ARITHMETIC DIFFERENTIAL OPERATORS ON A SEMISTABLE MODEL OF $\mathbb{P}^1$

DEEPAM PATEL, TOBIAS SCHMIDT, AND MATTHIAS STRAUCH

Abstract. In this paper we study sheaves of logarithmic arithmetic differential operators on a particular semistable model of the projective line. The main result here is that the first cohomology group of these sheaves contains a non-torsion element. This shows that the model is not D-affine for such differential operators.

Contents

1. Introduction

A fundamental result in the theory of classical complex D-modules is the D-affinity property of complex flag varieties, established by Beilinson-Bernstein [1]. It means that flag varieties behave for D-modules, as affine schemes do for quasi-coherent sheaves: objects are generated by global sections and the higher cohomology vanishes. The counterpart of complex D-modules in the setting of a p-adic field, is Berthelot's theory of arithmetic $\mathscr{D}$-modules [2, 3, 4]. In the pioneering papers [7, 10] Huyghe establishes the D-affinity

M. S. would like to acknowledge the support of the National Science Foundation (award DMS-1202303). T. S. would like to acknowledge support of the Heisenberg programme of Deutsche Forschungsgemeinschaft.

property for arithmetic $\mathscr{D}$-modules over the formal projective line and over general formal flag varieties, respectively.

On the other hand, we have started in [11, 12] to systematically investigate the relationship between equivariant arithmetic $\mathscr{D}$-modules on flag varieties and the theory of admissible representations of p-adic Lie groups [13, 14]. In this approach, a key point is to extend the $\mathscr{D}$-affinity property from the formal level to the level of rigid-analytic spaces. Since the latter are approximated by formal models through Raynaud's theory, it becomes a pressing question whether Huyghe's result extends from the smooth model to more general (semistable) models of the flag variety.

In this paper we approach this question in the simplest case of the group GL_2 over $\mathbb{Z}_p$. The corresponding flag variety is the projective line $\mathbb{X} = \mathbb{X}_0 = \mathbb{P}^1_{\mathbb{Z}_p}$. We study sheaves of logarithmic arithmetic differential operators on the simplest, so to speak, semistable (nonsmooth) model $\mathbb{X}_1$ of $\mathbb{P}^1_{\mathbb{Z}_p}$. This model is obtained by blowing up the reduced closed subscheme given by the set of $\mathbb{F}_p$-valued points of $\mathbb{X}$. We denote the corresponding formal schemes, the completions along the special fiber, by $\mathfrak{X}$ and $\mathfrak{X}_1$, respectively. The sheaf of logarithmic differential operators of level m, as defined in [12, sec. 5], will be denoted by $\mathcal{D}^{(m)}_{\mathfrak{X}_1}$, and its p-adic completion by $\mathscr{D}^{(m)}_{\mathfrak{X}_1}$. The formal scheme $\mathfrak{X}_1$ is the first member of a family of formal semistable models $\mathfrak{X}_n$ which we studied in [12]. In that paper, we obtained some results about the global sections of the sheaf of logarithmic arithmetic differential operators $\mathcal{D}^{(m)}_{\mathfrak{X}_n}$. One fundamental question that had not been treated there was the relation between $H^0(\mathfrak{X}_n, \mathcal{D}^{(m)}_{\mathfrak{X}_n})$ and $H^0(\mathfrak{X}_n, \mathscr{D}^{(m)}_{\mathfrak{X}_n})$. More precisely, one may ask if the natural inclusion

$$\widehat{H}^0(\mathfrak{X}_n, \mathcal{D}^{(m)}_{\mathfrak{X}_n}) \longrightarrow H^0(\mathfrak{X}_n, \mathscr{D}^{(m)}_{\mathfrak{X}_n})$$

is an isomorphism. On the left hand side $\widehat{H}^0(\mathfrak{X}_n, \mathcal{D}^{(m)}_{\mathfrak{X}_n})$ denotes the p-adic completion of $H^0(\mathfrak{X}_n, \mathcal{D}^{(m)}_{\mathfrak{X}_n})$. It is straightforward to see that there is a canonical exact sequence

$$0 \to \widehat{H}^0(\mathfrak{X}_n, \mathcal{D}^{(m)}_{\mathfrak{X}_n}) \to H^0(\mathfrak{X}_n, \mathscr{D}^{(m)}_{\mathfrak{X}_n}) \to T_p\left(H^1(\mathfrak{X}_n, \mathcal{D}^{(m)}_{\mathfrak{X}_n})\right) \to 0\ ,$$

where the group on the right is the p-adic Tate module

$$T_p\left(H^1(\mathfrak{X}_n, \mathcal{D}^{(m)}_{\mathfrak{X}_n})\right) = \varprojlim_k H^1(\mathfrak{X}_n, \mathcal{D}^{(m)}_{\mathfrak{X}_n})[p^k]\ ,$$

of $H^1(\mathfrak{X}_n, \mathcal{D}^{(m)}_{\mathfrak{X}_n})$. In this paper we only consider the case when $n = 1$, and the main results are summarized in the following theorem.

Theorem. *(i)* $T_p\Big(H^1(\mathfrak{X}_1, \mathcal{D}^{(m)}_{\mathfrak{X}_1})\Big) = 0$*, and the map*

$$\widehat{H}^0(\mathfrak{X}_1, \mathcal{D}^{(m)}_{\mathfrak{X}_1}) \longrightarrow H^0(\mathfrak{X}_1, \mathscr{D}^{(m)}_{\mathfrak{X}_1})$$

is therefore an isomorphism.

(ii) There is a canonical surjective homomorphism

$$H^1(\mathfrak{X}_1, \mathscr{D}^{(m)}_{\mathfrak{X}_1}) \longrightarrow \widehat{H}^1(\mathfrak{X}_1, \mathcal{D}^{(m)}_{\mathfrak{X}_1}) \ ,$$

and cohomology group on the right contains non-torsion elements. In particular, $H^1(\mathfrak{X}_1, \mathscr{D}^{(m)}_{\mathfrak{X}_1,\mathbb{Q}})$ *does not vanish.*

(iii) The cohomology group $H^1(\mathfrak{X}_1, \mathscr{D}^{\dagger}_{\mathfrak{X}_1,\mathbb{Q}})$ *does not vanish.*

The sheaf $\mathscr{D}^{\dagger}_{\mathfrak{X}_1,\mathbb{Q}}$ in (iii) is the inductive limit of the sheaves $\mathscr{D}^{(m)}_{\mathfrak{X}_1,\mathbb{Q}}$.

By considering a refinement of the order filtration on the sheaf $\mathcal{D}^{(m)}_{\mathfrak{X}_1}$ and computing the associated graded ring, one can show that $\widehat{H}^0(\mathfrak{X}_1, \mathcal{D}^{(m)}_{\mathfrak{X}_1})$ and $H^0(\mathfrak{X}_1, \mathscr{D}^{(m)}_{\mathfrak{X}_1})$ are noetherian rings. Details of this calculation will appear elsewhere.

As explained above, the investigations here and in [12] were motivated by the question if the formal models $\mathfrak{X}_n$ mentioned above are $\mathscr{D}^{\dagger}_{\mathfrak{X}_n,\mathbb{Q}}$-affine, and the non-vanishing of $H^1(\mathfrak{X}_1, \mathscr{D}^{\dagger}_{\mathfrak{X}_1,\mathbb{Q}})$ gives therefore a negative answer when $n = 1$. This has led us to consider in [11] a different family of sheaves $\widetilde{\mathscr{D}}^{(m)}_{n,k,\mathbb{Q}}$ of p-adically complete differential operators on $\mathfrak{X}_n$, and as it is shown there, $\mathfrak{X}_n$ turns out to be $\widetilde{\mathscr{D}}^{(m)}_{n,k,\mathbb{Q}}$-affine.

Although this paper is concerned with a very specific semistable model of the projective line $\mathbb{P}^1_{\mathbb{Z}_p}$, we believe that our methods and results generalize to more general semistable situations. To our knowledge, the attempt to calculate the cohomology of logarithmic arithmetic differential operators on semistable formal schemes has not yet been undertaken in the literature.

2. Global sections and cohomology of $\mathscr{D}^{(0)}$ on $\mathfrak{X}_1$

Let $\mathbb{X}_1$ be the blow-up of the projective line $\mathbb{X} = \mathbb{X}_0 = \mathbb{P}^1_{\mathbb{Z}_p}$ in the reduced closed subscheme given by the set of $\mathbb{F}_p$-valued points. For convenience of the reader, we recall some of the geometry of $\mathbb{X}_1$ from [12, sec. 4]. First of all, the irreducible components of the special fiber of $\mathbb{X}_1$ are projective lines over $\mathbb{F}_p$: besides the strict transform of $\mathbb{X}_{0,\mathbb{F}_p}$, there is for any $\mathbb{F}_p$-rational point a of $\mathbb{X}_0$ the corresponding component $E_a \simeq \mathbb{P}^1_{\mathbb{F}_p}$ of the exceptional divisor. Two different components of type E_a have empty intersection and any E_a intersects $\mathbb{X}_{0,\mathbb{F}_p}$ in the point corresponding to a.

We will work with the following open affine covering of $\mathbb{X}_1$ [12, 4.3.3]. Let $\mathcal{R} \subset \mathbb{Z}_p$ be any system of representatives for $\mathbb{Z}_p/p\mathbb{Z}_p$ and view $\mathcal{R}_\infty = \mathcal{R} \cup \{\infty\}$ as the set of $\mathbb{F}_p$-valued points on $\mathbb{X}_0$. Let $\mathbb{X}_0^\circ = \mathbb{X}_0 \backslash \mathcal{R}_\infty$ and view this as an open subscheme of $\mathbb{X}_1$. Let x_a be a local coordinate at $a \in \mathcal{R}_\infty$ and form

$$R_a^{(0)} = \mathbb{Z}_p[x_a]\left[\frac{1}{x_b} \;\middle|\; b \in \mathcal{R}, b \neq a\right].$$

Then $\mathbb{X}_1$ is obtained by blowing up the ideals $(p, x_a) \subset R_a^{(0)}$ for all $a \in \mathcal{R}_\infty$. We introduce new indeterminates z_a and $x_a^{(1)}$ with

$$x_a z_a = p \quad \text{and} \quad x_a^{(1)} z_a = 1\ .$$

Set also $x_{a,a_1}^{(1)} = x_a^{(1)} - a_1$ for $a_1 \in \mathcal{R}$. Then define

$$R_a^{(1)} = R_a^{(0)}[z_a]\left[\frac{1}{x_{a,a_1}^{(1)}} \;\middle|\; a_1 \in \mathcal{R}\right] \Big/ (x_a z_a - p)\ ,$$

and put $\mathbb{X}_a^{(1)} = \mathrm{Spec}(R_a^{(1)})$. This is an open affine neighbourhood in $\mathbb{X}_1$ of the singular point corresponding to a. Finally, set

$$R_{a,a_1}^{(1)} = R_a^{(0)}[x_{a,a_1}^{(1)}]\left[\frac{1}{x_{a,b}^{(1)}} \;\middle|\; b \in \mathcal{R}\backslash\{a_1\}\right]\ ,$$

and define

$$\mathbb{D}_{a,a_1}^{(1)} = \mathrm{Spec}\left(R_{a,a_1}^{(1)}\right)\ .$$

The special fiber of each $\mathbb{D}_{a,a_1}^{(1)}$ is isomorphic to an affine line over $\mathbb{F}_p$ all of whose $\mathbb{F}_p$-rational points have been removed, except the one given by $x_{a,a_1}^{(1)} = 0$. Let $\mathbb{X}_1^\circ$ be the union of the schemes $\mathbb{X}_a^{(1)}$, $a \in \mathcal{R}_\infty$, and $\mathbb{X}_0^\circ$. Then $\mathbb{X}_1$ is covered by $\mathbb{X}_1^\circ$ together with the 'residual disc schemes' $\mathbb{D}_{a,a_1}^{(1)}$ for $(a, a_1) \in \mathcal{R}_\infty \times \mathcal{R}$. Note that the singular points of $\mathbb{X}_1$ are contained in $\mathbb{X}_1^\circ$.

