{g}_+$ to the previous diagram 3.4.1. □

Remark: We have the identifications $\theta_{j'_{\mathbb{Q}}}(\mathcal{O}_{Y_{\mathbb{Q}}}) \simeq \mathcal{O}_{X'_{\mathbb{Q}}}$ and $\theta_{J'}\mathcal{O}_{\mathfrak{X}',\mathbb{Q}}({}^\dagger Z'_s) \simeq \mathcal{O}_{\mathfrak{X}',\mathbb{Q}}$, and that

$$\mathscr{D}^\dagger_{\mathfrak{X}'} \otimes_{\overline{\mathscr{D}}_{X'_{\mathbb{Q}}}} \overline{\mathcal{O}_{X'_{\mathbb{Q}}}} \simeq \mathcal{O}_{\mathfrak{X}',\mathbb{Q}}.$$

We have the constant overholonomic modules on $\mathbb{Y}$ resp. $\mathbb{Y}'$

$$\mathcal{O}_{\mathbb{Y}} = \mathbb{R}\Gamma_{\mathbb{Y}}(\mathcal{O}_{\mathcal{P},\mathbb{Q}})[d] \qquad \text{resp.} \qquad \mathcal{O}_{\mathbb{Y}'} = \mathbb{R}\Gamma_{\mathbb{Y}'}(\mathcal{O}_{\mathfrak{X}',\mathbb{Q}}) = \mathcal{O}_{\mathfrak{X}',\mathbb{Q}}({}^\dagger Z'_s)$$

as defined in 2.3.5, where $d = \dim P_s - \dim Y_s$.

Lemma 3.4.3. *There are canoncical isomorphisms*

(i) $u^!\mathcal{O}_{\mathbb{Y}} \simeq \mathcal{O}_{\mathbb{Y}'}$,

(ii) $u_+\mathcal{O}_{\mathbb{Y}'} \simeq \mathcal{O}_{\mathbb{Y}}$.

Proof. By [1, 1.2.8], $u^!$ and u_+ are exact functors of the categories $\mathrm{Ovhol}(\mathbb{Y}/L)$ and $\mathrm{Ovhol}(\mathbb{Y}'/L)$, and quasi-inverse, so that (ii) is a direct consequence of (i). Recall also that, by [14, 2.2.6.1,2.2.8,2.2.14], $\mathbb{R}\Gamma_{\mathbb{Y}'} \circ \mathbb{R}\Gamma_{\mathbb{Y}'} = \mathbb{R}\Gamma_{\mathbb{Y}'}$. We compute

$$\begin{aligned} u^!\mathcal{O}_{\mathbb{Y}} &= \mathbb{R}\Gamma_{\mathbb{Y}'} \circ \hat{g}^!(\mathbb{R}\Gamma^\dagger_{X_s}({}^\dagger Z_s)(\mathcal{O}_{\mathcal{P},\mathbb{Q}})[d]) \\ &\simeq \mathbb{R}\Gamma_{\mathbb{Y}'} \circ \mathbb{R}\Gamma^\dagger_{X'_s} \circ ({}^\dagger Z'_s)\hat{g}^!\mathcal{O}_{\mathcal{P},\mathbb{Q}}[d] \quad \text{[14, Théorème 2.2.18]} \\ &\simeq \mathbb{R}\Gamma_{\mathbb{Y}'} \circ \mathbb{R}\Gamma_{\mathbb{Y}'}\mathcal{O}_{\mathfrak{X}',\mathbb{Q}} \\ &\simeq \mathbb{R}\Gamma_{\mathbb{Y}'}\mathcal{O}_{\mathfrak{X}',\mathbb{Q}}. \end{aligned}$$

□

We come to the main result, which describes the relation between the classical intermediate extension $v_{\mathbb{Q}!+}\mathcal{O}_{Y_\mathbb{Q}}$ on the generic fibre and the Abe-Caro intermediate extension $v_{!+}\mathcal{O}_{\mathbb{Y}}$ on the special fibre.

Theorem 3.4.4. *There is a canonical isomorphism*

$$\mathscr{D}^\dagger_{\mathcal{P}} \otimes_{\overline{\mathscr{D}}_{P_\mathbb{Q}}} \overline{v_{\mathbb{Q}!+}(\mathcal{O}_{Y_\mathbb{Q}})} \simeq v_{!+}(\mathcal{O}_{\mathbb{Y}}).$$

Proof. Again, by [1, 1.2.8], $u_+ = u_!$, and $\theta_Q = \theta_v \circ u_+$. By previous lemma 3.4.3, $u_+\mathcal{O}_{\mathbb{Y}'} \simeq \mathcal{O}_{\mathbb{Y}}$ and we have a commutative diagram

$$\begin{tikzcd} v_+\mathcal{O}_{\mathbb{Y}} \arrow[r, "\theta_v"] \arrow[d, "\simeq"] & v_!\mathcal{O}_{\mathbb{Y}} \arrow[d, "\simeq"] \\ Q_+\mathcal{O}_{\mathbb{Y}'} \arrow[r, "\theta_Q"] & Q_!\mathcal{O}_{\mathbb{Y}'}. \end{tikzcd}$$

Now we have

$$\begin{aligned} v_{!+}(\mathcal{O}_{\mathbb{Y}}) &= \mathrm{im}(\theta_v) \\ &\simeq \mathrm{im}(\theta_Q) \\ &\simeq \mathscr{D}^\dagger_{\mathcal{P}} \otimes_{\overline{\mathscr{D}}_{P_\mathbb{Q}}} \mathrm{im}(\theta_v^{alg}) \text{ by 3.4.2}, \\ &\simeq \mathscr{D}^\dagger_{\mathcal{P}} \otimes_{\overline{\mathscr{D}}_{P_\mathbb{Q}}} \overline{v_{\mathbb{Q}!+}(\mathcal{O}_{Y_\mathbb{Q}})}. \end{aligned}$$

□

4. Localization theory on the flag variety

We specialize the above theory to the case where $\mathcal{P}$ is the (formal) flag variety of a connected split reductive group G over $\mathfrak{o}$. Such a space is coherently $\mathscr{D}^\dagger$-affine and its algebra of global differential operators $H^0(\mathcal{P}, \mathscr{D}^\dagger_{\mathcal{P}})$ identifies with (a central reduction of) the crystalline distribution algebra of G. Truly in the spirit of classical localization theory [3], this allows us to analyze geometrically the module theory of the distribution algebra.

4.1. **Crystalline distribution algebras.** In this subsection, G can be any connected affine smooth group scheme over $\mathfrak{o}$. Let I be the kernel of the morphism $\mathfrak{o}$-algebras $\varepsilon_G : \mathfrak{o}[G] \to \mathfrak{o}$ which represents $1 \in G$. Then I/I^2 is a free $\mathfrak{o} = \mathfrak{o}[G]/I$-module of finite rank. Let $t_1, \ldots, t_N \in I$ whose classes modulo I^2 form a base of I/I^2. The m-PD-envelope of I is denoted by $P_{(m)}(G)$. This algebra is a free $\mathfrak{o}$-module with basis

$$\underline{t}^{\{\underline{k}\}} = t_1^{\{k_1\}} \cdots t_N^{\{k_N\}},$$

where $q_i! t_i^{\{k_i\}} = t_i^{k_i}$ with $i = p^m q_i + r$ et $r < p^m$ [6, 1.5]. The algebra $P_{(m)}(G)$ has a descending filtration by the ideals

$$I^{\{n\}} = \bigoplus_{|\underline{k}| \geqslant n} \mathfrak{o} \cdot \underline{t}^{\{\underline{k}\}}.$$

The quotients $P^n_{(m)}(G) := P_{(m)}(G)/I^{\{n+1\}}$ are generated, as $\mathfrak{o}$-module, by the elements $\underline{t}^{\{k\}}$ where $|\underline{k}| \leqslant n$ and there is an isomorphism $P^n_{(m)}(G) \simeq \bigoplus_{|\underline{k}| \leqslant n} \mathfrak{o} \underline{t}^{\{\underline{k}\}}$ as $\mathfrak{o}$-modules. There are canonical surjections $pr^{n+1,n} : P^{n+1}_{(m)}(G) \twoheadrightarrow P^n_{(m)}(G)$.

We note

$$\mathrm{Lie}(G) := \mathrm{Hom}_{\mathfrak{o}}(I/I^2, \mathfrak{o}).$$

The Lie-algebra $\mathrm{Lie}(G)$ is a free $\mathfrak{o}$-module with basis $\xi_1, \ldots, \xi_N$ dual to $t_1, \ldots, t_N$. For $m' \geqslant m$, the universal property of divided power algebras gives homomorphismes of filtered algebras $\psi_{m,m'} : P_{(m')}(G) \to P_{(m)}(G)$ which induce on quotients homomorphismes of algebras $\psi^n_{m,m'} : P^n_{(m')}(G) \to P^n_{(m)}(G)$. *The module of distributions of level* m *and order* n is $D^{(m)}_n(G) := \mathrm{Hom}_{\mathfrak{o}}(P^n_{(m)}(G), \mathfrak{o})$ *The algebra of distributions of level* m is defined to be

$$D^{(m)}(G) := \varinjlim_n D^{(m)}_n(G)$$

where the limit is taken with respect to the maps $\mathrm{Hom}_{\mathfrak{o}}(pr^{n+1,n}, \mathfrak{o})$.

For $m' \geqslant m$, the algebra homomorphisms $\psi^n_{m,m'}$ give dually linear maps $\Phi^n_{m,m'} : D^{(m)}_n(G) \to D^{(m')}_n(G)$ and finally a morphism of filtered algebras $\Phi_{m,m'} : D^{(m)}(G) \to D^{(m')}(G)$. The direct limit

$$\mathrm{Dist}(G) = \varinjlim_m D^{(m)}(G)$$

equals the classical distribution algebra of the group scheme G [23, II.§4.6.1].