We denote the formal schemes corresponding to $\mathbb{X}$ and $\mathbb{X}_1$, i.e. their completions along the special fiber $\{p = 0\}$, by $\mathfrak{X}$ and $\mathfrak{X}_1$, respectively. The affine covering of $\mathbb{X}_1$ gives then rise, by completion, to an open affine covering of $\mathfrak{X}_1$. One has the simple descriptions [12, 4.4.1]

$$\widehat{\mathbb{X}}_a^{(1)} = \mathrm{Spf}\left(\mathbb{Z}_p\langle x_a, z_a\rangle\left[\frac{1}{(x_a)^{p-1}-1}, \frac{1}{(z_a)^{p-1}-1}\right] \Big/ (x_a z_a - p)\right)\ .$$

and

$$\widehat{\mathbb{D}}_{a,a_1}^{(1)} = \mathrm{Spf}\left(\mathbb{Z}_p\langle x_{a,a_1}^{(1)}\rangle\left[\frac{1}{(x_{a,a_1}^{(1)})^{p-1}-1}\right]\right)\ .$$

2.1. **Cohomology groups and their completions.** We view $\mathbb{X}_1$ as a semistable scheme with log structure defined by its normal crossings divisor $\{p = 0\}$. Generalities on logarithmic arithmetic differential operators can be found in [9]. However, here we will make everything explicit and work with an elementary description as in [12]. Denote by

$$pr : \mathbb{X}_1 \longrightarrow \mathbb{X}_0 = \mathbb{X}$$

the blow-up morphism. The logarithmic tangent sheaf $\mathcal{T}_{\mathbb{X}_1}$ coincides with the usual tangent sheaf on the smooth part of $\mathbb{X}_1$ and is locally on an open neighbourhood $\mathbb{X}_a^{(1)}$ of a singularity $a \in \mathcal{R}_\infty$ generated by $x_a\partial_{x_a}$. One has the relation $x_a\partial_{x_a} = -z_a\partial_{z_a}$. The sheaf $\mathcal{D}_{\mathbb{X}_1} = \mathscr{D}^{(0)}_{\mathfrak{X}_1}$ of logarithmic differential operators on $\mathbb{X}_1$ (of level zero) is then generated as a subalgebra of $pr^*(\mathcal{D}_{\mathbb{X}})$ by $\mathcal{T}_{\mathbb{X}_1}$ and the structure sheaf. On an open neighbourhood $\mathbb{X}_a^{(1)}$ it is therefore given as the module of all finite sums

$$D = \sum_{d \geqslant 0}^{<\infty} f_d (x_a\partial_{x_a})^d$$

with local sections f_d in $\mathcal{O}_{\mathbb{X}_a^{(1)}}$. We write $\mathcal{T}_{\mathfrak{X}_1}$ and $\mathcal{D}_{\mathfrak{X}_1}$ for the $\mathcal{O}_{\mathfrak{X}_1}$-modules generated by the restrictions of $\mathcal{T}_{\mathbb{X}_1}$ and $\mathcal{D}_{\mathbb{X}_1}$ to $\mathfrak{X}_1$ respectively. We finally let $\mathscr{D}_{\mathfrak{X}_1}$ be the p-adic completion of $\mathcal{D}_{\mathfrak{X}_1}$. On the formal completion $\widehat{\mathbb{X}}_a^{(1)}$ the sheaf $\mathscr{D}_{\mathfrak{X}_1}$ is given as the module of all p-adically convergent sums

$$D = \sum_{d \geqslant 0}^{\infty} f_d (x_a\partial_{x_a})^d, \;\; \mathcal{O}_{\widehat{\mathbb{X}}_a^{(1)}} \ni f_d \overset{p\text{-adically}}{\longrightarrow} 0 \text{ for } d \to \infty.$$

The submodule of all finite sums is equal to $\mathcal{D}_{\mathfrak{X}_1}|_{\widehat{\mathbb{X}}_a^{(1)}}$.

Lemma 2.1.1. *The canonical homomorphism*

$$H^i(\mathfrak{X}_1, \mathscr{D}_{\mathfrak{X}_1}) \longrightarrow \varprojlim_k H^i(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1}/p^k\mathcal{D}_{\mathfrak{X}_1})$$

is an isomorphism when $i = 0$ and surjective if $i = 1$. For $i > 1$ source and target of this map vanish.

Proof. For an inverse system of sheaves $(\mathcal{F}_k)_k$, the presheaf $U \mapsto \varprojlim_k \mathcal{F}_k(U)$ is actually a sheaf. This gives the statement for $i = 0$. For $i > 1$ the source and target of the map vanish because $\mathfrak{X}_1$ is a noetherian topological space of dimension one. In order to treat the case $i = 1$ we are going to use [5, ch. 0, Prop. 13.3.1]. The third condition of this proposition is fulfilled because the transition maps on the system of sheaves are obviously surjective. Let U be an affine open subset of $\mathfrak{X}_1$. Denote by $\mathfrak{X}_{1,k}$ the reduction of $\mathfrak{X}_1$ modulo p^k, and let $U_k = U \times_{\mathfrak{X}_1} \mathfrak{X}_{1,k}$ be the open affine subset of $\mathfrak{X}_{1,k}$. Then we have for all $i > 0$

$$H^i(U, \mathcal{D}_{\mathfrak{X}_1}/p^k\mathcal{D}_{\mathfrak{X}_1}) = H^i(U_k, \mathcal{D}_{\mathfrak{X}_1}/p^k\mathcal{D}_{\mathfrak{X}_1}) = 0 \;,$$

because $\mathcal{D}_{\mathfrak{X}_1}/p^k\mathcal{D}_{\mathfrak{X}_1}$ is a quasi-coherent sheaf on $\mathfrak{X}_{1,k}$. This shows that the second condition of loc.cit. is satisfied, and, for $i > 0$, also the first condition. Consider the exact sequence of quasi-coherent sheaves on $\mathfrak{X}_{1,k+1}$

$$0 \longrightarrow p^k\mathcal{D}_{\mathfrak{X}_1}/p^{k+1}\mathcal{D}_{\mathfrak{X}_1} \longrightarrow \mathcal{D}_{\mathfrak{X}_1}/p^{k+1}\mathcal{D}_{\mathfrak{X}_1} \longrightarrow \mathcal{D}_{\mathfrak{X}_1}/p^k\mathcal{D}_{\mathfrak{X}_1} \longrightarrow 0 \ .$$

Because $p^k\mathcal{D}_{\mathfrak{X}_1}/p^{k+1}\mathcal{D}_{\mathfrak{X}_1}$ has vanishing first cohomology on U_k, this sequence stays exact after applying $H^0(U_k, -)$, and this shows that the first condition of loc.cit. is fulfilled in the case $i = 0$. Hence we can conclude that the map in question is surjective for $i = 1$. $\square$

Next we consider the tautological exact sequence of sheaves on $\mathfrak{X}_1$

$$0 \to \mathcal{D}_{\mathfrak{X}_1} \xrightarrow{p^k} \mathcal{D}_{\mathfrak{X}_1} \to \mathcal{D}_{\mathfrak{X}_1}/p^k\mathcal{D}_{\mathfrak{X}_1} \to 0 \ .$$

The associated long exact cohomology sequence gives the exact sequence

$$H^i(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1}) \xrightarrow{p^k} H^i(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1}) \to H^i(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1}/p^k\mathcal{D}_{\mathfrak{X}_1}) \to H^{i+1}(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1}) \xrightarrow{p^k} H^{i+1}(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1}) \ .$$

We thus get an exact sequence

(2.1.2)
$$0 \to H^i(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1})\Big/p^k H^i(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1}) \to H^i(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1}/p^k\mathcal{D}_{\mathfrak{X}_1}) \to H^{i+1}(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1})\left[p^k\right] \to 0 \ ,$$

where $H^{i+1}(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1})\left[p^k\right]$ denotes the subgroup of elements annihilated by multiplication by p^k. Put

$$\widehat{H}^i(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1}) = \varprojlim_k \left(H^i(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1})\Big/p^k H^i(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1}) \right) \ ,$$

and

$$T_p\left(H^i(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1})\right) = \varprojlim_k H^i(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1})[p^k] \ ,$$

where the transition map $H^i(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1})[p^k] \to H^i(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1})[p^{k-1}]$ is the multiplication by p. We then have the

Proposition 2.1.3. *(a) For all $i \geqslant 0$ there is a natural exact sequence*

(2.1.4) $$0 \to \widehat{H}^i(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1}) \to \varprojlim_k H^i(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1}/p^k\mathcal{D}_{\mathfrak{X}_1}) \to T_p\left(H^{i+1}(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1})\right) \to 0 \ .$$

(b) For $i = 0$ the exact sequence in (a) becomes

$$0 \to \widehat{H}^0(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1}) \to H^0(\mathfrak{X}_1, \mathscr{D}_{\mathfrak{X}_1}) \to T_p\left(H^1(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1})\right) \to 0 \,. \tag{2.1.5}$$

(c) The cohomology group $H^2(\mathfrak{X}_1, \mathcal{D}_{\mathbb{X}_1})$ vanishes and the exact sequence in (a) gives therefore a canonical isomorphism

$$\widehat{H}^1(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1}) \simeq \varprojlim_k H^1(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1}/p^k\mathcal{D}_{\mathfrak{X}_1}) \,. \tag{2.1.6}$$

Proof. (a) For varying k the projective system

$$H^i(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1})\Big/ p^k H^i(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1})$$

has obviously surjective transition maps (hence satisfies the Mittag-Leffler condition). We can thus pass to the limit over k and using 2.1.1 we obtain the exact sequence 2.1.4.

(b) We use (a) in the case $i = 0$ and 2.1.1.

(c) $H^2(\mathfrak{X}_1, \mathcal{D}_{\mathbb{X}_1})$ vanishes because $\mathfrak{X}_1$ is a noetherian space of dimension one. The stated isomorphism follows then directly from (a). $\square$

2.2. **Vanishing of** $\mathrm{R}^1 pr_*(\mathcal{D}_{\mathfrak{X}_1})$**.** We use the Leray spectral sequence for the blow-up morphism

$$pr : \mathfrak{X}_1 \longrightarrow \mathfrak{X} = \mathfrak{X}_0 \,.$$

Applied to the sheaf $\mathcal{D}_{\mathfrak{X}_1}$ we get an exact sequence

$$0 \to H^1(\mathfrak{X}, pr_*(\mathcal{D}_{\mathfrak{X}_1})) \to H^1(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1}) \to H^0(\mathfrak{X}, \mathrm{R}^1 pr_*(\mathcal{D}_{\mathfrak{X}_1})) \to 0 \,. \tag{2.2.1}$$

Denote by $\mathcal{D}_{\mathfrak{X},d}$ and $\mathcal{D}_{\mathfrak{X}_1,d}$ the sheaves of differential operators of degree less or equal to d.