Let now $\mathcal{G}$ be the completion of G along its special fibre. We write $G_i = \mathrm{Spec}\ \mathfrak{o}[G]/\pi^{i+1}$. The morphism $G_{i+1} \hookrightarrow G_i$ induces $D^{(m)}(G_{i+1}) \to D^{(m)}(G_i)$. We put

$$\widehat{D}^{(m)}(\mathcal{G}) := \varprojlim_i D^{(m)}(G_i).$$

If $m' \geqslant m$, one has the morphisms $\hat{\Phi}_{m,m'} : \widehat{D}^{(m)}(\mathcal{G}) \to \widehat{D}^{(m')}(\mathcal{G})$ and the *crystalline distribution algebra* is defined to be

$$D^{\dagger}(\mathcal{G}) := \varinjlim_{m} \widehat{D}^{(m)}(\mathcal{G}) \otimes \mathbb{Q}.$$

Note, as for differential operators, that this dagger-algebra appears with coefficients tensored by $\mathbb{Q}$. For more details on the basic theory of the algebra $D^{\dagger}(\mathcal{G})$ we refer to [39, 40].

For a character $\theta : Z(\mathfrak{g}) \to L$ of the center $Z(\mathfrak{g})$ of the universal enveloping algebra of the L-Lie algebra $\mathfrak{g} = \mathrm{Lie}(G) \otimes \mathbb{Q}$, we will always denote by

$$D^{\dagger}(\mathcal{G})_{\theta} := D^{\dagger}(\mathcal{G})_{\theta} \otimes_{Z(\mathfrak{g}),\theta} L$$

the corresponding central reduction of $D^{\dagger}(\mathcal{G})$. The *trivial* character is the character θ_0 with $\ker \theta_0 = Z(\mathfrak{g}) \cap (U(\mathfrak{g})\mathfrak{g})$.

4.2. **The localization theorem and overholonomicity.** We keep the notation of the previous subsection, but specialize now to the case of a connected split reductive group scheme G over $\mathfrak{o}$. Let in the following $\theta = \theta_0$ be the trivial character. Our goal is to analyze the *central block* of the category of $D^{\dagger}(\mathcal{G})$-modules, i.e. the category of $D^{\dagger}(\mathcal{G})_{\theta_0}$-modules. We keep the notation from the preceding section.

We let $B \subset G$ be a Borel subgroup containing a maximal split torus T, with unipotent radical N. Denote by

$$P := G/B$$

the flag scheme. It is a smooth and projective scheme over $\mathfrak{o}$. We denote by $\mathcal{P}$ its formal completion. The G-action on P by translations endowes $\mathcal{P}$ with a $\mathcal{G}$-action. We recall the localization theorem for arithmetic $\mathscr{D}$-modules on the flag variety.

Theorem 4.2.1. (a) *The global section functor induces an equivalence of categories between coherent $\mathscr{D}^{\dagger}_{\mathcal{P}}$-modules and coherent $H^0(\mathcal{P}, \mathscr{D}^{\dagger}_{\mathcal{P}})$-modules. A quasi-inverse is given by the functor*

$$\mathscr{L}oc(M) = \mathscr{D}^{\dagger}_{\mathcal{P}} \otimes_{H^0(\mathcal{P},\mathscr{D}^{\dagger}_{\mathcal{P}})} M.$$

(b) *The $\mathcal{G}$-action on $\mathcal{P}$ induces an algebra isomorphism*

$$D^{\dagger}(\mathcal{G})_{\theta_0} \xrightarrow{\simeq} H^0(\mathcal{P}, \mathscr{D}^{\dagger}_{\mathcal{P}}).$$

Proof. This summarizes the main results of [40] and [45]. □

Remark: A. Sarrazola-Alzate has extended the above theorem to the case of an arbitrary central character θ using a twisted version of the sheaf $\mathscr{D}^{\dagger}_{\mathcal{P}}$, cf. [46].

Definition 4.2.2. A $D^{\dagger}(\mathcal{G})_{\theta_0}$-module M is called *geometrically (F-)overholonomic* if the coherent $\mathscr{D}^{\dagger}_{\mathcal{P}}$-module $\mathscr{L}oc(M)$ lies in the full subcategory $(F\text{-})\mathrm{Ovhol}(\mathbb{P}/L)$.

Remark: Already in the classical situation of algebraic D-modules on complex flag varieties, it is difficult in dimension > 1 to translate the condition of being a holonomic D-module to the algebraic side. Nevertheless, this condition cuts out an interesting abelian finite length subcategory inside all Lie algebra representations, which contains many examples (highest weight representations, Whittaker modules etc.) It is the cristalline analogue of this category which we propose to study.

Recall from 2.1 the set of equivalence classes of pairs $(Y, \mathcal{E})$ where $Y \subseteq \mathcal{P}_s$ is a connected smooth locally closed subvariety and $\mathcal{E}$ is an irreducible overconvergent isocrystal on $\mathbb{Y} = (Y, X)$, which is an object of $\mathrm{Ovhol}(\mathbb{Y}/L)$ (the category of overholonomic modules, stable by any base change). We put $\mathcal{L}(Y, \mathcal{E}) := v_{!+}(\mathcal{E}) \in \mathrm{Ovhol}(\mathbb{P}/L)$ where $v : \mathbb{Y} \to \mathbb{P}$ is the immersion of couples associated with Y.

Theorem 4.2.3. *The correspondence $(Y, \mathcal{E}) \mapsto H^0(\mathcal{P}, \mathcal{L}(Y, \mathcal{E}))$ induces a bijection*

$$\{\textit{pairs } (Y, \mathcal{E})\}/\sim \; \xrightarrow{\simeq} \{\textit{irreducible geometrically overholonomic } D^\dagger(\mathcal{G})_{\theta_0}\textit{-modules}\}/\simeq$$

Proof. This follows from the classification theorem 2.3.4 together with 4.2.1. □

We point out a related interesting property of the category of overholonomic $\mathscr{D}^\dagger_{\mathcal{P}}$-modules.

It is conjectured by de Jong that, if X is a connected smooth projective variety over an algebraically closed field of characteristic $p > 0$ with trivial étale fundamental group, then any isocrystal on X is constant. This conjecture is proved under certain additional assumptions by Esnault-Shiho in [27]. In our case, the fibration $G \to G/B = P$ is a separable proper morphism with geometrically connected fibre between locally noetherian connected schemes. To compute the fundamental group of $\mathcal{P}_s$, we may pass to a simply connected cover of the semisimple derived group of $\mathcal{G}_s$. The homotopy exact sequence [31, Exp. 10 Cor. 1.4] implies then that étale fundamental group of $\mathcal{P}_s$ is trivial. Here is a short representation-theoretic proof of de Jong's conjecture for the flag variety $\mathcal{P}_s$.[2]

Proposition 4.2.4. *Any convergent isocrystal on $\mathcal{P}_s$ is constant.*

Proof. Any convergent isocrystal $\mathcal{E}$ may be viewed as a coherent $\mathscr{D}^\dagger_{\mathcal{P}}$-module which is coherent over $\mathcal{O}_{\mathcal{P},\mathbb{Q}}$ [6, Prop. (4.1.4)]. Then $H^0(\mathcal{P}, \mathcal{E})$ is a finite dimensional representation of the reductive L-Lie algebra $\mathfrak{g}$ and hence completely reducible (semisimple). In addition, it has central character θ_0. But the trivial one dimensional representation is the only irreducible $\mathfrak{g}$-representation of finite dimension and with central character θ_0. Since the trivial representation localizes to the trivial connection $\mathcal{O}_{\mathcal{P},\mathbb{Q}}$ and since localization commutes with direct sums, the isocrystal $\mathcal{E}$ must be constant. □

[2]The homotopy exact sequence implies in the same manner that the generic fibre P_L has trivial étale fundamental group. By Chern-Weil theory and Grothendieck's theorem on formal functions, the de Rham Chern classes on P_L become trivial after tensoring with $\mathbb{Q}$. But these classes correspond to the rational crystalline classes on $\mathcal{P}_s$ via the comparison theorem between de Rham and crystalline cohomology, from which one may deduce the conjecture. We thank H. Esnault for explaining this general argument to us.

4.3. **Link to locally analytic representations.** In this subsection, we explain how the methods and results of the present paper might ultimately have applications to locally analytic representations. Although this is more of a speculation and we do not prove any substantial result in this subsection, we like to include it in this paper, as it has been a major source of motivation for us in writing this paper.

To put everything in context, we briefly recall some classical results from the geometric representation theory of non-compact real Lie groups [32, 47]. Let $G_{\mathbb{R}}$ be a non-compact connected reductive real Lie group ($GL_n(\mathbb{R})$ is a first example) and let $\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} Lie(G_{\mathbb{R}})$ be its complexified Lie algebra. Let $K_{\mathbb{R}} \subset G_{\mathbb{R}}$ be a maximal compact subgroup with complexification K. Given an admissible $G_{\mathbb{R}}$-representation[3] V, we denote by $V_{K_{\mathbb{R}}-\mathrm{fin}} \subseteq V$ its subspace of $K_{\mathbb{R}}$-finite vectors. It is dense in V and naturally equipped with the structure of a Harish-Chandra $(\mathfrak{g}, K)$-module. The formation

$$V \mapsto HC(V) := V_{K_{\mathbb{R}}-\mathrm{fin}}$$

is a covariant, exact and faithful functor from admissible $G_{\mathbb{R}}$-representations of finite length to Harish-Chandra $(\mathfrak{g}, K)$-modules. One calls two finite length representations V_1 and V_2 *infinitesimally equivalent* if $HC(V_1) \simeq HC(V_2)$. Classifying irreducible representations according to infinitesimal equivalence is a first step towards a full classification of irreducible representations.[4] Any Harish-Chandra module M admits a globalization (i.e. a finite length representation V with $HC(V) \simeq M$) which implies, by its functorial properties, that HC even preserves irreducibility. Hence, classifying irreducible representations up to infinitesimal equivalence is the same as classifying irreducible Harish-Chandra modules for the pair $(\mathfrak{g}, K)$. If $G_{\mathbb{R}}$ admits a connected complexification ($GL_n(\mathbb{R})$ is a first example), then irreducible Harish-Chandra modules have infinitesimal characters. In this case, the classification of modules with fixed character θ is equivalent, using Beilinson-Bernstein localization, to the classification of irreducible twisted Harish-Chandra sheaves on the complex flag variety of $\mathfrak{g}$. The latter, in turn, is a special case of the general classification of holonomic twisted D-modules as intermediate extensions over locally closed subvarieties [3].