Lemma 2.2.2. *(a) For all $d \geqslant 0$ one has $\mathrm{R}^1 pr_*(\mathcal{D}_{\mathfrak{X}_1,d}) = 0$.*

(b) $\mathrm{R}^1 pr_*(\mathcal{D}_{\mathfrak{X}_1}) = 0$.

(c) $H^1(\mathfrak{X}, pr_*(\mathcal{D}_{\mathfrak{X}_1})) = H^1(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1})$.

Proof. (a) *Reduction: passage to the graded sheaves.* We have

$$\mathcal{T}_{\mathfrak{X}_1}^{\otimes d} = \mathcal{D}_{\mathfrak{X}_1,d}/\mathcal{D}_{\mathfrak{X}_1,d-1} \,,$$

and we consider the tautological exact sequence

$$0 \longrightarrow \mathcal{D}_{\mathfrak{X}_1,d-1} \longrightarrow \mathcal{D}_{\mathfrak{X}_1,d} \longrightarrow \mathcal{T}_{\mathfrak{X}_1}^{\otimes d} \longrightarrow 0 \; . \tag{2.2.3}$$

For $d = 0$ we have $\mathcal{D}_{\mathfrak{X}_1,0} = \mathcal{T}_{\mathfrak{X}_1}^{\otimes 0} = \mathcal{O}_{\mathfrak{X}_1}$. Therefore, if we show

$$\mathrm{R}^1 pr_*(\mathcal{T}_{\mathfrak{X}_1}^{\otimes d}) = 0$$

for all $d \geqslant 0$, then we can argue by induction and get $\mathrm{R}^1 pr_*(\mathcal{D}_{\mathfrak{X}_1,d}) = 0$ for all d. Using that taking higher direct images commutes with inductive limits we get

$$\mathrm{R}^1 pr_*(\mathcal{D}_{\mathfrak{X}_1}) = 0 \; .$$

Working with local coordinates. Over the complement of $pr^{-1}(\mathbb{X}(\mathbb{F}_p))$ the blow-up morphism is an isomorphism, and the stalk of the sheaf $\mathrm{R}^1 pr_*(\mathcal{D}_{\mathfrak{X}_1})$ vanishes thus outside $\mathfrak{X}(\mathbb{F}_p)$. Consider a point $P \in \mathfrak{X}(\mathbb{F}_p)$. Choosing a local coordinate at P we may assume that P corresponds to the point given by the ideal (x, p) of the ring

$$R = \mathbb{Z}_p\langle x\rangle\left[\frac{1}{x^{p-1}-1}\right] \; .$$

Then $\mathrm{Spf}(R)$ is an open neighborhood of P in $\mathfrak{X}$. Put

$$R' = \mathbb{Z}_p\langle x, z\rangle\left[\frac{1}{x^{p-1}-1}, \frac{1}{z^{p-1}-1}\right] \Big/ (xz - p) \; ,$$

and $R'' = \mathbb{Z}_p\langle t\rangle$, and identify the open subsets $\mathrm{Spf}(R')\left[\frac{1}{z}\right] \subset \mathrm{Spf}(R')$ and $\mathrm{Spf}(R'')\left[\frac{1}{t}\right] \subset \mathrm{Spf}(R'')$ via the relation $zt = 1$. Then

$$pr^{-1}(\mathrm{Spf}(R)) = \mathrm{Spf}(R') \cup \mathrm{Spf}(R'')$$

is an open neighborhood of the fiber $pr^{-1}(P)$. To show that the stalk of $\mathrm{R}^1 pr_*(\mathcal{T}_{\mathfrak{X}_1}^{\otimes d})$ at P vanishes it suffices to show that

$$H^1(pr^{-1}(U), \mathcal{T}_{\mathfrak{X}_1}^{\otimes d}) = 0$$

for all affine open subsets $U \subset \mathrm{Spf}(R) \subset \mathfrak{X}$ containing P. Identify $\mathrm{Spf}(R)$ with a closed subset of $\mathrm{Spf}(R')$. Then we have $pr^{-1}(U) = U \cup \mathrm{Spf}(R'')$. Hence it suffices to show that

$$H^1(V \cup \mathrm{Spf}(R''), \mathcal{T}_{\mathfrak{X}_1}^{\otimes d}) = 0$$

for all affine open subsets $V \subset \mathrm{Spf}(R') \subset \mathfrak{X}_1$ containing P (which we also consider as a point of $\mathfrak{X}_1$).

Using Čech cohomology. For such a V the open subset $V \cup \mathrm{Spf}(R'')$ always contains $\mathrm{Spf}(R')\left[\frac{1}{z}\right] = \mathrm{Spf}(R'')\left[\frac{1}{t}\right]$ and we may thus assume $\mathrm{Spf}(R')\left[\frac{1}{z}\right] \subset V$. Then we have

$$V \cap \mathrm{Spf}(R'') = \mathrm{Spf}(R')\left[\frac{1}{z}\right] = \mathrm{Spf}(R'')\left[\frac{1}{t}\right] .$$

Then $H^1(V \cup \mathrm{Spf}(R''), \mathcal{T}_{\mathfrak{X}_1}^{\otimes d})$ is equal to the cokernel of the map

$$H^0\left(V, \mathcal{T}_{\mathfrak{X}_1}^{\otimes d}\right) \oplus H^0\left(\mathrm{Spf}(R''), \mathcal{T}_{\mathfrak{X}_1}^{\otimes d}\right) \longrightarrow H^0\left(\mathrm{Spf}(R'')\left[\frac{1}{t}\right], \mathcal{T}_{\mathfrak{X}_1}^{\otimes d}\right) .$$

which sends (s_1, s_2) to the difference of these sections when restricted to $\mathrm{Spf}(R'')\left[\frac{1}{t}\right]$. Any element in

$$H^0\left(\mathrm{Spf}(R'')\left[\frac{1}{t}\right], \mathcal{T}_{\mathfrak{X}_1}^{\otimes d}\right)$$

has the form $\left(\sum_{i\in\mathbb{Z}} a_i t^i\right)\partial_t^{\otimes d}$. The sum $\left(\sum_{i\geqslant 0} a_i t^i\right)\partial_t^{\otimes d}$ clearly extends to a section over $\mathrm{Spf}(R'')$. Note that we have in $\mathcal{T}_{\mathfrak{X}_1}^{\otimes d}$

$$\partial_t^{\otimes d} = (-z^2\partial_z)^{\otimes d} = (-1)^d z^{2d}\partial_z^{\otimes d}$$

and therefore

$$\left(\sum_{i<0} a_i t^i\right)\partial_t^{\otimes d} = (-1)^d\left(\sum_{i<0} a_i z^{-i+d}\right) z^d \partial_z^{\otimes d} ,$$

and this extends to a section over V.

(b) This follows from (a) and the fact that the higher direct image functor commutes with inductive limits.

(c) This is an immediate consequence of (b) and 2.2.1. □

2.3. **The cohomology group $H^1(\mathfrak{X}, pr_*(\mathcal{D}_{\mathfrak{X}_1}))$.** Consider the exact sequence 2.2.3 and the corresponding sequence of direct images on $\mathfrak{X}$

$$0 \longrightarrow pr_*\left(\mathcal{D}_{\mathfrak{X}_1,d-1}\right) \longrightarrow pr_*\left(\mathcal{D}_{\mathfrak{X}_1,d}\right) \longrightarrow pr_*\left(\mathcal{T}_{\mathfrak{X}_1}^{\otimes d}\right) \longrightarrow \mathrm{R}^1 pr_*\left(\mathcal{D}_{\mathfrak{X}_1,d-1}\right) = 0 , \tag{2.3.1}$$

where we have used 2.2.2 (a). We have

$$H^1(\mathfrak{X}, pr_*(\mathcal{D}_{\mathfrak{X}_1})) = \varinjlim_d H^1(\mathfrak{X}, pr_*(\mathcal{D}_{\mathfrak{X}_1,d})) .$$

Because $\mathcal{D}_{\mathfrak{X}_1,d}$ is coherent and pr is projective, the sheaf $pr_*(\mathcal{D}_{\mathfrak{X}_1,d})$ is coherent and $H^1(\mathfrak{X}, pr_*(\mathcal{D}_{\mathfrak{X}_1,d}))$ is thus a finitely generated $\mathbb{Z}_p$-module. Since the corresponding cohomology group on the generic fiber (in the sense of rigid geometry) vanishes (by GAGA and [1]), we see that $H^1(\mathfrak{X}, pr_*(\mathcal{D}_{\mathfrak{X}_1,d}))$ is annihilated by a finite power of p. (We will give below a more precise description of $H^1(\mathfrak{X}, pr_*(\mathcal{D}_{\mathfrak{X}_1,d}))$ which shows directly that it is annihilated by a finite power of p.) In the proof of theorem 2.3.4 we will need the following elementary

Lemma 2.3.2. *Let x, y be the standard coordinates on $\mathbb{P}^1$ satisfying $xy = 1$. Then we have $\partial_y = -x^2\partial_x$ and, more generally, for any $s \in \mathbb{Z}_{\geqslant 1}$*

$$\partial_y^s = (-1)^s \sum_{t=1}^{s} a_{s,t} x^{s+t} \partial_x^t \,,$$

where for all $s \geqslant 1$ and $1 \leqslant t \leqslant s$

$$a_{s,t} = \binom{s}{t} \frac{(s-1)!}{(t-1)!} \,, \quad \textit{in particular}\,, \;\; a_{s,1} = s! \quad \textit{and} \quad a_{s,s} = 1 \,. \tag{2.3.3}$$

Proof. We prove this by induction on s. The formula holds obviously in the case $s = 1$. Assuming the formula to be correct for a given s, we have

$$\begin{aligned}
\partial_y^{s+1} &= (-x^2\partial_x)(-1)^s \left(\textstyle\sum_{t=1}^s a_{s,t} x^{s+t} \partial_x^t\right) \\
&= (-1)^{s+1} \textstyle\sum_{t=1}^s \left(a_{s,t} x^2 (x^{s+t}\partial_x + (s+t)x^{s+t-1})\partial_x^t\right) \\
&= (-1)^{s+1} \textstyle\sum_{t=1}^s \left(a_{s,t} x^{s+t+2}\partial_x^{t+1} + a_{s,t}(s+t)x^{s+t+1}\partial_x^t\right) \\
&= (-1)^{s+1} \left(a_{s,1}(s+1)x^{s+2}\partial_x\right. \\
&\quad + \left[\textstyle\sum_{t=2}^s \left(a_{s,t-1} + a_{s,t}(s+t)\right) x^{s+1+t}\partial_x^{t+1}\right] + \left. a_{s,s} x^{2s+2} \partial_x^{s+1}\right)
\end{aligned}$$

Using 2.3.3 we then get for $2 \leqslant t \leqslant s$ that

$$\begin{aligned}
a_{s,t-1}+a_{s,t}(s+t) &= \tbinom{s}{t-1}\tfrac{(s-1)!}{(t-2)!}+\tbinom{s}{t}\tfrac{(s-1)!}{(t-1)!}(s+t)\\
&= \tfrac{s!}{(s-t+1)!(t-1)!}\tfrac{(s-1)!}{(t-2)!}+\tfrac{s!}{(s-t)!t!}\tfrac{(s-1)!(s+t)}{(t-1)!}\\
&= \tfrac{s!(s-1)!}{(t-1)!}\left[\tfrac{1}{(t-2)!(s-t+1)}+\tfrac{s+t}{(s-t)!t!}\right]\\
&= \tfrac{s!(s-1)!}{(t-1)!}\left[\tfrac{(t-1)t+(s-t+1)(s+t)}{t!(s-t+1)!}\right]\\
&= \tfrac{s!(s-1)!}{(t-1)!}\tfrac{s(s+1)}{t!(s+1-t)!}=\tbinom{s+1}{t}\tfrac{s!}{(t-1)!}=a_{s+1,t}.
\end{aligned}$$