We like to speculate how this might generalize from real-analytic to p-adic analytic Lie groups. Concretely, let $G(L)$ be the group of L-valued points of our split connected reductive group G. This is a non-compact locally L-analytic group whose basic theory of admissible locally analytic representations (in certain complete locally convex Hausdorff spaces over L) has been developed by Schneider-Teitelbaum in a series of papers [50, 51, 52]). We have the n-th congruence subgroup scheme $G(n)$ of the $\mathfrak{o}$-group scheme G. It is a smooth affine group scheme over $\mathfrak{o}$. We denote by $\widehat{G(n)}$ the completion of $G(n)$ along the unit section $1 \in G(n)_k$ in its special fibre and by $G(n)^{\circ} := \widehat{G(n)}^{\mathrm{rig}}$ its rigid-analytic generic

[3] *Representation* here means a jointly continuous linear action of $G_{\mathbb{R}}$ on a complete locally convex Hausdorff space V over $\mathbb{C}$. For the notion of *admissibility*, see for example [47, 3.1].

[4] For unitary representations, infinitesimal equivalence already implies equivalence, a famous result of Harish-Chandra [32].

fibre (in the sense of Berthelot, cf. [4, 22]). By construction, $G(n)^\circ$ is a rigid-analytic group over L, whose underlying space is strictly quasi-Stein (in the sense of [26, 2.1.17]), and which comes with an associated rigid-analytic distribution algebra $\mathcal{D}^{an}(G(n)^\circ)$, cf. [26, 5.2]. The latter naturally contains $U(\mathfrak{g})$ and one has the central reduction

$$\mathcal{D}^{an}(G(n)^\circ)_\theta := \mathcal{D}^{an}(G(n)^\circ) \otimes_{Z(\mathfrak{g}),\theta} L$$

for each character θ of the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$.

We need the following basic lemma. For the notion of a *good analytic open subgroup*, we refer to [26, 5.2]. Let e be the ramification index of the extension $L/\mathbb{Q}_p$.

Lemma 4.3.1. *The congruence subgroup* $\ker(G(\mathfrak{o}) \to G(\mathfrak{o}/\pi^n))$ *is a good analytic open subgroup of* $G(L)$ *for* n *larger than* $\frac{e}{p-1}$.

Proof. Choose a closed embedding of the group scheme G into some $GL_{m/\mathfrak{o}}$ [10, Prop. 13.2] and identify $Lie(GL_{m/\mathfrak{o}}) = Mat_m(\mathfrak{o})$. For n larger than $\frac{e}{p-1}$, the $\mathfrak{o}$-Lie lattice $\pi^n Mat_m(\mathfrak{o})$ exponentiates to the open subgroup $1 + \pi^n Mat_m(\mathfrak{o})$ of $GL_m(L)$. Hence the $\mathfrak{o}$-Lie lattice $\pi^n Lie(G) \subseteq \mathfrak{g}$ exponentiates to the congruence subgroup $\ker(G(\mathfrak{o}) \to G(\mathfrak{o}/\pi^n))$. By definition, the latter is thus a good analytic open subgroup of $G(L)$.

□

The lemma and [26, 5.3] imply that the ring $\mathcal{D}^{an}(G(n)^\circ)$ is coherent for any n larger than $\frac{e}{p-1}$. The corresponding category $\mathrm{Mod}^{\mathrm{fp}}(\mathcal{D}^{an}(G(n)^\circ)$ of finitely presented $\mathcal{D}^{an}(G(n)^\circ)$-modules is then abelian.

Let V be an admissible $G(L)$-representation. Let $V_{G(n)^\circ-\mathrm{an}} \subseteq V$ be its subspace of $G(n)^\circ$-analytic vectors [26, 3.4.1]. The latter is naturally a module over $\mathcal{D}^{an}(G(n)^\circ)$, cf. [26, 5.1.8] (adapted to the σ-affinoid rigid group $G(n)^\circ$). For fixed n and nonzero V, the subspace $V_{G(n)^\circ-\mathrm{an}}$ may be zero, which is why we introduce the subcategory $\mathrm{Rep}^{(n)}(G(L))$ of representations V which are topologically generated, as $G(L)$-representations, by their $G(n)^\circ$-analytic vectors. Any topologically irreducible G-representation lies in $\mathrm{Rep}^{(n)}(G(L))$, for sufficiently large n and, moreover, admits an infinitesimal character [24].

Proposition 4.3.2. *For sufficiently large* n, *the formation* $HC^{(n)}_{p-\mathrm{adic}}(V) := (V_{G(n)^\circ-\mathrm{an}})'$ *defines a contravariant exact and faithful functor*

$$HC^{(n)}_{p-\mathrm{adic}} : \mathrm{Rep}^{(n)}(G(L)) \longrightarrow \mathrm{Mod}^{\mathrm{fp}}(\mathcal{D}^{an}(G(n)^\circ).$$

Proof. Let n be larger than $\frac{e}{p-1}$. The lemma and [25, A.13/14] show that the functor is well-defined and exact. The property $HC^{(n)}_{p-\mathrm{adic}}(V) = 0$ obviously implies $V = 0$. Since $HC^{n}_{p-\mathrm{adic}}$ is exact, this gives faithfulness. □

We believe that the functors $HC^{(n)}_{p-\mathrm{adic}}$ are the correct p-adic analogue for the functor HC in the real setting. Let V_1 and V_2 be two topologically irreducible admissible $G(L)$-representations. We say V_1 and V_2 are *infinitesimally equivalent* if $HC^{(n)}_{p-\mathrm{adic}}(V_1) \simeq$

$HC^{(n)}_{p-\mathrm{adic}}(V_2)$ for some sufficiently large n, such that $V_1, V_2 \in \mathrm{Rep}^{(n)}(G(L))$. It would be interesting to classify topologically irreducible representations up to infinitesimal equivalence. At the moment, a p-adic analogue of Casselman's globalization result is not known (at least to our knowledge), but it still seems relevant to obtain more information on the categories $\mathrm{Mod}^{\mathrm{fp}}(\mathcal{D}^{an}(G(n)^{\circ}_{\theta})$ and their irreducible modules. At first step, one might want to consider only the trivial character θ_0 and might ask, analogous to the real-analytic setting, for a geometric approach via some sort of p-adic Beilinson-Bernstein localization.

A first basic result in this direction is [39, 5.3.1] which provides a canonical isomorphism

$$\mathcal{D}^{an}(G(n)^{\circ}) \simeq D^{\dagger}(\mathcal{G}(n))$$

with the crystalline distribution algebra $D^{\dagger}(\mathcal{G}(n))$ of the p-adic completion $\mathcal{G}(n)$ of $G(n)$. Moreover, in [38] we have introduced the sheaf of arithmetic differential operators $\mathscr{D}^{\dagger}_{\mathcal{P},n}$ of congruence level n on the formal flag scheme $\mathcal{P}$ of G. We proved that the global sections of $\mathscr{D}^{\dagger}_{\mathcal{P},n}$ give back $D^{\dagger}(\mathcal{G}(n))_{\theta_0}$ and that $\mathcal{P}$ is coherently $\mathscr{D}^{\dagger}_{\mathcal{P},n}$-affine. We therefore expect that the category $\mathrm{Mod}^{\mathrm{fp}}(\mathcal{D}^{an}(G(n)^{\circ}_{\theta_0})$ and its irreducible modules can effectively be studied through the geometry of arithmetic $\mathscr{D}$-modules with congruence level structures on $\mathcal{P}$. Even if we are ultimately only interested in results up to sufficiently large n, we expect that the methods for higher n will strongly be inspired by the classical case $n = 0$. The methods and results in the present article in the case $n = 0$ thus form a first step in this programme.

5. Highest weight representations and the rank one case

We keep the notation from the preceding section.

5.1. **Highest weight representations.** We assume in this section that the field L is *locally compact*. This allows us to make use of the results in [48]. Under this hypothesis, we establish a crystalline version of the central block of the classical BGG category $\mathcal{O}$. We go on and show that its irreducible objects are geometrically overholonomic and compute their associated parameters $(Y, \mathcal{E})$ in the geometric classification 4.2.3.

Let Δ be the set of simple roots in Φ^+. We fix a (Chevalley) basis for $\mathrm{Lie}(G)$ compatible with its root space decomposition. In particular, we obtain a $\mathfrak{o}$-basis $t_1, ..., t_n$ of $\mathrm{Lie}(T)$ which is made up from a L-basis of the center of $\mathfrak{g}$ and finitely many elements t_α, indexed by $\alpha \in \Delta$, such that $\beta(t_\alpha) \in \mathbb{Z}$ for all $\beta \in \Phi$. Let $\Gamma := \mathbb{Z}_{\geqslant 0}\Phi^+ \subset \mathbb{Q}\Phi =: \Lambda_r \subseteq \Lambda$ where Λ_r and Λ are the root lattice and the integral weight lattice respectively.

For $w \in W$ we let $\lambda_w = -w(\rho) - \rho$. These are $|W|$ pairwise different elements of Λ_r.

Let $\mathcal{O}_0$ be the central block of the classical BGG category, e.g. [36]. This is a full abelian subcategory of finitely generated $U(\mathfrak{g})_{\theta_0}$-modules which is noetherian and artinian. Its irreducible objects are given by the unique irreducible quotients $M(\lambda_w) \to L(\lambda_w)$ where

$$M(\lambda_w) := U(\mathfrak{g}) \otimes_{U(\mathfrak{t}),\lambda_w} L$$

is the Verma module with highest weight λ_w for $w \in W$.

To define a crystalline variant of the category $\mathcal{O}_0$ we follow the constructions given in [48] in the case of the Arens-Michael envelope of $U(\mathfrak{g})$. In order to do so, we need the field L to be locally compact.

By the discussion in [39, 5.3] the algebra $D^\dagger(\mathcal{G}) = \varinjlim_m \widehat{D}^{(m)}(\mathcal{G}) \otimes \mathbb{Q}$ is an inductive limit of Hausdorff locally convex L-vector spaces with injective and compact transition maps. According to [49, 7.19/16.9/16.10] it is therefore Hausdorff, complete and barrelled.

The framework of diagonalisable modules over suitable commutative topological L-algebras as described in [48, sec. 2] applies therefore to the L-algebra $D^\dagger(\mathcal{T})$. Note that it contains the universal enveloping algebra $U(\mathfrak{t})$ as a dense subalgebra. A *L-valued weight* λ of $D^\dagger(\mathcal{T})$ is a L-algebra homomorphism $D^\dagger(\mathcal{T}) \to L$. A set of weights Y is called *relatively compact* if its image under the injective map $\lambda \mapsto (\lambda(t_1), ..., \lambda(t_n))$ has a compact closure in L^n. Let λ be weight and M some topological $D^\dagger(\mathcal{T})$-module. A nonzero $m \in M$ is called a *λ-weight vector* if $h.m = \lambda(h).m$ for all $h \in D^\dagger(\mathcal{T})$. In this case λ is called a *weight of M*. The closure M_λ in M of the L-vector space generated by all λ-weight vectors is called the *λ-weight space* of M. The module M is called *$D^\dagger(\mathcal{T})$-diagonalisable* if there is a set of weights $\Pi(M)$ with the property: to every $m \in M$ there exists a family $\{m_\lambda \in M_\lambda\}_{\lambda \in \Pi(M)}$ converging cofinitely against zero in M and satisfying

$$m = \sum_{\lambda \in \Pi(M)} m_\lambda.$$

Given a diagonalisable module M we may form $M^{ss} = \oplus_{\lambda \in \Pi(M)} M_\lambda$ (depending on the choice of $\Pi(M)$).

Definition 5.1.1. The category $\mathcal{O}_0^\dagger$ equals the full subcategory of $D^\dagger(\mathcal{G})_{\theta_0}$-modules M satisfying:

(1) M is a coherent $D^\dagger(\mathcal{G})_{\theta_0}$-module
(2) M is $D^\dagger(\mathcal{T})$-diagonalisable with $\Pi(M)$ contained in the union of the cosets $\lambda_w - \Gamma$
(3) All weight spaces $M_\lambda, \lambda \in \Pi(M)$, are finite dimensional over L.

By definition, given $M \in \mathcal{O}_0^\dagger$, then any finitely generated $U(\mathfrak{g})$-submodule of M^{ss} lies in $\mathcal{O}_0$. In particular, M^{ss} contains a *maximal vector*, i.e. a nonzero $m \in M_\lambda$ (of some weight λ) such that $\mathfrak{n}.m = 0$. We will make precise the relation between the two categories $\mathcal{O}_0$ and $\mathcal{O}_0^\dagger$ below.

We list some basic properties of the category $\mathcal{O}_0^\dagger$.

Proposition 5.1.2. (i) *The direct sum of two modules of $\mathcal{O}_0^\dagger$ is in $\mathcal{O}_0^\dagger$*

(ii) *the (co)kernel and (co)image of an arbitrary $D^\dagger(\mathcal{G})_{\theta_0}$-linear map between objects in $\mathcal{O}_0^\dagger$ is in $\mathcal{O}_0^\dagger$*
(iii) *the sum of two coherent submodules of an object in $\mathcal{O}_0^\dagger$ is in $\mathcal{O}_0^\dagger$*
(iv) *any finitely generated submodule of an object in $\mathcal{O}_0^\dagger$ is in $\mathcal{O}_0^\dagger$*

(v) $\mathcal{O}_0^\dagger$ *is an abelian category.*

Proof. This can be proved using a variant of the proof of [48, Prop. 3.6.3]. Note that any $\Pi(M)$ which is contained in the union of the cosets $\lambda_w - \Gamma$ is relatively compact. Indeed, Γ is relatively compact its closure being contained in the compact subset $\mathbb{Z}_p^{|\Delta|}$ of L^n, cf. [48, Lem. 3.6.1]. □

We exhibit Verma type modules in $\mathcal{O}_0^\dagger$. The main difference between the case of the crystalline distribution algebra and the case of the Arens-Michael envelope treated in [48] is that *not* every weight $\mathfrak{t} \to L$ extends to a weight of $D^\dagger(\mathcal{T})$. The following lemma is sufficient for our purposes.

Lemma 5.1.3. *Any linear form $\lambda : \mathrm{Lie}(T) \to \mathfrak{o}$ such that $\lambda(h_i) \in \mathbb{Z}_p$ for all $i = 1, ..., n$ extends canonically to a L-algebra homomorphism $D^\dagger(\mathcal{T}) \to L$.*

Proof. Recall that the distribution algebra $\mathrm{Dist}(\mathbb{G}_m)$ of the $\mathfrak{o}$-group scheme $\mathbb{G}_m$ is generated as an $\mathfrak{o}$-module by the elements $\binom{\delta_1}{k}$ for $k \in \mathbb{N}$ where δ_1 is a generator of $\mathrm{Lie}(\mathbb{G}_m)$, cf. [41, Part I.7.8]. Our choice of Chevalley basis implies an isomorphism of group schemes $T \simeq \prod_{i=1,...,n} \mathbb{G}_m$ such that the basis element h_i becomes the generator of the i-th copy $\mathrm{Lie}(\mathbb{G}_m)$. Since $\binom{\lambda(h_i)}{k} \in \mathbb{Z}_p$, the associated L-algebra homomorphism $\lambda : U(\mathfrak{t}) \to L$ restricts to an $\mathfrak{o}$-algebra homomorphism $\mathrm{Dist}(T) \to \mathfrak{o}$. Since $\mathrm{Dist}(T) = \varinjlim_m D^{(m)}(T)$, this extends then to a L-algebra homomorphism $D^\dagger(\mathcal{T}) \to L$, □

We may apply the lemma to any weight λ_w and hence consider the $D^\dagger(\mathcal{G})$-module

$$M^\dagger(\lambda_w) := D^\dagger(\mathcal{G}) \otimes_{D^\dagger(\mathcal{T}),\lambda_w} L.$$

Proposition 5.1.4. *The module $M^\dagger(\lambda_w)$ lies in $\mathcal{O}_0^\dagger$. We have*

$$M^\dagger(\lambda_w)^{ss} = M(\lambda_w) \quad \textit{and} \quad M^\dagger(\lambda_w) = D^\dagger(\mathcal{G}) \otimes_{U(\mathfrak{g})} M(\lambda_w).$$

There is a canonical inclusion preserving bijection between subobjects of $M^\dagger(\lambda_w)$ and abstract $U(\mathfrak{g})$-submodules of $M(\lambda_w)$. In particular, $M^\dagger(\lambda_w)$ admits a unique maximal subobject and hence a unique irreducible quotient $L^\dagger(\lambda_w)$. The latter satisfies $L^\dagger(\lambda_w)^{ss} = L(\lambda_w)$.

Proof. This can be proved as in [48, Prop. 3.7.1]. Note that the triangular decomposition

$$D^{(m)}(G) = D^{(m)}(N^-) \otimes_{\mathfrak{o}} D^{(m)}(T) \otimes_{\mathfrak{o}} D^{(m)}(N),$$

cf. [40, 2.2], implies that $M^\dagger(w) \simeq D^\dagger(\mathcal{N}^-)$ as a left $D^\dagger(\mathcal{N}^-)$-module. This implies the first displayed identity. Moreover, $M^\dagger(\lambda_w)$ equals the quotient of $D^\dagger(\mathcal{G})$ by the left ideal generated by $\ker(\lambda_w)$, which implies the second displayed identity. Note also that the nonzero quotient morphism $M^\dagger(\lambda_w) \to L^\dagger(\lambda_w)$ yields a nonzero quotient morphism $M^\dagger(\lambda_w)^{ss} \to L^\dagger(\lambda_w)^{ss}$ since $(-)^{ss}$ is faithful and exact [48, Prop. 2.0.2]. Hence $M^\dagger(\lambda_w)^{ss} = M(\lambda_w)$ implies $L^\dagger(\lambda_w)^{ss} = L(\lambda_w)$. □

Corollary 5.1.5. *The modules $L^\dagger(\lambda_w)$ exhaust, up to isomorphism, all the irreducible objects in $\mathcal{O}_0^\dagger$.*

Proof. Let L be an irreducible object in $\mathcal{O}_0^\dagger$. Take a maximal vector $m \in L^{ss}$ of some weight λ. Then $U(\mathfrak{g})m$ is a highest weight module in $\mathcal{O}$ of weight λ, cf. [36, 1.2]. Hence $Z(\mathfrak{g})$ acts on the maximal vector m via the central character θ_λ associated to λ via the Harish-Chandra homomorphism [36, 1.7]. But $U(\mathfrak{g})m \subset L$ whence $\theta_\lambda = \theta_0$ and so $\lambda = \lambda_w$ for some $w \in W$. We obtain a nonzero $D^\dagger(\mathcal{G})$-linear map $M^\dagger(\lambda_w) \to L, 1 \otimes 1 \mapsto m$. So L is an irreducible quotient of $M^\dagger(\lambda_w)$, i.e. $L \simeq L^\dagger(\lambda_w)$. $\square$

Corollary 5.1.6. *The category $\mathcal{O}_0^\dagger$ is artinian and noetherian.*

Proof. This can be deduced similarly to [48, Prop.4.2.2]. In fact, let $M \in \mathcal{O}_0^\dagger$ and consider the finite-dimensional L-vector space $V := \sum_w M_{\lambda_w}$. Suppose $N' \subsetneq N \subseteq M$ are two subobjects. Let $m \in N \backslash N'$ be a maximal vector of some weight λ. As in the preceding proof we deduce from the action of $Z(\mathfrak{g})$ on m that $\lambda = \lambda_w$ for some $w \in W$. So $m \in N \cap V$ whence $\dim_L N \cap V > \dim_L N' \cap V$. This implies that M has finite length. $\square$

Given a module $M \in \mathcal{O}_0$, we can define the coherent $D^\dagger(\mathcal{G})_{\theta_0}$-module

$$M^\dagger := D^\dagger(\mathcal{G}) \otimes_{U(\mathfrak{g})} M.$$

Theorem 5.1.7. *The functor $F : M \rightsquigarrow M^\dagger$ is exact and induces an equivalence of abelian categories*

$$\mathcal{O}_0 \xrightarrow{\simeq} \mathcal{O}_0^\dagger.$$

A quasi-inverse is given by the functor $(-)^{ss}$.