And finally one has $a_{s,1}(s+1)=s!(s+1)=(s+1)!=a_{s+1,1}$. $\square$

Theorem 2.3.4. *For all $d\geqslant 1$ the canonical map*

$$H^1(\mathfrak{X},pr_*(\mathcal{D}_{\mathfrak{X}_1,d-1}))\to H^1(\mathfrak{X},pr_*(\mathcal{D}_{\mathfrak{X}_1,d})) \tag{2.3.5}$$

coming from the long exact cohomology sequence associated to 2.3.1 is injective and embeds $H^1(\mathfrak{X},pr_*(\mathcal{D}_{\mathfrak{X}_1,d-1}))$ as a direct summand of $H^1(\mathfrak{X},pr_*(\mathcal{D}_{\mathfrak{X}_1,d}))$. Therefore, there is a splitting:

$$H^1(\mathfrak{X},pr_*(\mathcal{D}_{\mathfrak{X}_1,d}))=H^1(\mathfrak{X},pr_*(\mathcal{D}_{\mathfrak{X}_1,d-1}))\oplus H^1(\mathfrak{X},pr_*(\mathcal{T}_{\mathfrak{X}_1}^{\otimes d}))\ . \tag{2.3.6}$$

Proof. (i) We start with some preliminary considerations. The sheaf $pr_*(\mathcal{D}_{\mathfrak{X}_1,d})$ (resp. $pr_*(\mathcal{T}_{\mathfrak{X}_1}^{\otimes d})$) is naturally a subsheaf of $\mathcal{D}_{\mathfrak{X},d}$ (resp. $\mathcal{T}_{\mathfrak{X}}^{\otimes d}$), cf. [12, 5.2], and we denote by $Q_{\leqslant d}$ (resp. Q_d) the quotient sheaf. Consider the commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & pr_*(\mathcal{D}_{\mathfrak{X}_1,d-1}) & \to & \mathcal{D}_{\mathfrak{X},d-1} & \to & Q_{\leqslant d-1} & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & pr_*(\mathcal{D}_{\mathfrak{X}_1,d}) & \to & \mathcal{D}_{\mathfrak{X},d} & \to & Q_{\leqslant d} & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & pr_*(\mathcal{T}_{\mathfrak{X}_1}^{\otimes d}) & \to & \mathcal{T}_{\mathfrak{X}}^{\otimes d} & \to & Q_d & \to & 0
\end{array}$$

where the horizontal sequences are the tautological exact sequences. The corresponding long exact sequences give rise to the commutative diagram

(2.3.7)

$$\begin{array}{ccccccc}
H^0(\mathfrak{X}, pr_*(\mathcal{D}_{\mathfrak{X}_1,d-1})) & \hookrightarrow & H^0(\mathfrak{X}, \mathcal{D}_{\mathfrak{X},d-1}) & \to & H^0(\mathfrak{X}, Q_{\leqslant d-1}) & \twoheadrightarrow & H^1(\mathfrak{X}, pr_*(\mathcal{D}_{\mathfrak{X}_1,d-1})) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^0(\mathfrak{X}, pr_*(\mathcal{D}_{\mathfrak{X}_1,d})) & \hookrightarrow & H^0(\mathfrak{X}, \mathcal{D}_{\mathfrak{X},d}) & \to & H^0(\mathfrak{X}, Q_{\leqslant d}) & \twoheadrightarrow & H^1(\mathfrak{X}, pr_*(\mathcal{D}_{\mathfrak{X}_1,d})) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^0(\mathfrak{X}, pr_*(\mathcal{T}_{\mathfrak{X}_1}^{\otimes d})) & \hookrightarrow & H^0(\mathfrak{X}, \mathcal{T}_{\mathfrak{X}}^{\otimes d}) & \to & H^0(\mathfrak{X}, Q_d) & \twoheadrightarrow & H^1(\mathfrak{X}, pr_*(\mathcal{T}_{\mathfrak{X}_1}^{\otimes d})).
\end{array}$$

Note that the horizontal arrows on the right are surjections, since $\mathcal{T}_{\mathbb{X}} = \mathcal{O}(2)$ and hence $H^1(\mathfrak{X}, \mathcal{T}_{\mathfrak{X}}^{\otimes d}) = 0$ and then $H^1(\mathfrak{X}, \mathcal{D}_{\mathfrak{X},d}) = 0$ by induction on d. The sheaves $Q_{\leqslant d-1}$, $Q_{\leqslant d}$ and Q_d are skyscraper sheaves with support in $\mathfrak{X}(\mathbb{F}_p)$. Let x_a be a local coordinate at $a \in \mathfrak{X}(\mathbb{F}_p)$. Then, cf. [12, 5.2 (c)],

$$Q_{\leqslant d} = \bigoplus_{a \in \mathfrak{X}(\mathbb{F}_p)} \bigoplus_{k=1}^{d} \bigoplus_{i=0}^{k-1} \left(\mathbb{Z}/p^{k-i}\right) \cdot x_a^i \partial_{x_a}^k , \tag{2.3.8}$$

$$Q_{\leqslant d-1} = \bigoplus_{a \in \mathfrak{X}(\mathbb{F}_p)} \bigoplus_{k=1}^{d-1} \bigoplus_{i=0}^{k-1} \left(\mathbb{Z}/p^{k-i}\right) \cdot x_a^i \partial_{x_a}^k , \tag{2.3.9}$$

and

$$Q_d = \bigoplus_{a \in \mathfrak{X}(\mathbb{F}_p)} \bigoplus_{i=0}^{d-1} \left(\mathbb{Z}/p^{d-i}\right) \cdot x_a^i \partial_{x_a}^d . \tag{2.3.10}$$

Hence there is a splitting

$$Q_{\leqslant d} = Q_{\leqslant d-1} \oplus Q_d . \tag{2.3.11}$$

We introduce the following notation and terminology. For a global section $\delta \in H^0(\mathfrak{X}, \mathcal{D}_{\mathfrak{X},d})$ we denote its image in $Q_{\leqslant d}$ by $Q_{\leqslant d}(\delta)$. The component of this element in Q_d, according to the splitting 2.3.11, will be denoted by $Q_d(\delta)$, and we denote the components in $\bigoplus_{i=0}^{d-1} \left(\mathbb{Z}/p^{d-i}\right) \cdot x_a^i \partial_{x_a}^d$ corresponding to $a \in \mathfrak{X}(\mathbb{F}_p)$ by $Q_{d,a}(\delta)$. We call $Q_d(\delta)$ (resp. $Q_{\leqslant d}(\delta)$) the *local data in degree* d (resp. *in degree less or equal to* d) of δ. Similarly we call $Q_{d,a}(\delta)$ the *local data in degree* d *at* a of δ.

(ii) Now we prove the injectivity of the map 2.3.5. The injectivity of this map is equivalent, by the long exact cohomology sequence attached to 2.3.1, to the surjectivity of the map

$$H^0(\mathfrak{X}, pr_*(\mathcal{D}_{\mathfrak{X}_1,d})) \to H^0(\mathfrak{X}, pr_*(\mathcal{T}_{\mathfrak{X}_1}^{\otimes d})) \tag{2.3.12}$$

which appears on the right hand side of 2.3.7. We are going to prove that 2.3.12 is surjective as follows: consider $\delta_1 \in H^0(\mathfrak{X}, pr_*(\mathcal{T}_{\mathfrak{X}_1}^{\otimes d}))$ and let $\delta \in H^0(\mathfrak{X}, \mathcal{T}_{\mathfrak{X}}^{\otimes d})$ be its image. Then $Q_d(\delta) = 0$. The crucial step is to lift δ to an element $\widetilde{\delta} \in H^0(\mathfrak{X}, \mathcal{D}_{\mathfrak{X},d})$ in such a way that $Q_{\leqslant d}(\widetilde{\delta}) = 0$. This implies that $\widetilde{\delta}$ does in fact come from an element (necessarily unique) $\widetilde{\delta}_1 \in H^0(\mathfrak{X}, pr_*(\mathcal{D}_{\mathfrak{X}_1,d}))$ which is a preimage of δ_1 under the map 2.3.12.

We let $x = x_0$ and $y = x_\infty$. Then $\delta \in H^0(\mathfrak{X}, \mathcal{T}_{\mathfrak{X}}^{\otimes d})$ can be written as

$$\delta = \sum_{s=0}^{d-1} A_s y^s \partial_y^{\otimes d} + \sum_{s'=0}^{d} B_{s'} x^{s'} \partial_x^{\otimes d} \in H^0(\mathfrak{X}, \mathcal{T}_{\mathfrak{X}}^{\otimes d})$$

with coefficients $A_s, B_{s'} \in \mathbb{Z}_p$. Let us consider in detail what it means that $Q_d(\delta) = 0$. For instance, if we write δ in terms of ∂_y, we have to use the transformation formula (in $H^0(\mathfrak{X}, \mathcal{T}_{\mathfrak{X}}^{\otimes d})$): $x^{s'} \partial_x^{\otimes d} = \pm y^{2d-s'} \partial_y^{\otimes d}$, and

$$\delta = \sum_{s=0}^{d-1} A_s y^s \partial_y^{\otimes d} + \sum_{s'=0}^{d} (\pm B_{s'}) y^{2d-s'} \partial_y^{\otimes d} \ .$$

Since $s' \leqslant d$ we have $2d - s' \geqslant d$, we see that the vanishing of the local data of δ in degree d at ∞ imposes the condition that $p^{d-s} | A_s$ for $0 \leqslant s \leqslant d-1$. Similarly we find $p^{d-s'} | B_{s'}$ for $0 \leqslant s' \leqslant d$.

We are looking for a preimage $\widetilde{\delta} \in H^0(\mathfrak{X}, \mathcal{D}_{\mathfrak{X},d})$ of δ whose image in $H^0(\mathfrak{X}, Q_{\leqslant d})$ vanishes. We start by taking as a candidate the element $\widetilde{\delta}_d$ which is given by the same formula as δ, but now the summands are considered to be global sections of $\mathcal{D}_{\mathfrak{X},d}$, i.e.,

$$\widetilde{\delta}_d = \sum_{s=0}^{d-1} A_s y^s \partial_y^{d} + \sum_{s'=0}^{d} B_{s'} x^{s'} \partial_x^{d} \in H^0(\mathfrak{X}, \mathcal{D}_{\mathfrak{X},d}) \ .$$

(We write $\partial_x^{\otimes d}$ when we consider it as a section of $\mathcal{T}_{\mathfrak{X}}^{\otimes d}$, and we write ∂_x^d when we consider it as a section of $\mathcal{D}_{\mathfrak{X},d}$.) By 2.3.2 this is indeed a global section of $\mathcal{D}_{\mathfrak{X},d}$.

The problem that we are facing now is this: while the local data of $\widetilde{\delta}_d$ in degree d vanish (by assumption), it will in general not be the case that the local data of $\widetilde{\delta}_d$ in degree $< d$ vanish as well. Our aim is to modify $\widetilde{\delta}_d$ by adding a global section of $\mathcal{D}_{\mathfrak{X},d-1}$ to it, such that the difference has vanishing local data in all degrees, hence comes from an element in $pr_*\mathcal{D}_{\mathfrak{X}_1,d}$.

In order to do so, we determine the local data of $\widetilde{\delta}_d$ at infinity in all degrees. Using 2.3.2 we write

$$\begin{aligned}\widetilde{\delta}_d = & \textstyle\sum_{s=0}^{d-1} A_s y^s \partial_y^d + \sum_{s'=0}^{d} B_{s'} x^{s'} \partial_x^d \\ = & \textstyle\sum_{s=0}^{d-1} A_s y^s \partial_y^d + \sum_{s'=0}^{d} B_{s'} (-1)^d \left(\sum_{e=1}^{d} a_{d,e} y^{d+e-s'} \partial_y^e \right) \\ = & \textstyle\sum_{s=0}^{d-1} A_s y^s \partial_y^d + (-1)^d \sum_{e=1}^{d} a_{d,e} \left(\sum_{s'=0}^{d} B_{s'} y^{d+e-s'} \right) \partial_y^e \,.\end{aligned}$$

Because $d + e - s' \geqslant e$ the term $y^{d+e-s'} \partial_y^e$ does not contribute to local data at infinity. So, in fact, $\widetilde{\delta}_d$ has vanishing local data at infinity in all degrees less or equal to d.

Now we analyze the local data at points $a \in \mathfrak{X}(\mathbb{F}_p) \backslash \{\infty\} = \mathbb{F}_p$. Let $\xi_a \in \mathbb{Z}_p$ be a lift of a. We use 2.3.2 again and write

$$\begin{aligned}\widetilde{\delta}_d &= \sum_{s=0}^{d-1} A_s y^s \partial_y^d + \sum_{s'=0}^{d} B_{s'} x^{s'} \partial_x^d \\ &= \sum_{s'=0}^{d} B_{s'} x^{s'} \partial_x^d + \sum_{s=0}^{d-1} A_s (-1)^d \left(\sum_{e=1}^{d} a_{d,e} x^{d+e-s} \partial_x^e \right) \\ &= \sum_{s'=0}^{d} B_{s'} x^{s'} \partial_x^d + (-1)^d \sum_{e=1}^{d} a_{d,e} \left(\sum_{s=0}^{d-1} A_s x^{d+e-s} \right) \partial_x^e \\ &\overset{(*)}{=} \left(\sum_{s'=0}^{d} B_{s'} x^{s'} + (-1)^d \sum_{s=0}^{d-1} A_s x^{2d-s} \right) \partial_x^d + (-1)^d \sum_{e=1}^{d-1} a_{d,e} \left(\sum_{s=0}^{d-1} A_s x^{d+e-s} \right) \partial_x^e \\ &= \left(\sum_{s'=0}^{d} B_{s'} x^{s'} + (-1)^d \sum_{s=0}^{d-1} A_s x^{2d-s} \right) \partial_x^d \\ &\quad + (-1)^d \sum_{e=1}^{d-1} a_{d,e} \left(\sum_{s=0}^{d-1} A_s (x_a + \xi_a)^{d+e-s} \right) \partial_{x_a}^e\end{aligned}$$

$$= \left(\sum_{s'=0}^{d} B_{s'} x^{s'} + (-1)^d \sum_{s=0}^{d-1} A_s x^{2d-s} \right) \partial_x^d$$

$$+ (-1)^d \sum_{e=1}^{d-1} a_{d,e} \left(\sum_{s=0}^{d-1} \sum_{k=0}^{d+e-s} \binom{d+e-s}{k} \xi_a^k A_s x_a^{d+e-s-k} \partial_{x_a}^e \right) .$$

The term $\binom{d+e-s}{k}\xi_a^k A_s x_a^{d+e-s-k}\partial_{x_a}^e$ gives a non-zero contribution to the local data at a in degree e only if $d+e-s-k < e$, i.e., $d < s+k$, and in this case the contribution is modulo $p^{e-(d+e-s-k)} = p^{s+k-d}$. Since $s+k-d \leqslant s+(d+e-s)-d = e$ and because $p^{d-s}|A_s$ we find that the contribution of $\binom{d+e-s}{k}\xi_a^k A_s x_a^{d+e-s-k}\partial_{x_a}^e$ vanishes if $d-s \geqslant e$. So we only need to pay attention to those terms for which $d-s < e$ or, equivalently, $d-e < s$.

Noting that $\sum_{1<s<d} A_s y^{s-1} \partial_y^{d-1}$ is a global section of $\mathcal{D}_{\mathfrak{X},d-1}$ we now consider

$$\widetilde{\delta}_{d,d-1} \stackrel{\text{def}}{=} \widetilde{\delta}_d + a_{d,d-1} \left(\sum_{1<s<d} A_s y^{s-1} \right) \partial_y^{d-1} .$$

Because $d-1-(s-1) = d-s$ and because $p^{d-s}|A_s$ this differential operator has vanishing local data at infinity in degree $d-1$ (and in degree d). We write $\widetilde{\delta}_{d,d-1}$ in terms of powers of ∂_x (using equation (*) above for $\widetilde{\delta}_d$) and find:

$$\widetilde{\delta}_{d,d-1} = \left(\sum_{s'=0}^{d} B_{s'} x^{s'} + (-1)^d \sum_{s=0}^{d-1} A_s x^{2d-s} \right) \partial_x^d + (-1)^d \sum_{e=1}^{d-1} a_{d,e} \left(\sum_{s=0}^{d-e} A_s x^{d+e-s} \right) \partial_x^e$$

$$+ (-1)^d \sum_{e=1}^{d-1} a_{d,e} \left(\sum_{d-e<s<d} A_s x^{d+e-s} \right) \partial_x^e$$

$$+ (-1)^{d-1} a_{d,d-1} \sum_{1<s<d} A_s \left(\sum_{e=1}^{d-1} a_{d-1,e} x^{d-1+e-(s-1)} \partial_x^e \right) .$$

As mentioned above, the terms in $\sum_{e=1}^{d-1} a_{d,e} \left(\sum_{s=0}^{d-e} A_s x^{d+e-s} \right) \partial_x^e$ do not contribute to the local data in degrees less than d. We continue our calculation and find:

$$\begin{aligned}
\widetilde{\delta}_{d,d-1} &= \left(\sum_{s'=0}^{d} B_{s'} x^{s'} + (-1)^d \sum_{s=0}^{d-1} A_s x^{2d-s}\right) \partial_x^d + (-1)^d \sum_{e=1}^{d-1} a_{d,e} \left(\sum_{s=0}^{d-e} A_s x^{d+e-s}\right) \partial_x^e \\
&\quad + (-1)^d \sum_{e=1}^{d-2} a_{d,e} \left(\sum_{d-e<s<d} A_s x^{d+e-s}\right) \partial_x^e + (-1)^d a_{d,d-1} \left(\sum_{1<s<d} A_s x^{2d-1-s}\right) \partial_x^{d-1} \\
&\quad + (-1)^{d-1} a_{d,d-1} \sum_{1<s<d} A_s \left(\sum_{e=1}^{d-1} a_{d-1,e} x^{d+e-s} \partial_x^e\right) \\
&= \left(\sum_{s'=0}^{d} B_{s'} x^{s'} + (-1)^d \sum_{s=0}^{d-1} A_s x^{2d-s}\right) \partial_x^d + (-1)^d \sum_{e=1}^{d-1} a_{d,e} \left(\sum_{s=0}^{d-e} A_s x^{d+e-s}\right) \partial_x^e \\
&\quad + (-1)^d \sum_{e=1}^{d-2} a_{d,e} \left(\sum_{d-e<s<d} A_s x^{d+e-s}\right) \partial_x^e + (-1)^d a_{d,d-1} \left(\sum_{1<s<d} A_s x^{2d-1-s}\right) \partial_x^{d-1} \\
&\quad + (-1)^{d-1} a_{d,d-1} \sum_{1<s<d} A_s \left(\sum_{e=1}^{d-2} a_{d-1,e} x^{d+e-s} \partial_x^e\right) \\
&\quad + (-1)^{d-1} a_{d,d-1} \sum_{1<s<d} A_s a_{d-1,d-1} x^{2d-1-s} \partial_x^{d-1} \\
&= \left(\sum_{s'=0}^{d} B_{s'} x^{s'} + (-1)^d \sum_{s=0}^{d-1} A_s x^{2d-s}\right) \partial_x^d + (-1)^d \sum_{e=1}^{d-1} a_{d,e} \left(\sum_{s=0}^{d-e} A_s x^{d+e-s}\right) \partial_x^e \\
&\quad + (-1)^d \sum_{e=1}^{d-2} a_{d,e} \left(\sum_{d-e<s<d} A_s x^{d+e-s}\right) \partial_x^e \\
&\quad + (-1)^{d-1} a_{d,d-1} \sum_{e=1}^{d-2} a_{d-1,e} \left(\sum_{1<s<d} A_s x^{d+e-s}\right) \partial_x^e \,.
\end{aligned}$$

We therefore see that $\widetilde{\delta}_{d,d-1}$ has vanishing local data in degrees d and $d-1$. As above, in the last sum $\sum_{e=1}^{d-2} a_{d-1,e}\left(\sum_{1<s<d} A_s x^{d+e-s}\right)\partial_x^e$ all those terms with $d-s \geqslant e$ do not contribute local data, so we write $\widetilde{\delta}_{d,d-1}$ as the sum of

$$\left(\sum_{s'=0}^{d} B_{s'}x^{s'} + (-1)^d \sum_{s=0}^{d-1} A_s x^{2d-s}\right)\partial_x^d$$

and

$$(-1)^d \sum_{e=1}^{d-1} a_{d,e}\left(\sum_{s=0}^{d-e} A_s x^{d+e-s}\right)\partial_x^e + (-1)^{d-1}a_{d,d-1}\sum_{e=1}^{d-2} a_{d-1,e}\left(\sum_{s=2}^{d-e} A_s x^{d+e-s}\right)\partial_x^e$$

and

$$(-1)^d \sum_{e=1}^{d-2}(a_{d,e} - a_{d,d-1}a_{d-1,e})\left(\sum_{d-e<s<d} A_s x^{d+e-s}\right)\partial_x^e$$

Now we define

$$\widetilde{\delta}_{d,d-1,d-2} = \widetilde{\delta}_{d,d-1} - (-1)^d(a_{d,d-2} - a_{d,d-1}a_{d-1,d-2})\left(\sum_{2<s<d} A_s y^{s-2}\right)\partial_y^{d-2}\ .$$

Continuing in this manner shows that we eventually find $\widetilde{\delta} \stackrel{\text{def}}{=} \widetilde{\delta}_{d,\ldots,1} \in H^0(\mathfrak{X}, \mathcal{D}_{\mathfrak{X},d})$ which has vanishing local data in all degrees less or equal to d, and its projection to $H^0(\mathfrak{X}, \mathcal{T}_{\mathfrak{X}}^{\otimes d})$ is equal to δ. This finishes the proof of the injectivity of the map 2.3.5.

(iii) Now we prove the splitting 2.3.6. We start by making the following general remark: if H_1 is a subgroup of a finite abelian p-group H, then H_1 is a direct summand of H if (and only if) $pH \cap H_1 = pH_1$.