Proof. The ring extension $U(\mathfrak{g}) \to D^\dagger(\mathcal{G})$ is flat [40, Lem. 4.1]. We already now that $F(M(\lambda_w)) = M^\dagger(\lambda_w)$. Since any object $M \in \mathcal{O}_0$ admits a finite composition series with irreducible constituents of the form $L(w)$, there is a surjection $\oplus_w M(\lambda_w) \to M$. Since F commutes with direct sums, we see that $F(M)$ equals the quotient of $\oplus_w M^\dagger(\lambda_w)$ modulo a finitely generated submodule and so lies in $\mathcal{O}_0^\dagger$, according to parts (iii)-(v) of 5.1.2. We therefore have an exact functor $F : \mathcal{O}_0 \to \mathcal{O}_0^\dagger$. Given $M \in \mathcal{O}_0^\dagger$ we have a functorial morphism $M \to F(M)^{ss}, m \mapsto 1 \otimes m$ which is bijective for irreducible M according to 5.1.4. By dévissage, we obtain $M \simeq F(M)^{ss}$ in general. Let $M \in \mathcal{O}_0^\dagger$. To obtain $M^{ss} \in \mathcal{O}_0$ we use induction on the length of M and suppose that $N \subset M$ is a maximal submodule, i.e. $M/N \simeq L^\dagger(\lambda_w)$ for some w, such that $N^{ss} \in \mathcal{O}_0$. Exactness of $(-)^{ss}$ and $L^\dagger(\lambda_w)^{ss} = L(\lambda_w)$ implies that M^{ss} is an extension of two finitely generated $U(\mathfrak{g})$-modules and hence itself finitely generated. So $M^{ss} \in \mathcal{O}_0$. We may now deduce that $(-)^{ss}$ is also a right quasi-inverse to F. Indeed, for any $M \in \mathcal{O}_0^\dagger$, there is a natural morphism $F(M^{ss}) \to M$ in $\mathcal{O}_0^\dagger$ which is bijective for irreducible M according to 5.1.4. By dévissage, we obtain $F(M^{ss}) \simeq M$ in general. $\square$

To conclude, we will show that the irreducible modules $L^\dagger(\lambda_w)$ are all geometrically F-overholonomic.[5] To do this, fix $w \in W$ and let

$$Y_w := BwB/B \subset P = G/B$$

[5] The local compactness assumption on L is not necessary for this.

be the Bruhat cell in P associated with $w \in W$. Let $v : Y_w \hookrightarrow P$ be the corresponding immersion over $\mathfrak{o}$ and let $v_{\mathbb{Q}} : Y_{w\mathbb{Q}} \hookrightarrow P_{\mathbb{Q}}$ be the corresponding immersion on the level of L-algebraic varieties. It is well-known (e.g. [35, Prop. 12.3.2]) that there is a canonical isomorphism of $\mathscr{D}_{P_{\mathbb{Q}}}$-modules

$$\mathrm{Loc}(L(\lambda_w)) := \mathscr{D}_{P_{\mathbb{Q}}} \otimes_{U(\mathfrak{g})} L(\lambda_w) \simeq v_{\mathbb{Q}!+}(\mathcal{O}_{Y_{w\mathbb{Q}}}).$$

Let $X_w \subseteq P$ be the Zariski closure of the Bruhat cell Y_w in P, a Schubert scheme. We let

$$X'_w \longrightarrow X_w$$

be its *Demazure desingularization*, which is defined at the level of $\mathfrak{o}$-schemes [41, II, 13.6]. We are then in the axiomatic situation (S), the point of departure for subsection 3.4, so that all the results of this subsection apply. In particular, we have the frame $\mathbb{Y}_w = (Y_{w,s}, X_{w,s}, \mathcal{P})$ together with its c-locally closed immersion

$$v : \mathbb{Y}_w \longrightarrow \mathbb{P}$$

and the constant overholonomic module $\mathcal{O}_{\mathbb{Y}_w}$ on $\mathbb{Y}_w$. Its intermediate extension $v_{!+}(\mathcal{O}_{\mathbb{Y}_w})$ is an overholonomic F-$\mathscr{D}^{\dagger}_{\mathcal{P}}$-module, cf. 2.3.7. In this situation, the main theorem 3.4.4 implies directly the following result.

Proposition 5.1.8. *There is a canonical isomorphism of $\mathscr{D}^{\dagger}_{\mathcal{P}}$-modules*

$$\mathscr{D}^{\dagger}_{\mathcal{P}} \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} \overline{v_{\mathbb{Q}!+}(\mathcal{O}_{Y_{w\mathbb{Q}}})} \simeq v_{!+}(\mathcal{O}_{\mathbb{Y}_w}).$$

Now consider the localization

$$\mathscr{L}oc(L^{\dagger}(\lambda_w)) = \mathscr{D}^{\dagger}_{\mathcal{P}} \otimes_{D^{\dagger}(\mathcal{G})} L(\lambda_w)^{\dagger}.$$

Theorem 5.1.9. *Let $w \in W$. There is a $\mathscr{D}^{\dagger}_{\mathcal{P}}$-linear isomorphism*

$$\mathscr{L}oc(L^{\dagger}(\lambda_w)) \simeq v_{!+}(\mathcal{O}_{\mathbb{Y}_w}).$$

The crystalline highest weight module $L^{\dagger}(\lambda_w)$ is geometrically F-overholonomic.

Proof. We write $L(w)$ resp. $L^{\dagger}(w)$ for $L(\lambda_w)$ resp. $L^{\dagger}(\lambda_w)$. Since $L^{\dagger}(w) = D^{\dagger}(\mathcal{G}) \otimes_{U(\mathfrak{g})} L(w)$, associativity of tensor products yields a canonical isomorphism

$$\mathscr{D}^{\dagger}_{\mathcal{P}} \otimes_{D^{\dagger}(\mathcal{G})} L(w)^{\dagger} \simeq \mathscr{D}^{\dagger}_{\mathcal{P}} \otimes_{U(\mathfrak{g})} L(w) \simeq \mathscr{D}^{\dagger}_{\mathcal{P}} \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} (\overline{\mathscr{D}}_{P_{\mathbb{Q}}} \otimes_{U(\mathfrak{g})} L(w)) \simeq \mathscr{D}^{\dagger}_{\mathcal{P}} \otimes_{\overline{\mathscr{D}}_{P_{\mathbb{Q}}}} \overline{\mathrm{Loc}(L(w))}.$$

Since $\mathrm{Loc}(L(w)) \simeq v_{\mathbb{Q}!+}(\mathcal{O}_{Y_{w\mathbb{Q}}})$, the asserted isomorphism follows now in combination with 5.1.8. Since $v_{!+}(\mathcal{O}_{\mathbb{Y}_w})$ is a overholonomic F-$\mathscr{D}^{\dagger}_{\mathcal{P}}$-module, the module $L^{\dagger}(w)$ is seen to be geometrically F-overholonomic. □

5.2. **The SL_2-case.** We first establish some general results for curves. We also assume, in the case $p = 2$ that $\mathfrak{o}$ is an unramified extension of $\mathbb{Z}_2$. If $\mathfrak{o}$ has ramification index equal to e, one can choose an integer h satisfying

$$\frac{e}{p-1} < h \leqslant e.$$

Then $p \in \varpi^h\mathfrak{o}$. Denote by σ the p-power map $\mathfrak{o}/\varpi^h\mathfrak{o} \to \mathfrak{o}/\varpi^h\mathfrak{o}$ and by $\sigma^{(s)}$ the composite $\sigma \circ \ldots \circ \sigma$ (s times). Let $\mathfrak{X}$ be a smooth $\mathfrak{o}$-formal scheme of relative dimension 1, $\mathfrak{X}'$ a smooth formal scheme lifting $X^{(s)} = X \times_{\sigma^{(s)}} \mathrm{Spec}\,(\mathfrak{o}/\varpi^h\mathfrak{o})$ where X denotes the special fiber of $\mathfrak{X}$. Let F be the relative Frobenius $X \to X^{(s)}$, such that the composite map $X \to X$ given by $p_1 \circ F$ equals the map $x \mapsto x^{p^s}$ (where p_1 denotes the first projection $X^{(s)} \to X$), and $F : \mathfrak{X} \to \mathfrak{X}'$ a lifting of the relative Frobenius. Such a lifting F always exists, if $\mathfrak{X}$ is affine or equal to the formal projective line. In [8], Berthelot proved, for any $l \in \mathbb{N}$, that the $\mathcal{O}$-module inverse image F^* induces an equivalence of categories between coherent $\widehat{\mathcal{D}}^{(l)}_{\mathfrak{X}',\mathbb{Q}}$-modules (resp. $\mathscr{D}^{\dagger}_{\mathfrak{X}'}$-coherent modules) and $\widehat{\mathcal{D}}^{(l+s)}_{\mathfrak{X},\mathbb{Q}}$-modules (resp. $\mathscr{D}^{\dagger}_{\mathfrak{X}}$-coherent modules). Denote

$$F^{\flat}_d(\widehat{\mathcal{D}}^{(l)}_{\mathfrak{X}',\mathbb{Q}}) := \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}'}}(\mathcal{O}_{\mathfrak{X}}, \widehat{\mathcal{D}}^{(l)}_{\mathfrak{X}',\mathbb{Q}}),$$

where the $\mathcal{H}om$ is taken for the right $\mathcal{O}_{\mathfrak{X}'}$-module structure of $\widehat{\mathcal{D}}^{(l)}_{\mathfrak{X}',\mathbb{Q}}$. This module is a $(\widehat{\mathcal{D}}^{(l)}_{\mathfrak{X}',\mathbb{Q}}, \widehat{\mathcal{D}}^{(l+s)}_{\mathfrak{X},\mathbb{Q}})$-bimodule and a quasi-inverse functor for F^* is given by

$$\mathcal{E}' \mapsto F^{\flat}_d(\widehat{\mathcal{D}}^{(l)}_{\mathfrak{X}',\mathbb{Q}}) \otimes_{\widehat{\mathcal{D}}^{(l+s)}_{\mathfrak{X},\mathbb{Q}}} \mathcal{E}.$$

In the same manner, a quasi-inverse for F^*, on the level of $\mathscr{D}^{\dagger}_{\mathfrak{X}}$-modules, is constructed via the $(\mathscr{D}^{\dagger}_{\mathfrak{X}'}, \mathscr{D}^{\dagger}_{\mathfrak{X}})$-bimodule $F^{\flat}_d(\mathscr{D}^{\dagger}_{\mathfrak{X}'})$, which is defined analogously. Moreover, if $\mathfrak{X}$ is affine and endowed with a local coordinate t, then $F^{\flat}_d(\widehat{\mathcal{D}}^{(0)}_{\mathfrak{X}',\mathbb{Q}})$, is a finite free left $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{X}',\mathbb{Q}}$-module of rank p. The left $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{X}',\mathbb{Q}}$-module structure of $F^{\flat}_d(\widehat{\mathcal{D}}^{(0)}_{\mathfrak{X}',\mathbb{Q}})$ is given by $(P \cdot u)(x) := P \cdot u(x)$, and we have an explicit isomorphism of $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{X}',\mathbb{Q}}$-modules $F^{\flat}_d(\widehat{\mathcal{D}}^{(0)}_{\mathfrak{X}',\mathbb{Q}}) \simeq (\widehat{\mathcal{D}}^{(0)}_{\mathfrak{X}',\mathbb{Q}})^{\oplus p}$ given by $u \mapsto (u(1), u(t), \ldots, u(t^{p-1})) \in (\widehat{\mathcal{D}}^{(0)}_{\mathfrak{X}',\mathbb{Q}})^{\oplus p}$.