Now let $c_{\leqslant d} \in H^0(\mathfrak{X}, Q_{\leqslant d})$ be any element and let $[c_{\leqslant d}] \in H^1(\mathfrak{X}, pr_*\mathcal{D}_{\mathfrak{X},d})$ be its image. Suppose $p[c_{\leqslant d}] = [pc_{\leqslant d}]$ lies in $H^1(\mathfrak{X}, pr_*\mathcal{D}_{\mathfrak{X},d-1})$, and write $[pc_{\leqslant d}] = [c_{\leqslant d-1}]$ for some element $c_{\leqslant d-1} \in H^0(\mathfrak{X}, Q_{\leqslant d-1})$. Then there is $\delta_{\leqslant d} \in H^0(\mathfrak{X}, \mathcal{D}_{\mathfrak{X},d})$ such that $Q_{\leqslant d}(\delta_{\leqslant d}) = pc_{\leqslant d} - c_{\leqslant d-1}$. This is implies that

$$Q_d(\delta_{\leqslant d}) \in H^0(\mathfrak{X}, Q_d) = \bigoplus_{a\in\mathfrak{X}(\mathbb{F}_p)} \bigoplus_{i=0}^{d-1} \left(\mathbb{Z}/p^{d-i}\right)\cdot x_a^i \partial_{x_a}^d$$

is such that all its local data in the various groups $\mathbb{Z}/p^{d-i}$ are divisible by p. Write $\delta_{\leqslant d} = \delta_d + \delta_{\leqslant d-1}$ with

$$\delta_d = \sum_{s=0}^{d-1} A_s y^s \partial_y^d + \sum_{s'=0}^{d} B_{s'} x^{s'} \partial_x^d \ ,$$

and with $\delta_{\leqslant d-1} \in H^0(\mathfrak{X}, \mathcal{D}_{\mathfrak{X},d-1})$. The local data in degree d of $\delta_{\leqslant d}$ (or, equivalently, δ_d) at $a = \infty$ and $a = 0$ can be read off immediately from this expression for δ_d and it follows that all coefficients A_s, $0 \leqslant s \leqslant d-1$, and $B_{s'}$, $0 \leqslant s' \leqslant d$, are divisible by p. So we can write $\delta_d = p\delta'_d$, and hence $\delta_{\leqslant d} = p\delta'_d + \delta_{\leqslant d-1}$. We then have $Q_{\leqslant d}(\delta_{\leqslant d}) = pQ_{\leqslant d}(\delta'_d) + Q_{\leqslant d-1}(\delta_{\leqslant d-1})$. From $Q_{\leqslant d}(\delta_{\leqslant d}) = pc_{\leqslant d} - c_{\leqslant d-1}$ we thus get

$$p(Q_{\leqslant d}(\delta'_d) - c_{\leqslant d}) = -Q_{\leqslant d-1}(\delta_{\leqslant d-1}) - c_{\leqslant d-1} \ .$$

Write $Q_{\leqslant d}(\delta'_d) - c_{\leqslant d} = c_d + c'_{\leqslant d-1}$ with $c'_d \in Q_d$ and $c'_{\leqslant d-1} \in Q_{\leqslant d-1}$ and we find:

$$pc'_{\leqslant d-1} = -Q_{\leqslant d-1}(\delta_{\leqslant d-1}) - c_{\leqslant d-1} \ ,$$

and thus $[c_{\leqslant d-1}] = p[c'_{\leqslant d-1}]$. □

In order to estimate the exponent of $H^1(\mathfrak{X}, pr_*(\mathcal{D}_{\mathfrak{X}_1,d}))$ we need the following elementary lemma.

Lemma 2.3.13. *Let $A = \mathbb{Z}/p^{n_1} \oplus \cdots \oplus \mathbb{Z}/p^{n_r}$ be an abelian torsion group with $0 < n_1 \leqslant n_2 \leqslant \ldots \leqslant n_r$. Let $a \in A$ be an arbitrary element. Then $A/\langle a \rangle$ surjects onto $\mathbb{Z}/p^{n_1} \oplus \cdots \oplus \mathbb{Z}/p^{n_{r-1}}$.*

Proof. Write $a = (a_1, \ldots, a_r)$, and choose $i \in \{1, \ldots, r\}$ such that

$$\mathrm{ord}(a_i) = \max\{\mathrm{ord}(a_j) \mid j = 1, \ldots, r\} \ .$$

If now $b = (b_1, \ldots, b_r) \in \langle a \rangle$ is such that $b_i = 0$, then $b = 0$. Therefore, the map

$$\mathbb{Z}/p^{n_1} \oplus \cdots \oplus \mathbb{Z}/p^{n_{i-1}} \oplus \mathbb{Z}/p^{n_{i+1}} \oplus \mathbb{Z}/p^{n_r} \hookrightarrow \mathbb{Z}/p^{n_1} \oplus \cdots \oplus \mathbb{Z}/p^{n_r} = A \twoheadrightarrow A/\langle a \rangle$$

is injective. Because finite-abelian groups are self-dual (non-canonically), we see that there is a surjection

$$A/\langle a \rangle \twoheadrightarrow \mathbb{Z}/p^{n_1} \oplus \cdots \oplus \mathbb{Z}/p^{n_{i-1}} \oplus \mathbb{Z}/p^{n_{i+1}} \oplus \mathbb{Z}/p^{n_r} \ .$$

But the group on the right clearly surjects onto $\mathbb{Z}/p^{n_1} \oplus \cdots \oplus \mathbb{Z}/p^{n_{r-1}}$. □

Proposition 2.3.14. *For any $d \geqslant 1$ the cohomology group $H^1(\mathfrak{X}, pr_*(\mathcal{T}_{\mathfrak{X}_1}^{\otimes d}))$ contains elements of order p^e where $e = \lfloor \frac{p-1}{p+1}(d+1) \rfloor$. In particular, as d tends to infinity, the exponents of $H^1(\mathfrak{X}, pr_*(\mathcal{T}_{\mathfrak{X}_1}^{\otimes d}))$ and of $H^1(\mathfrak{X}, pr_*(\mathcal{D}_{\mathfrak{X}_1,d}))$ tend to infinity.*

Proof. By 2.3.6 we have

$$H^1(\mathfrak{X}, pr_*(\mathcal{D}_{\mathfrak{X}_1,d})) = H^1(\mathfrak{X}, pr_*(\mathcal{D}_{\mathfrak{X}_1,d-1})) \oplus H^1(\mathfrak{X}, pr_*(\mathcal{T}_{\mathfrak{X}_1}^{\otimes d})) \; .$$

Furthermore, $H^1(\mathfrak{X}, pr_*(\mathcal{T}_{\mathfrak{X}_1}^{\otimes d}))$ is the quotient of

$$Q_d = \bigoplus_{a\in\mathfrak{X}(\mathbb{F}_p)} \bigoplus_{i=0}^{d-1} \left(\mathbb{Z}/p^{d-i}\right) \cdot x_a^i \partial_{x_a}^d \simeq \left(\bigoplus_{i=0}^{d-1} \left(\mathbb{Z}/p^{d-i}\right)\right)^{\oplus(p+1)} ,$$

cf. 2.3.10, by the image of $H^0(\mathfrak{X}, \mathcal{T}_{\mathfrak{X}}^{\otimes d})$ which is a free $\mathbb{Z}_p$-module of rank $2d+1$. Write $2d+1 = k(p+1)+r$ with $0 \leqslant r \leqslant p$, so that $k = \frac{2d+1}{p+1} - \frac{r}{p+1}$. Then, by applying 2.3.13 repeatedly we see that $H^1(\mathfrak{X}, pr_*(\mathcal{T}_{\mathfrak{X}_1}^{\otimes d}))$ must be of exponent at least p^e where

$$e = d-k = \frac{p-1}{p+1}d + \frac{r-1}{p+1} = \frac{p-1}{p+1}(d+1) - \frac{p-r}{p+1} = \left\lfloor \frac{p-1}{p+1}(d+1) \right\rfloor \; .$$

$\square$

Remark 2.3.15. With some more work it should also be possible to explicitly determine the structure of $H^1(\mathfrak{X}, pr_*(\mathcal{D}_{\mathfrak{X}_1,d}))$.

2.4. $\widehat{H}^0(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1}) = H^0(\mathfrak{X}_1, \mathscr{D}_{\mathfrak{X}_1})$ and $H^1(\mathfrak{X}_1, \mathscr{D}_{\mathfrak{X}_1})$ is non-torsion.

Theorem 2.4.1. *(a)* $T_pH^1(\mathfrak{X}, pr_*(\mathcal{D}_{\mathfrak{X}_1})) = 0$.

(b) $T_pH^1(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1}) = 0$.

(c) $\widehat{H}^0(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1}) = H^0(\mathfrak{X}_1, \mathscr{D}_{\mathfrak{X}_1})$.

(d) $\widehat{H}^1(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1})$ *contains a non-torsion element.*

(e) $H^1(\mathfrak{X}_1, \mathscr{D}_{\mathfrak{X}_1})$ *contains a non-torsion element.*

Proof. (a) We have $H^1(\mathfrak{X}, pr_*(\mathcal{D}_{\mathfrak{X}_1})) = \varinjlim_d H^1(\mathfrak{X}, pr_*(\mathcal{D}_{\mathfrak{X}_1,d}))$. Using 2.3.6 we see that

$$H^1(\mathfrak{X}, pr_*(\mathcal{D}_{\mathfrak{X}_1})) \simeq \bigoplus_{d=1}^{\infty} H^1(\mathfrak{X}, pr_*(\mathcal{T}_{\mathfrak{X}_1}^{\otimes d})) \; . \tag{2.4.2}$$

Because each group $H^1(\mathfrak{X}, pr_*(\mathcal{T}_{\mathfrak{X}_1}^{\otimes d}))$ is a finite p-group, the p-adic Tate module

$$T_pH^1(\mathfrak{X}, pr_*(\mathcal{D}_{\mathfrak{X}_1}))$$

must vanish.

(b) This follows from (a) and 2.2.2.

(c) This follows from (b) and 2.1.5.

(d) For $d \geqslant 1$ put $e_d = \lfloor \frac{p-1}{p+1}(d+1) \rfloor$. Let $c_d \in H^1(\mathfrak{X}, pr_*(\mathcal{T}_{\mathfrak{X}_1}^{\otimes d}))$ be an element of order e_d, cf. 2.3.14. It follows from 2.4.2 that $H^1(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1}) = H^1(\mathfrak{X}, pr_*(\mathcal{D}_{\mathfrak{X}_1}))$ contains a subgroup isomorphic to $\bigoplus_{d \geqslant 1} \langle c_d \rangle$. Let $(n_d)_{d \geqslant 1}$ be an increasing sequence of non-negative integers $n_d \leqslant e_d$ such that $\lim_{d \to \infty} n_d = \infty$ and $\lim_{d \to \infty}(e_d - n_d) = \infty$. Then $c = \sum_{d \geqslant 1} p^{n_d} c_d$ converges in the p-adic completion $\widehat{H}^1(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1})$ of $H^1(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1})$. Moreover, c is clearly not a torsion element.

(e) This follows from the fact that the map

$$H^1(\mathfrak{X}_1, \mathscr{D}_{\mathfrak{X}_1}) \longrightarrow \varprojlim_k H^1(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1}/p^k \mathcal{D}_{\mathfrak{X}_1}) = \widehat{H}^1(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1}) \ ,$$

cf. 2.1.1, is surjective. (The equality sign on the right is 2.1.6.) $\square$

3. Global sections and cohomology of $\mathscr{D}^{(m)}$ on $\mathfrak{X}_1$

In this section we consider the sheaves of differential operators $\mathscr{D}_{\mathfrak{X}_1}^{(m)}$ on $\mathfrak{X}_1$ of level $m \geqslant 0$. The discussion is along the same lines as in section 2, with a few modifications which we are going to point out as we proceed.