We will need the following lemma. Recall that a smooth formal $\mathfrak{o}$-scheme $\mathfrak{Y}$ is called *coherently $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y},\mathbb{Q}}$-affine*, if the global section functor induces an equivalence between coherent modules over $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y},\mathbb{Q}}$ and over its ring of global sections respectively.

Lemma 5.2.1. *Let $\mathfrak{Y}$ be an irreducible smooth formal curve over $\mathfrak{o}$, which is coherently $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y},\mathbb{Q}}$-affine. If $\mathcal{F}$ is a coherent $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y},\mathbb{Q}}$-module, such that $\mathcal{F}$ has no subquotient isomorphic to $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y},\mathbb{Q}}$, then $\mathcal{F}$ is holonomic.*

Proof. Let us prove the lemma. Using Caro's criterion [17, Théorème 2.6, Proposition 2.8], it suffices to show that $\mathcal{H}om_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y},\mathbb{Q}}}(\mathcal{F}, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y},\mathbb{Q}}) = 0$. Denote by $D = \Gamma(\mathfrak{Y}, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y},\mathbb{Q}})$, which is a coherent ring, by hypothesis. There is an open affine $\mathfrak{V} \subset \mathfrak{Y}$, such that the ring $E = \Gamma(\mathfrak{V}, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y},\mathbb{Q}})$ has no zero divisors [28].

Fact. Restriction to $\mathfrak{V} \subset \mathfrak{Y}$ gives an inclusion $D \subset E$.

Proof of the fact. The inclusion comes, after taking global sections over $\mathfrak{Y}$, from the inclusion of sheaves $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y},\mathbb{Q}} \hookrightarrow j_*\widehat{\mathcal{D}}^{(0)}_{\mathfrak{V},\mathbb{Q}}$, where j denotes the inclusion $\mathfrak{V} \subset \mathfrak{Y}$. The statement is local over $\mathfrak{Y}$, so that we can assume that $\mathfrak{Y}$ is irreducible and endowed with a local coordinate and associated derivation ∂, so that $\mathcal{D}^{(0)}_{\mathfrak{Y}}$ is $\mathcal{O}_{\mathfrak{Y}}$-free on the basis $\{\partial^{\nu}\}_{\nu \geqslant 0}$. The inclusion $\mathcal{O}_{\mathfrak{Y}} \hookrightarrow j_*\mathcal{O}_{\mathfrak{V}}$ therefore implies the inclusion $\mathcal{D}^{(0)}_{\mathfrak{Y}} \hookrightarrow j_*\mathcal{D}^{(0)}_{\mathfrak{V}}$. Passing to p-adic completions and inverting p gives the inclusion $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y},\mathbb{Q}} \hookrightarrow j_*\widehat{\mathcal{D}}^{(0)}_{\mathfrak{V},\mathbb{Q}}$, as claimed.

Returning to the proof of the lemma, the inclusion $D \subset E$, shows that also D has no zero divisors. Moreover, $F = \Gamma(\mathfrak{Y}, \mathcal{F})$ is a coherent D-module. If $x_1, \ldots, x_r$ denotes a finite set of D-module generators for F, then let $N_i := \sum_{k=1}^{i} D \cdot x_k \subset F$ for $i > 0$ and $N_0 := 0$. Any N_i, being a finitely generated submodule of F, is coherent over D and therefore $\mathcal{N}_i = \widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y},\mathbb{Q}} \otimes_D N_i$ is coherent over $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y},\mathbb{Q}}$. By construction, we have the finite filtration

$$0 = \mathcal{N}_0 \subset \mathcal{N}_1 \subset \ldots \subset \mathcal{N}_{r-1} \subset \mathcal{N}_r.$$

We will prove by finite induction on $i \geqslant 0$, that $\mathcal{H}om_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y},\mathbb{Q}}}(\mathcal{N}_i, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y},\mathbb{Q}}) = 0$. This is trivial for $i = 0$. Assume that this is true for some i. We have an exact sequence of coherent $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y},\mathbb{Q}}$-modules

$$0 \to \mathcal{N}_i \to \mathcal{N}_{i+1} \to \mathcal{N}_{i+1}/\mathcal{N}_i \to 0,$$

thus an exact sequence

$$0 \to \mathcal{H}om_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y},\mathbb{Q}}}(\mathcal{N}_{i+1}/\mathcal{N}_i, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y},\mathbb{Q}}) \to \mathcal{H}om_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y},\mathbb{Q}}}(\mathcal{N}_{i+1}, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y},\mathbb{Q}}) \to \mathcal{H}om_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y},\mathbb{Q}}}(\mathcal{N}_i, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y},\mathbb{Q}}).$$

The coherent left $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y},\mathbb{Q}}$-module $\mathcal{N}_{i+1}/\mathcal{N}_i$ is generated by a single element x_{i+1} and is thus isomorphic to $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y},\mathbb{Q}}/\mathcal{J}$ with $\mathcal{J}$ some finitely generated left ideal of $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y},\mathbb{Q}}$. By hypothesis $\mathcal{J} \neq 0$, so that $J = \Gamma(\mathfrak{Y}, \mathcal{J}) \neq 0$. Take $f \in J$ a non zero element. Consider $u \in \mathcal{H}om_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y},\mathbb{Q}}}(\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y},\mathbb{Q}}/\mathcal{J}, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y},\mathbb{Q}})$ and $\overline{1}$ the class modulo J of $1 \in D$. Writing also u for its global sections, we have $fu(\overline{1}) = u(f \cdot \overline{1}) = u(\overline{0}) = 0$. Since D has no zero divisor, this means $u(\overline{1}) = 0$ and $u = 0$. Thus, $\mathcal{H}om_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y},\mathbb{Q}}}(\mathcal{N}_{i+1}/\mathcal{N}_i, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y},\mathbb{Q}}) = 0$, and this proves the assertion for $i + 1$. $\square$

Then we have the following

Proposition 5.2.2. *Assume that $\mathfrak{X}$ is affine or equal to the formal projective line over $\mathfrak{o}$. Any irreducible coherent $\mathscr{D}^{\dagger}_{\mathfrak{X}}$-module is holonomic.*

Note that we do not assume any Frobenius structure here.

Proof. By [17, Théorème 2.6, Proposition 2.8], it is enough to show $\mathcal{H}om_{\mathscr{D}^\dagger_{\mathfrak{X}}}(\mathcal{E}, \mathscr{D}^\dagger_{\mathfrak{X}}) = 0$. Let $m \in \mathbb{N}$, and $\mathcal{E}^{(m)}$ be a coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\mathbb{Q}}$-module such that

$$\mathcal{E} \simeq \mathscr{D}^\dagger_{\mathfrak{X}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\mathbb{Q}}} \mathcal{E}^{(m)}.$$

As

$$\mathcal{H}om_{\mathscr{D}^\dagger_{\mathfrak{X}}}(\mathcal{E}, \mathscr{D}^\dagger_{\mathfrak{X}}) \simeq \mathcal{H}om_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\mathbb{Q}}}(\mathcal{E}^{(m)}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\mathbb{Q}}) \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\mathbb{Q}}} \mathscr{D}^\dagger_{\mathfrak{X}},$$

it is enough to prove that

$$\mathcal{H}om_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\mathbb{Q}}}(\mathcal{E}^{(m)}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\mathbb{Q}}) = 0.$$

As we have remarked above, the relative Frobenius $X \to X^{(m)}$ admits a lifting F. By Frobenius descent applied to F, we have

$$\mathcal{H}om_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\mathbb{Q}}}(\mathcal{E}^{(m)}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\mathbb{Q}}) \simeq \mathcal{H}om_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{X}',\mathbb{Q}}}(\mathcal{F}, F^\flat_d(\widehat{\mathcal{D}}^{(0)}_{\mathfrak{X}',\mathbb{Q}})),$$

where

$$\mathcal{F} = F^\flat_d(\widehat{\mathcal{D}}^{(0)}_{\mathfrak{X}',\mathbb{Q}}) \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\mathbb{Q}}} \mathcal{E},$$

is a coherent (left) $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{X}',\mathbb{Q}}$-module. We will now prove that $\mathcal{F}$ is holonomic.
If $\mathfrak{X}$ is affine, then $\mathfrak{X}'$ is again affine. In the case of the projective line, $\mathfrak{X}'$ is again isomorphic to the projective line. In both cases, $\mathfrak{X}'$ is coherently $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{X}',\mathbb{Q}}$-affine (for the case of the projective line, see [37]). In order to apply the previous lemma, we have to prove that $\mathcal{F}$ has no subquotient isomorphic to $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{X}',\mathbb{Q}}$. Assume, for a contradiction, that $\mathcal{F}$ admits such a subquotient. Then $F^*\widehat{\mathcal{D}}^{(0)}_{\mathfrak{X}',\mathbb{Q}}$ is a subquotient of $F^*\mathcal{F} \simeq \mathcal{E}^{(m)}$, and

$$\mathscr{D}^\dagger_{\mathfrak{X}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\mathbb{Q}}} F^*\widehat{\mathcal{D}}^{(0)}_{\mathfrak{X}',\mathbb{Q}}$$

is a subquotient of $\mathscr{D}^\dagger_{\mathfrak{X}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\mathbb{Q}}} \mathcal{E}^{(m)} \simeq \mathcal{E}$. By compatibility of the Frobenius with tensor product [8, Chapitre 3], this is equivalent to saying that $F^*\mathscr{D}^\dagger_{\mathfrak{X}'}$ is a subquotient of $\mathcal{E}$. Using the equivalence of categories [8, Théorème 4.2.4], we get that

$$F^\flat\mathscr{D}^\dagger_{\mathfrak{X}'} \otimes_{\mathscr{D}^\dagger_{\mathfrak{X}}} F^*\mathscr{D}^\dagger_{\mathfrak{X}'} \text{ is a subquotient of } F^\flat\mathscr{D}^\dagger_{\mathfrak{X}'} \otimes_{\mathscr{D}^\dagger_{\mathfrak{X}}} \mathcal{E}.$$

By [8, Proposition 4.2.2] the left-hand side of the previous formula is isomorphic to $\mathscr{D}^\dagger_{\mathfrak{X}'}$ as bi-$\mathscr{D}^\dagger_{\mathfrak{X}'}$-module. Moreover, since F^* establishes an equivalence of categories, the right-hand side of the previous formula is an irreducible $\mathscr{D}^\dagger_{\mathfrak{X}'}$-module. We finally arrive at the fact that $\mathscr{D}^\dagger_{\mathfrak{X}'}$ is a subquotient of the irreducible module $F^\flat\mathscr{D}^\dagger_{\mathfrak{X}'}\otimes\mathcal{E}$, which gives a contradiction. Hence the lemma applies and proves that $\mathcal{F}$ is holonomic.
Consider now an open $\mathfrak{V} \subset \mathfrak{X}$, endowed with a local coordinate. Then $F^\flat\widehat{\mathcal{D}}^{(0)}_{\mathfrak{V}',\mathbb{Q}}$ is a free left $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{V}',\mathbb{Q}}$-module of rank p. As $\mathcal{F}$ is holonomic,

$$\mathcal{H}om_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{V}',\mathbb{Q}}}(\mathcal{F}|_{\mathfrak{V}'}, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{V}',\mathbb{Q}}) = 0,$$

which implies that

$$\mathcal{H}om_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{V}',\mathbb{Q}}}(\mathcal{F}|_{\mathfrak{V}'}, F^{\flat}\widehat{\mathcal{D}}^{(0)}_{\mathfrak{V}',\mathbb{Q}}) = 0.$$

Using a covering of $\mathfrak{X}$ by such $\mathfrak{V}$, one arrives at

$$\mathcal{H}om_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\mathbb{Q}}}(\mathcal{E}^{(m)}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\mathbb{Q}}) = \mathcal{H}om_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{X}',\mathbb{Q}}}(\mathcal{F}, F^{\flat}\widehat{\mathcal{D}}^{(0)}_{\mathfrak{X}',\mathbb{Q}}) = 0.$$

This implies $\mathcal{H}om_{\mathscr{D}^{\dagger}_{\mathfrak{X}}}(\mathcal{E}, \mathscr{D}^{\dagger}_{\mathfrak{X}}) = 0$ and shows that $\mathcal{E}$ is holonomic. $\square$

After these more general results, we return to the setting of 4.2 in the case of $G = \mathrm{SL}_2$. We let B be the subgroup of upper triangular matrices and $T \subset B$ be the subgroup of diagonal matrices. We identify $\Lambda = \mathbb{Z}$ so that $\Delta = \{\alpha\}$ with $\alpha = 2$. We identify

$$P = G/B = \mathbb{P}^1_{\mathfrak{o}}$$

with the projective line $\mathbb{P}^1_{\mathfrak{o}}$ over $\mathfrak{o}$. We choose an affine coordinate t around zero. The group G acts by fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} . (t) = \left(\frac{at+b}{ct+d}\right)$$

in the usual way. The stabiliser of the point $\infty \in \mathbb{P}^1_{\mathfrak{o}}$ is B.

Remark: In this setting, proposition 5.2.2 shows that *any* irreducible $D^{\dagger}(\mathcal{G})_{\theta_0}$-module is geometrically holonomic, i.e. localizes to a holonomic module. Under the presence of Frobenius structures, one knows that for coherent modules on quasi-projective varieties, the notions of holonomicity and overholonomicity are equivalent [19]. In general, the implication "holonomic $\Longrightarrow$ overholonomic" for coherent modules on curves is an open question[6].

In this setting, the theorem 4.2.3 gives a classification of in terms of irreducible overconvergent isocrystals $\mathcal{E}$ on couples $\mathbb{Y} = (Y, X)$ where Y is either :

(1) a closed point of $\mathbb{P}^1_k$ or

(2) an open complement of finitely many closed points $Z = \{y_1, ..., y_n\}$ of $\mathbb{P}^1_k$.

In case (1), the point is a complete invariant, since we have necessarily $\mathcal{E} = \mathcal{O}_{\mathbb{Y}}$ in this case. Suppose that the point is k-rational. Since the (finitely many) k-rational points $\mathbb{P}^1(k)$ of $\mathbb{P}^1_k$ form a single orbit under the natural action of the (finite) group $G(k)$ of k-rational points of G, it suffices to consider the point

$$\{\infty\} = Y_{1,s} = X_{1,s}$$

[6]The key problem is to verify whether an overconvergent isocrystal which is coherent and holonomic comes - up to alteration of the curve - from a convergent log-isocrystal. We thank Daniel Caro for explaining this point to us.

in $\mathbb{P}^1_k$. According to 5.1.9, the global sections of $v_{!+}(\mathcal{O}_{\mathbb{Y}})$ are equal to the $D^\dagger(\mathcal{G})_{\theta_0}$-module $L^\dagger(-2)$, the crystalline version of the classical anti-dominant Verma module $M(-2) = L(-2)$.
Suppose now that the point is k'-rational for a finite extension field k'/k. Let $M = H^0(\mathcal{P}, v_{!+}(\mathcal{O}_{\mathbb{Y}}))$. Let $\mathfrak{o}'$ be a finite extension of $\mathfrak{o}$ with residue field k' and quotient field L'. The base change $M_{L'} = M \otimes_L L'$ has the same geometric parameter, but now considered a rational point of the special fibre of $\mathcal{P} \times_{\mathfrak{o}} \mathfrak{o}'$. This means that M is a twisted form of the module $L^\dagger(-2)$, with respect to the field extension L'/L.

We come to case (2). For $Z = \varnothing$ and hence $Y = \mathbb{P}^1_k$ we obtain the trivial representation, i.e. the augmentation character $D^\dagger(\mathcal{G}) \to L$. Indeed, there are no convergent isocrystals on $\mathcal{P}$ besides the constant one, cf. 4.2.4. Let $n > 0$. Modulo the appearance of twisted forms (see the above argument), we may assume that all points $y_1, ..., y_n$ are k-rational and $y_1 = \infty$. There are then two extreme cases

$$Y = \mathbb{A}^1_k \quad \text{resp.} \quad Y = \mathbb{P}^1_k \backslash \mathbb{P}^1(k),$$

the affine line and so-called Drinfeld's upper half plane, respectively.

We discuss an interesting example in the case $Y = \mathbb{A}^1_k$. For this, we assume that L contains the p-th roots of unity μ_p and we choose an element $\pi \in \mathfrak{o}$ with

$$\mathrm{ord}_p(\pi) = 1/(p-1).$$

We have the affine coordinate t on $\mathbb{A}^1_{\mathfrak{o}}$ and we let $\partial = d/dt$. We let $\mathscr{L}_\varpi$ be the coherent $\mathscr{D}^\dagger_{\mathcal{P}}$-module defined by the *Dwork overconvergent F-isocrystal* L_π on $\mathbb{Y}$ associated with π, i.e. $\mathscr{L}_\pi = v_{!+}L_\pi$ where $v : \mathbb{Y} \to \mathbb{P}$. Recall that the underlying $\mathcal{O}_{\mathcal{P},\mathbb{Q}}$-module of $\mathscr{L}_\pi$ is $\mathcal{O}_{\mathcal{P},\mathbb{Q}}(\infty)$, endowed with a compatible $\mathscr{D}^\dagger_{\mathcal{P}}$-module structure for which $\partial(1) = -\pi$, [9, 4.5.5].

Write $\mathfrak{n} = L.e$ with $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Let $\eta : \mathfrak{n} \to L$ be a nonzero character and consider Kostant's *standard Whittaker module*

$$W_{\theta_0,\eta} := U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})\otimes U(\mathfrak{n})} L_{\theta_0,\eta}$$

with character η and central character θ_0, cf. [42, (3.6.1)] for its original definition over the complex numbers. It is an irreducible $U(\mathfrak{g})$-module, cf. [11, Lem. 5.3] which holds over any field of characteristic zero (note that in Block's notation $g(q) = q - \eta(e) \neq q$ here), but does not lie in $\mathcal{O}_0$. In fact, the restriction of the $\mathscr{D}_{\mathbb{P}^1_L}$-module $\mathrm{Loc}(W_{\theta_0,\eta})$ to $\mathbb{A}^1_L$ has an irregular singularity at ∞ [43, 4.4].

Let $W^\dagger_{\theta_0,\eta} := D^\dagger(\mathcal{G}) \otimes_{U(\mathfrak{g})} W_{\theta_0,\eta}$.

Theorem 5.2.3. *Let $\eta(e) := \pi$. There is a canonical $\mathscr{D}^\dagger_{\mathcal{P}}$-linear isomorphism*

$$\mathscr{L}oc(W^\dagger_{\theta_0,\eta}) \xrightarrow{\simeq} \mathscr{L}_\pi.$$

The crystalline Whittaker module $W^\dagger_{\theta_0,\eta}$ is geometrically F-overholonomic.

Proof. For any character η, the module $W_{\theta_0,\eta}$ admits the presentation

$$W_{\theta_0,\eta} = U(\mathfrak{g})/U(\mathfrak{g})(e - \eta(e)).$$

For our particular choice, one finds $W^{\dagger}_{\theta_0,\eta} = D^{\dagger}(\mathcal{G})/D^{\dagger}(\mathcal{G})(e-\pi)$. The canonical morphism $U(\mathfrak{g}) \to \mathscr{D}_{\mathbb{P}^1_L}$ maps e to $-\partial$, cf. [35, 11.2.1], and the isomorphism of part (b) in theorem 4.2.1 is compatible with this morphism. We obtain

$$\mathscr{L}oc(W^{\dagger}_{\theta_0,\eta}) = \mathscr{D}^{\dagger}_{\mathcal{P}}/\mathscr{D}^{\dagger}_{\mathcal{P}}(\partial + \pi)$$

which coincides with the standard presentation of the $\mathscr{D}^{\dagger}_{\mathcal{P}}$-module $\mathscr{L}_{\pi}$ [5, Prop. 5.2.3]. $\square$

Remark: It is interesting to note that the Dwork isocrystal $\mathscr{L}_{\pi}$ is *algebraic* in the sense that it comes from an algebraic $\mathscr{D}_{\mathbb{P}^1_L}$-module, namely $\mathrm{Loc}(W_{\theta_0,\eta})$, by extension of scalars $\overline{\mathscr{D}}_{\mathbb{P}^1_L} \to \mathscr{D}^{\dagger}_{\mathcal{P}}$.