3.1. **Comparing the cohomology of $\mathcal{D}^{(m)}$ and $\mathscr{D}^{(m)}$.** Let $\mathcal{D}_{\mathbb{X}_1}^{(m)} = \mathcal{D}_{\mathbb{X}_1, \log}^{(m)}$ be the sheaf of logarithmic differential operators on $\mathbb{X}_1$ of level m [12, 5.6]. For a local description let $(a, a_1) \in \mathcal{R}_\infty \times \mathcal{R}$. For $m, d \geqslant 0$ we let $q_d^{(m)}$ be defined as usual by $d = q_d^{(m)} p^m + r$ with $0 \leqslant r < p^m$. On an open neighbourhood $\mathbb{X}_a^{(1)} \subset \mathbb{X}_1$ of the singularity corresponding to a, the $\mathcal{O}_{\mathbb{X}_1}$-module $\mathcal{D}_{\mathbb{X}_1}^{(m)}$ is generated by operators of the form

$$q_d^{(m)}! \binom{D}{d}$$

where $D = x_a \partial_{x_a} = -z_a \partial_{z_a}$ is a local section of $\mathcal{T}_{\mathbb{X}_1}$. On the 'residual disc scheme' $\mathbb{D}_{a,a_1}^{(1)}$ with coordinate function $x_{a,a_1}^{(1)}$, the module $\mathcal{D}_{\mathbb{X}_1}^{(m)}$ is generated by the usual divided powers

$$\frac{q_d^{(m)}!}{d!} \partial^d_{x_{a,a_1}^{(1)}} \ .$$

We write $\mathcal{D}_{\mathfrak{X}_1}^{(m)}$ for the $\mathcal{O}_{\mathfrak{X}_1}$-module generated by the restriction of $\mathcal{D}_{\mathbb{X}_1}^{(m)}$ to $\mathfrak{X}_1$. Let $\mathscr{D}_{\mathfrak{X}_1}^{(m)}$ be the p-adic completion $\mathcal{D}_{\mathfrak{X}_1}^{(m)}$. The first lemma is exactly as 2.1.1.

Lemma 3.1.1. *The canonical homomorphism*

$$H^i(\mathfrak{X}_1, \mathscr{D}_{\mathfrak{X}_1}^{(m)}) \longrightarrow \varprojlim_k H^i(\mathfrak{X}_1, \mathcal{D}_{\mathfrak{X}_1}^{(m)}/p^k \mathcal{D}_{\mathfrak{X}_1}^{(m)}) \ .$$

$\square$

is an isomorphism for $i=0$ and is surjective for $i=1$. For $i>1$ source and target of this map vanish.

And also the next result goes over without any changes.

Proposition 3.1.2. *(a) For all $i \geqslant 0$ there is a canonical exact sequence*

$$0 \to \widehat{H}^i(\mathfrak{X}_1, \mathcal{D}^{(m)}_{\mathfrak{X}_1}) \to \varprojlim_k H^i(\mathfrak{X}_1, \mathcal{D}^{(m)}_{\mathfrak{X}_1}/\mathcal{D}^{(m)}_{\mathfrak{X}_1}) \to T_p\left(H^{i+1}(\mathfrak{X}_1, \mathcal{D}^{(m)}_{\mathfrak{X}_1})\right) \to 0 \ . \tag{3.1.3}$$

(b) For $i=0$ the exact sequence in (a) is

$$0 \to \widehat{H}^0(\mathfrak{X}_1, \mathcal{D}^{(m)}_{\mathfrak{X}_1}) \to H^0(\mathfrak{X}_1, \mathscr{D}^{(m)}_{\mathfrak{X}_1}) \to T_p\left(H^1(\mathfrak{X}_1, \mathcal{D}^{(m)}_{\mathfrak{X}_1})\right) \to 0 \ . \tag{3.1.4}$$

(c) The cohomology group $H^2\left(\mathfrak{X}_1, \mathcal{D}^{(m)}_{\mathfrak{X}_1}\right)$ vanishes and the exact sequence in (a) gives therefore a canonical isomorphism

$$\widehat{H}^1(\mathfrak{X}_1, \mathcal{D}^{(m)}_{\mathfrak{X}_1}) \simeq \varprojlim_k H^1(\mathfrak{X}_1, \mathcal{D}^{(m)}_{\mathfrak{X}_1}/p^k\mathcal{D}^{(m)}_{\mathfrak{X}_1}) \ . \tag{3.1.5}$$

$\square$

3.2. **Vanishing of** $\mathrm{R}^1 pr_*(\mathcal{D}^{(m)}_{\mathfrak{X}_1})$**.** As above we use the Leray spectral sequence for the blow-up morphism

$$pr : \mathfrak{X}_1 \longrightarrow \mathfrak{X} = \mathfrak{X}_0 \ .$$

Applied to the sheaf $\mathcal{D}^{(m)}_{\mathfrak{X}_1}$ we get an exact sequence

$$0 \to H^1(\mathfrak{X}, pr_*(\mathcal{D}^{(m)}_{\mathfrak{X}_1})) \to H^1(\mathfrak{X}_1, \mathcal{D}^{(m)}_{\mathfrak{X}_1}) \to H^0(\mathfrak{X}, \mathrm{R}^1 pr_*(\mathcal{D}^{(m)}_{\mathfrak{X}_1})) \to 0 \ . \tag{3.2.1}$$

Denote by $\mathcal{D}^{(m)}_{\mathfrak{X},d}$ and $\mathcal{D}^{(m)}_{\mathfrak{X}_1,d}$ the sheaves of differential operators of degree less or equal to d. Note also that

$$(\mathcal{T}^{\otimes d}_{\mathfrak{X}_1})^{(m)} = \frac{q^{(m)}_d!}{d!}\mathcal{T}^{\otimes d}_{\mathfrak{X}_1} \subset \mathcal{T}^{\otimes d}_{\mathfrak{X}_1} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \ , \tag{3.2.2}$$

cf. [12, 3.2]. Then, similar to 2.2.3, we have an exact sequence

$$0 \longrightarrow \mathcal{D}^{(m)}_{\mathfrak{X}_1,d-1} \longrightarrow \mathcal{D}^{(m)}_{\mathfrak{X}_1,d} \longrightarrow (\mathcal{T}^{\otimes d}_{\mathfrak{X}_1})^{(m)} \longrightarrow 0 \ . \tag{3.2.3}$$

Lemma 3.2.4. *(a) For all $d \geqslant 0$ one has* $\mathrm{R}^1 pr_*(\mathcal{D}^{(m)}_{\mathfrak{X}_1,d}) = 0$.

(b) $\mathrm{R}^1 pr_*(\mathcal{D}^{(m)}_{\mathfrak{X}_1}) = 0$.

(c) $H^1(\mathfrak{X}, pr_*(\mathcal{D}^{(m)}_{\mathfrak{X}_1})) = H^1(\mathfrak{X}_1, \mathcal{D}^{(m)}_{\mathfrak{X}_1})$.

Proof. (a) This follows as in 2.2.2 (a) using 3.2.2 in the Cech cohomology argument.

(b) Follows from (a) by passing to the limit.

(c) Follows from (b) and 3.2.1. $\square$

3.3. **The cohomology group $H^1(\mathfrak{X}, pr_*(\mathcal{D}^{(m)}_{\mathfrak{X}_1}))$.** Consider the exact sequence 3.2.3 and the corresponding sequence of direct images on $\mathfrak{X}$

$$0 \longrightarrow pr_*\left(\mathcal{D}^{(m)}_{\mathfrak{X}_1,d-1}\right) \longrightarrow pr_*\left(\mathcal{D}^{(m)}_{\mathfrak{X}_1,d}\right) \longrightarrow pr_*\left(\mathcal{T}^{\otimes d}_{\mathfrak{X}_1}\right) \longrightarrow \mathrm{R}^1 pr_*\left(\mathcal{D}^{(m)}_{\mathfrak{X}_1,d-1}\right) = 0 \ , \tag{3.3.1}$$

where we have used 3.2.4 (a). We have

$$H^1(\mathfrak{X}, pr_*(\mathcal{D}^{(m)}_{\mathfrak{X}_1})) = \varinjlim_d H^1(\mathfrak{X}, pr_*(\mathcal{D}^{(m)}_{\mathfrak{X}_1,d})) \ .$$

We put $\partial_x^{\langle d \rangle_{(m)}} = \frac{q_d^{(m)}!}{d!}\partial_x^d$, and similarly for ∂_y^d (and also for $\partial_{x_a}^d$). With this notation we deduce from 2.3.2 the following

Lemma 3.3.2. *Let x, y be the standard coordinates on $\mathbb{P}^1$ satisfying $xy = 1$. Then we have for any $s \in \mathbb{Z}_{\geqslant 1}$*

$$\partial_y^{\langle s \rangle_{(m)}} = (-1)^s \sum_{t=1}^{s} a^{(m)}_{s,t} x^{s+t} \partial_x^{\langle t \rangle_{(m)}} \ ,$$

where for all $s \geqslant 1$ and $1 \leqslant t \leqslant s$

$$a^{(m)}_{s,t} = \binom{s}{t} \frac{(s-1)!}{(t-1)!} \frac{q_s^{(m)}!}{s!} \left(\frac{q_t^{(m)}!}{t!}\right)^{-1} = \binom{s-1}{t-1} \frac{q_s^{(m)}!}{q_t^{(m)}!} \ . \tag{3.3.3}$$

These numbers are always integers, and we have, in particular,

$$a_{s,1} = q_s^{(m)}! \quad \textit{and} \quad a_{s,s} = 1 \ .$$

$\square$

Theorem 3.3.4. *For all $d \geqslant 1$ the canonical map*

$$H^1(\mathfrak{X}, pr_*(\mathcal{D}^{(m)}_{\mathfrak{X}_1,d-1})) \to H^1(\mathfrak{X}, pr_*(\mathcal{D}^{(m)}_{\mathfrak{X}_1,d})) \tag{3.3.5}$$

coming from the long exact cohomology sequence associated to 3.3.1 is injective and embeds $H^1(\mathfrak{X}, pr_*(\mathcal{D}^{(m)}_{\mathfrak{X}_1,d-1}))$ as a direct summand of $H^1(\mathfrak{X}, pr_*(\mathcal{D}^{(m)}_{\mathfrak{X}_1,d}))$. Therefore, there is a splitting:

$$H^1\left(\mathfrak{X}, pr_*(\mathcal{D}^{(m)}_{\mathfrak{X}_1,d})\right) = H^1\left(\mathfrak{X}, pr_*(\mathcal{D}^{(m)}_{\mathfrak{X}_1,d-1})\right) \oplus H^1\left(\mathfrak{X}, pr_*((\mathcal{T}^{\otimes d}_{\mathfrak{X}_1})^{(m)})\right) . \tag{3.3.6}$$

□

Proof. The proof proceeds along the lines of 2.3.4 taking into account the following points.

(i) The skyscraper sheaf $Q^{(m)}_d)$ (resp. $Q^{(m)}_{\leqslant d}$) is defined, similar as before, as the quotient of $(\mathcal{T}^{\otimes d}_{\mathfrak{X}})^{(m)}$ (resp. $\mathcal{D}^{(m)}_{\mathfrak{X},d}$) by $pr_*((\mathcal{T}^{\otimes d}_{\mathfrak{X}_1})^{(m)})$ (resp. $pr_*(\mathcal{D}^{(m)}_{\mathfrak{X}_1,d})$). By 3.2.2, the sheaf $Q^{(m)}_{\leqslant d}$ (resp. $Q^{(m)}_d$) is actually isomorphic to the sheaf $Q_{\leqslant d}$ (resp. Q_d).