We discuss an example in the second case, where $Y = \mathbb{P}^1_k\backslash\mathbb{P}^1(k)$. We identify $k = \mathbb{F}_q$. We assume that L contains the cyclic group μ_{q+1} of $(q+1)$-th roots of unity. We consider the so-called *Drinfeld curve*

$$Y' = \Big\{(x,y) \in \mathbb{A}^2_k \mid xy^q - x^q y = 1\Big\}.$$

It is an affine smooth irreducible curve and the map $(x,y) \mapsto [x : y]$ is an unramified Galois covering

$$u : Y' \longrightarrow Y$$

with Galois group μ_{q+1}. The group μ_{q+1} acts by homotheties $\zeta.(x,y) = (\zeta.x, \zeta.y)$. We have a smooth projective compactification

$$\overline{Y'} = \Big\{[x : y : z] \in \mathbb{P}^2_k \mid xy^q - x^q y = z^{q+1}\Big\}$$

and the covering extends to a smooth (and tamely ramified) morphism

$$u : \overline{Y'} \longrightarrow \mathbb{P}^1_k,$$

given by $[x : y : z] \mapsto [x : y]$. The boundary $Z' = \overline{Y'}\backslash Y'$ is mapped bijectively to $Z = \mathbb{P}^1(k)$ and the ramification index at each point in Z is $q+1$. For more details the reader may consult [12, chap. 2]. We denote by $u : \mathbb{Y}' \to \mathbb{Y}$ the morphism of couples induced by u. We let $\mathcal{E} = \mathbb{R}^{\bullet}u_{\mathrm{rig},*}\mathcal{O}_{\mathbb{Y}'}$ be the relative rigid cohomology sheaf which, in our situation, is just the direct image of $\mathcal{O}_{\mathbb{Y}}$ under the morphism u endowed with the Gauss-Manin connection.

Proposition 5.2.4. *The relative rigid cohomology sheaf, as an overconvergent F-isocrystal on $\mathbb{Y}$, admits a decomposition $\mathcal{E} = \oplus_{j=0,\dots,q}\mathcal{E}(j)$, where $\mathcal{E}(j)$ is the isotypic subspace (of rank one) on which μ_{q+1} acts by the character $\zeta \mapsto \zeta^j$. In particular, each pair $(Y, \mathcal{E}(j))$ corresponds to an irreducible geometrically overholonomic $D^{\dagger}(\mathcal{G})_{\theta_0}$-module $H^0(\mathcal{P}, v_{!+}\mathcal{E}(j))$.*

Proof. The cover $u : Y' \to Y$ is an abelian prime-to-p Galois covering as considered in [29]. The relative rigid cohomology, as an overconvergent F-isocrystal on the base Y (denoted there by $E^\dagger$) together with its decomposition $E^\dagger = \oplus_j E^\dagger(j)$ is constructed in [29, sec. 2]. Note that $u : Y' \to Y$ is even equal to (one of the $q-1$ connected components of) the Deligne-Lusztig torsor for the nonsplit torus μ_{q+1} in the finite group $G(\mathbb{F}_q)$, a special situation considered in [29, sec. 4]. $\square$

Are the modules $H^0(\mathcal{P}, v_{!+}\mathcal{E}(j))$ *algebraic* in the sense that they arise from irreducible $U(\mathfrak{g})$-modules, by extension of scalars $U(\mathfrak{g}) \to D^\dagger(\mathcal{G})$? Let us remark that the *théorème d'algébrisation* of Christol-Mebkhout [21, thm. 5.0-10] implies that any overconvergent F-isocrystal on the open Y is algebraic, i.e. comes from an algebraic connection on a characteristic zero lift of Y. However, this does not imply (at least a priori) that the intermediate extensions preserve this algebraicity. To our knowledge, the most general result in this direction at the moment is our theorem 3.4.4 above. Actually, to prove this algebraicity result, it would be enough to prove an analogous statement to 3.4.4 in the case where the map b (using notations of 3.4) induces a finite étale cover $Y' := b^{-1}Y \to Y$. To prove 3.4.4, we use the equivalence of categories [1, 1.2.8]. We need to replace this argument in the case where b induces a finite étale cover $Y' := b^{-1}Y \to Y$.

If the modules $H^0(\mathcal{P}, v_{!+}\mathcal{E}(j))$ are algebraic, to which class do they belong? We recall that irreducible $U(\mathfrak{g})$-modules fall into three classes: highest weight modules, Whittaker modules and a third class whose objects (with a fixed central character) are in bijective correspondence with similarity classes of irreducible elements of a certain localization of the first Weyl algebra [11]. We plan to come back to these question in future work.

We finish this paper with the remark, still in the case (2), that if we concentrate on the subcategory of overconvergent F-isocrystals on $Y = \mathbb{P}^1_k \backslash Z$ which are *unit-root*, then work of Tsuzuki [54, Thm. 7.2.3] shows that this category is equivalent to the category of p-adic representations of the étale fundamental group $\pi_1^{et}(Y)$ with finite monodromy (i.e. representations such that for each $y \in Z$ the inertia subgroup at y acts through a finite quotient). Of course, the trivial representation corresponds to the constant isocrystal $\mathcal{O}_{\mathbb{Y}}$.

6. Appendix: Complements on divisors

Lemma 6.0.1. *Let P be a normal, irreducible, separated noetherian scheme, Z a proper closed subset of P, $x \notin Z$, then there exists an effective Cartier divisor D of P such that $x \notin D$ and Z_{red} is a subscheme of D_{red}, where Z_{red} and D_{red} are the reduced schemes associated to Z and D respectively. Moreover $U = P \backslash D$ can be chosen to be an affine subset of P.*

Proof. Let $V = P \backslash Z$ and U an affine subscheme of V containing x. Then as P is irreducible, U is dense and since P is normal, we can apply [53, tag 0EGJ] to see that $P \backslash U$ is the support of a Cartier divisor D of P. We finally get the inclusion of topological spaces $|Z| \subset |P|$. Let $\mathcal{J}_Z$ and $\mathcal{J}_D$ be the sheaves of ideals defining the closed subschemes D and Z, the inclusion of topological spaces implies that on any affine subset of P, there exists

n such that $\mathcal{J}_D^n \subset \mathcal{J}_Z$. As P is noetherian, there exists N such that $\mathcal{J}_D^N \subset \mathcal{J}_Z$, which implies that there is a closed immersion $Z_{red} \hookrightarrow D_{red}$. □

Lemma 6.0.2. *Let P be a normal and irreducible separated noetherian scheme, X a proper closed subset of P, $X_1, \dots, X_r$ its irreducible components.*

(i) There exist effective Cartier divisors $D_1, \dots, D_N$ such that $X = \bigcap_{i=1}^N D_i$.
(ii) There exists an effective Cartier divisor D containing $X_2 \bigcup \dots \bigcup X_r$ and such that $X_1 \bigcap (P \backslash D)$ is dense in X_1. Moreover $U = P \backslash D$ can be chosen to be an affine subset of P.

Proof. Let us prove (i). Let $\mathcal{D}$ be the set of effective Cartier divisors containing X, which is not empty by the previous lemma. Note that $X = \bigcap_{D \in \mathcal{D}} D$. Indeed, if not, there exists some $x \in \bigcap_{D \in \mathcal{D}} D$ such that $x \notin X$. But, by the previous lemma we can find some divisor in $\mathcal{D}$ not containing x, which is a contradiction. It remains to show that this intersection is finite. If not, there is a sequence $(D_i)_{i \in \mathbb{N}}$ of divisors of $\mathcal{D}$ such that $Z_j := \bigcap_{i \leqslant j} D_i \subsetneq Z_{j-1}$. Denote by $\mathcal{I}_i$ the sheaf of ideals defining D_i, so that $Z_j = V(\mathcal{I}_1 + \dots + \mathcal{I}_j)$ and consider $U = \operatorname{Spec} A \subset P$ an affine open. As A is noetherian, the sequence $\mathcal{I}_1(U) + \dots + \mathcal{I}_j(U)$ is stationary, meaning that there exists n such that $Z_n \bigcap U = Z_m \bigcap U$, for any $m \geqslant n$. As P is noetherian, we see that there exists N such that $Z_m = Z_N$ for any $m \geqslant N$. This is a contradiction and the intersection is finite.
Let us prove (ii). Let $Y = X_2 \bigcup \dots \bigcup X_r$ and $x \in X_1 \backslash Y$. Applying 6.0.1 we see that there exists a divisor D containing Y such that $x \notin D$. Then $(P \backslash D) \bigcap X_1$ is dense since X_1 is irreducible. □

Lemma 6.0.3. *Let Y be an open dense normal subscheme of an irreducible scheme X, D a divisor of Y, then $\overline{D} \bigcap Y = D$.*

Proof. The question is local over X and we can assume that X is affine equal to $\operatorname{Spec} A$ and $Y = D(h)$. Localizing again on X and using the normality of Y, we can assume that the divisor is given by a single equation, i.e. $D = V(g)$ for some $g \in A$. Then $\overline{D} = V(A \bigcap A[1/h]g)$, so that $gA \subset A \bigcap A[1/h]g$ and $\overline{D} \bigcap Y \subset D$, which proves the claim as obviously $D \subset \overline{D} \bigcap Y$. □

From now on we consider varieties over some field k.

Lemma 6.0.4. *Let P be an irreducible, normal, quasi-projective variety and $X \subsetneq P$ a closed subvariety with irreducible components $X_1, \dots, X_r$. There exist reduced effective Cartier divisors $D_1, \dots, D_r$ in P such that*

$$\forall i \in \{1, \dots, r\},\ X_i \subset D_i \text{ and } \forall j \neq i, (P \backslash D_i) \bigcap X_j \text{ is dense in } X_j.$$

Proof. Pick a point x_i of X_i for each irreducible component X_i, then as $P \backslash X_i$ is quasi-projective, there is a dense affine open subset $U_i \subset P \backslash X_i$ containing $x_1, \dots, x_{i-1}, x_{i+1}, \dots x_r$ ([53, tag 01ZY]). And as P is normal, again by [53, tag 0EGJ], $P \backslash U_i$ is the support of some effective Cartier divisor D_i containing X_i and such that $(P \backslash D_i) \bigcap X_j$ is dense in X_j for $j \neq i$. □

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