(ii) The subtle part is the proof of the injectivity. As in the proof of 2.3.4 consider an element δ of $H^0\left(\mathfrak{X}, (\mathcal{T}^{\otimes d}_{\mathfrak{X}})^{(m)}\right)$ whose image in the group $H^0(\mathfrak{X}, Q^{(m)}_d)$ vanishes. Then we want to lift it to an element $\widetilde{\delta} \in H^0\left(\mathfrak{X}, \mathcal{D}^{(m)}_{\mathfrak{X},d}\right)$ such that the image of $\widetilde{\delta}$ in $H^0(\mathfrak{X}, Q^{(m)}_{\leqslant d})$ vanishes. The discussion now proceeds along exactly the same lines as before. The difference is that one has to use the transformation formula in 3.3.2. This does not affect the arguments because the coefficients $a^{(m)}_{s,t}$ are integral. □

Proposition 3.3.7. *For any $d \geqslant 1$ the cohomology group $H^1\left(\mathfrak{X}, pr_*((\mathcal{T}^{\otimes d}_{\mathfrak{X}_1})^{(m)})\right)$ contains elements of order p^e where $e = \lfloor \frac{p-1}{p+1}(d+1) \rfloor$. In particular, as d tends to infinity, the exponents of $H^1\left(\mathfrak{X}, pr_*((\mathcal{T}^{\otimes d}_{\mathfrak{X}_1})^{(m)})\right)$ and of $H^1\left(\mathfrak{X}, pr_*(\mathcal{D}^{(m)}_{\mathfrak{X}_1,d})\right)$ tend to infinity.*

Proof. The proof of 2.3.14 carries over to the case $m > 0$. □

Theorem 3.3.8. *(a) $T_pH^1(\mathfrak{X}, pr_*(\mathcal{D}^{(m)}_{\mathfrak{X}_1})) = 0$.*

(b) $T_pH^1(\mathfrak{X}_1, \mathcal{D}^{(m)}_{\mathfrak{X}_1}) = 0$.

(c) $H^0(\mathfrak{X}_1, \mathcal{D}^{(m)}_{\mathfrak{X}_1})^\wedge = H^0(\mathfrak{X}_1, \mathscr{D}^{(m)}_{\mathfrak{X}_1})$.

(d) $H^1(\mathfrak{X}_1, \mathscr{D}^{(m)}_{\mathfrak{X}_1})$ contains non-torsion elements.

Proof. The proof of 2.4.1 carries over to the case when $m > 0$. □

3.4. $H^1(\mathfrak{X}_1, \mathscr{D}^\dagger_{\mathfrak{X}_1,\mathbb{Q}})$ does not vanish for $p > 2$.

Theorem 3.4.1. *(i) The inductive limit*

$$\varinjlim_m \widehat{H}^1(\mathfrak{X}_1, \mathcal{D}^{(m)}_{\mathfrak{X}_1}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

does not vanish when $p > 2$.

(ii) $H^1(\mathfrak{X}_1, \mathscr{D}^\dagger_{\mathfrak{X}_1,\mathbb{Q}})$ does not vanish when $p > 2$.

Proof. (i) Let us consider the transition map

$$\widehat{H}^1(\mathfrak{X}_1, \mathcal{D}^{(0)}_{\mathfrak{X}_1}) \longrightarrow \widehat{H}^1(\mathfrak{X}_1, \mathcal{D}^{(m)}_{\mathfrak{X}_1}) \ . \tag{3.4.2}$$

Using 2.4.2 and its analogues in level m, together with 3.2.4 and 3.1.5, we rewrite 3.4.2 as

$$\left[\bigoplus_{d=1}^{\infty} H^1(\mathfrak{X}, pr_*(\mathcal{T}^{\otimes d}_{\mathfrak{X}_1}))\right]^\wedge \longrightarrow \left[\bigoplus_{d=1}^{\infty} H^1\left(\mathfrak{X}, pr_*((\mathcal{T}^{\otimes d}_{\mathfrak{X}_1})^{(m)})\right)\right]^\wedge \ , \tag{3.4.3}$$

where $[-]^\wedge$ denotes the p-adic completion of $[-]$. Because of 3.2.2 we can formally write the right hand side of 3.4.3 as

$$\left[\bigoplus_{d=1}^{\infty} \frac{q_d^{(m)}!}{d!} H^1\left(\mathfrak{X}, pr_*(\mathcal{T}^{\otimes d}_{\mathfrak{X}_1})\right)\right]^\wedge .$$

By definition of the topology, the latter p-adic completion is, as topological $\mathbb{Z}_p$-module, canonically isomorphic to

$$\left[\bigoplus_{d=1}^{\infty} H^1\left(\mathfrak{X}, pr_*(\mathcal{T}^{\otimes d}_{\mathfrak{X}_1})\right)\right]^\wedge$$

via mapping, in the d-th component, $\frac{q_d^{(m)}!}{d!} \mapsto 1$. With this identification, the map in 3.4.3 assumes the following explicit form

$$(c_d)_{d\geqslant 1} \mapsto \left(\frac{d!}{q_d^{(m)}!} \cdot c_d\right)_{d\geqslant 1} \in \left[\bigoplus_{d=1}^{\infty} H^1\left(\mathfrak{X}, pr_*(\mathcal{T}^{\otimes d}_{\mathfrak{X}_1})\right)\right]^\wedge \ ,$$

where $c_d \in H^1\left(\mathfrak{X}, pr_*(\mathcal{T}^{\otimes d}_{\mathfrak{X}_1})\right)$. Now let c_d be a cohomology class of order p^{e_d} where $e_d = \lfloor \frac{p-1}{p+1}(d+1) \rfloor$, cf. 3.3.7. Denote by v_p the (logarithmic) normalized p-adic valuation. Then we have

$$v_p\left(\frac{d!}{q_d^{(m)}!}\right) \leqslant \frac{d}{p-1} - \left(\frac{\lfloor\frac{d}{p^m}\rfloor}{p-1} - \log_p\left(\left\lfloor\frac{d}{p^m}\right\rfloor\right)\right) \leqslant \frac{d}{p-1} - \frac{d}{(p-1)p^m} + \log_p(d) + 1\ .$$

Let n_d be a non-negative integer, and denote by ord the order of an element. Then

$$\begin{aligned} v_p\left(\mathrm{ord}\left(\tfrac{d!}{q_d^{(m)}!}\cdot p^{n_d}c_d\right)\right) &\geqslant \tfrac{p-1}{p+1}(d+1) - 1 - n_d - \left(\tfrac{d}{p-1} - \tfrac{d}{(p-1)p^m} + \log_p(d) + 1\right) \\ &= \left(\tfrac{p-1}{p+1} - \tfrac{1}{p-1} + \tfrac{1}{(p-1)p^m}\right) d - n_d - \log_p(d) - 2 \\ &= \left(\tfrac{p^2-3p}{p^2-1} + \tfrac{1}{(p-1)p^m}\right) d - n_d - \log_p(d) - 2\ . \end{aligned}$$

If $p \geqslant 3$ and if we put, for instance, $n_d = \lfloor\sqrt{d}\rfloor$, $d \geqslant 1$, then we have, for any m,

$$\lim_{d\to\infty}\left[\left(\frac{p^2-3p}{p^2-1} + \frac{1}{(p-1)p^m}\right) d - n_d - \log_p(d) - 2\right] = \infty\ .$$

This means that the sequence of elements $(p^{n_d}c_d)$ defines an element c of $\widehat{H}^1(\mathfrak{X}_1, \mathcal{D}^{(0)}_{\mathfrak{X}_1})$, and this element c has the property that its image in any group $\widehat{H}^1(\mathfrak{X}_1, \mathcal{D}^{(m)}_{\mathfrak{X}_1})$ is non-torsion. The image of c in

$$\varinjlim_m \widehat{H}^1(\mathfrak{X}_1, \mathcal{D}^{(m)}_{\mathfrak{X}_1})$$

will also not be a torsion element.

(ii) Because the maps

$$H^1(\mathfrak{X}_1, \mathscr{D}^{(m)}_{\mathfrak{X}_1}) \longrightarrow \varprojlim_k H^1(\mathfrak{X}_1, \mathcal{D}^{(m)}_{\mathfrak{X}_1}/p^k\mathcal{D}^{(m)}_{\mathfrak{X}_1}) = \widehat{H}^1(\mathfrak{X}_1, \mathcal{D}^{(m)}_{\mathfrak{X}_1})$$

are surjective, cf. 3.1.1, the same is true after tensoring with $\mathbb{Q}$ and taking the limit for $m \to \infty$. $\square$

References

[1] A. Beĭlinson and J. Bernstein. Localisation de $\mathfrak{g}$-modules. *C. R. Acad. Sci. Paris Sér. I Math.*, 292(1):15–18, 1981.

[2] P. Berthelot. Cohomologie rigide et théorie des $\mathcal{D}$-modules. In *p-adic analysis (Trento, 1989)*, volume 1454 of *Lecture Notes in Math.*, pages 80–124. Springer, Berlin, 1990.

[3] P. Berthelot. $\mathcal{D}$-modules arithmétiques I. Opérateurs différentiels de niveau fini. *Ann. Sci. E.N.S*, 29:185–272, 1996.

[4] Pierre Berthelot. $\mathcal{D}$-modules arithmétiques. II. Descente par Frobenius. *Mém. Soc. Math. Fr. (N.S.)*, (81):vi+136, 2000.
[5] A. Grothendieck. Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I. *Inst. Hautes Études Sci. Publ. Math.*, (11):167, 1961.
[6] R. Hartshorne. *Algebraic geometry.* Springer-Verlag, New York, 1977. Graduate Texts in Math., No. 52.
[7] C. Huyghe. $\mathscr{D}^\dagger$-affinité de l'espace projectif. *Compositio Math.*, 108(3):277–318, 1997. With an appendix by P. Berthelot.
[8] J. C. McConnell and J. C. Robson. *Noncommutative Noetherian rings.* Pure and Applied Mathematics (New York). John Wiley & Sons Ltd., Chichester, 1987.
[9] C. Montagnon. Généralisation de la théorie arithmétique des $\mathscr{D}$-modules à la géométrie logarithmique. Thesis, Université de Rennes.
[10] Christine Noot-Huyghe. Un théorème de Beilinson-Bernstein pour les $\mathscr{D}$-modules arithmétiques. *Bull. Soc. Math. France*, 137(2):159–183, 2009.
[11] D. Patel, T. Schmidt, and M. Strauch. Locally analytic representations of $\mathrm{GL}(2, L)$ via semistable models of $\mathbb{P}^1$. *Journal of the Institute of Mathematics of Jussieu* (to appear).
[12] D. Patel, T. Schmidt, and M. Strauch. Integral models of $\mathbb{P}^1$ and analytic distribution algebras for GL(2). *Münster J. Math.*, 7:241–271, 2014.
[13] P. Schneider. Continuous representation theory of p-adic Lie groups. In *International Congress of Mathematicians. Vol. II*, pages 1261–1282. Eur. Math. Soc., Zürich, 2006.
[14] P. Schneider and J. Teitelbaum. Algebras of p-adic distributions and admissible representations. *Invent. Math.*, 153(1):145–196, 2003.

Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, IN 47907, U.S.A.
E-mail address: `deeppatel1981@gmail.com`

IRMAR, Université de Rennes 1, Campus Beaulieu, 35042 Rennes cedex, France
E-mail address: `Tobias.Schmidt@univ-rennes1.fr`

Indiana University, Department of Mathematics, Rawles Hall, Bloomington, IN 47405, U.S.A.
E-mail address: `mstrauch@indiana.edu`