

FRÉCHET VALUED REAL ANALYTIC FUNCTIONS

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Abstract

We characterize those Fréchet spaces E for which $\text{Proj}^1 A(\Omega, E) = 0$, where $A(\Omega, E)$ is the (PLB)-space of E -valued real analytic functions on the open set $\Omega \subset \mathbb{R}^d$. This has various consequences, among those a characterization of all (LB)-spaces X for which $\text{Ext}^1(X, A(\Omega)) = 0$ in the category of (PLB)-spaces, which means that every exact sequence

$$0 \longrightarrow A(\Omega) \xrightarrow{i} Y \xrightarrow{q} X \longrightarrow 0$$

where Y is of type (PLB) splits.

Let E be a Fréchet space, $\| \cdot \|_1 \leq \| \cdot \|_2 \leq \dots$ a fundamental system of seminorms, $\Omega \subset \mathbb{R}^d$ open and $A(\Omega, E)$ the space of E -valued real analytic functions on Ω , i.e. of those E -valued functions f on Ω for which $y \circ f \in A(\Omega)$ for all $y \in E'$. By E_k , $k \in \mathbb{N}$ we denote the local Banach spaces and by $j_n^k : E_n \rightarrow E_k$ for $n \geq k$, $j^k : E \rightarrow E_k$ the canonical linking maps. Then we can describe $A(\Omega, E)$ also as the space of all E -valued functions f on Ω so that $j^k \circ f \in A(\Omega, E_k)$ for all $k \in \mathbb{N}$. Here $A(\Omega, E_k)$ is the space of all E_k -valued functions which can be expanded onto their Taylor series in a neighborhood of any point of Ω . Therefore $A(\Omega, E)$ can be written in a natural way as projective limit

$$A(\Omega, E) = \lim \text{proj}_n H(K_n, E_n)$$

where $K_1 \subset \overset{\circ}{K}_2 \subset K_2 \subset \dots$ is a compact exhaustion of Ω and $H(K_n, E_n)$ denotes the (LB)-space of germs of E_n -valued holomorphic functions on K_n . So $A(\Omega, E)$ is a (PLB)-space, i.e. a countable projective limit of (LB)-spaces, and we may ask the question, when $\text{Proj}^1 A(\Omega, E) = 0$.

Let us recall that for a (PLB)-space $Y = \lim \text{proj}_n Y_n$ all reduced defining spectra $(Y_n)_n$ of (LB)-spaces are equivalent. Therefore $\text{Proj}^1 Y = \text{Proj}_n^1 Y_n$ does not depend on the (reduced) spectrum $(Y_n)_n$ and is one of the most important invariants of such a space. Its vanishing is, at least in the case of a (PLS)-space, equivalent to Y being

2000 Mathematics Subject Classification. Primary: 46E40, 46M18, 46M40, 46A63. Secondary: 46A04, 46A13, 46E10.

Key words and phrases: space of real analytic functions, Fréchet valued real analytic functions, (LB)-space, (PLB)-space, Proj^1 , Ext^1 , linear topological invariants, splitting of exact sequences.

The paper was completed during a stay at the University of Liège supported by an A. v. Humboldt award. The author gratefully acknowledges the support.

ultrabornological. Its main significance is that it is the “cohomology of Y ”. $\text{Proj}^1 Y = 0$ for a (PLB)-space Y allows to conclude from “local surjectivity” to surjectivity for linear maps where Y appears as kernel. Spaces like $\mathcal{D}'(\Omega)$ of distributions, $A(\Omega)$ of real analytic functions or $L(E, F)$ of continuous linear maps, where E and F are Fréchet spaces, are (PLB)-spaces. For the first two we have $\text{Proj}^1 Y = 0$. The fact that $\text{Proj}^1 A(\Omega) = 0$ or, equivalently, that $A(\Omega)$ is ultrabornological played an important role in the proof in [9] that $A(\Omega)$ does not have a basis. For nuclear Fréchet spaces we have $\text{Proj}^1 L(E, F) = \text{Ext}^1(E, F)$, hence $\text{Proj}^1 L(E, F) = 0$ means that every exact sequence $0 \rightarrow F \rightarrow G \rightarrow E \rightarrow 0$ of Fréchet spaces splits. A similar connection we will use below to give a characterization of those (LB)-spaces for which $\text{Ext}_{PLB}^1(X, A(\Omega)) = 0$. See for that also the final remarks of §3.

Throughout the paper we use common terminology for locally convex spaces, in particular Fréchet spaces. For all this we refer to [24], for homological concepts to [32].

The author thanks P. Domański for useful conversations and remarks.

1 Main theorem

The answer to our problem is given in the first of our two main theorems. Throughout this paper E will always denote a Fréchet space.

Theorem 1.1 $\text{Proj}^1 A(\Omega, E) = 0$ if and only if E has property $(\overline{\overline{\Omega}})$.

A Fréchet space with a fundamental sequence of seminorms $(\|\cdot\|_n)$ defining the topology is said to have property $(\overline{\overline{\Omega}})$ if

$$\forall k \exists l \forall n, \vartheta \in]0, 1[\exists C \forall u \in E' \quad \|u\|_l^* \leq C \|u\|_k^{*\vartheta} \|u\|_n^{*1-\vartheta}.$$

Here $\|\cdot\|^*$ denotes the dual norm for $\|\cdot\|$. Property $(\overline{\overline{\Omega}})$ is a linear topological invariant. For its role see [29], [9], [2], [6], [23]. From [24, Lemma 29.13] we obtain the following equivalence.

Remark E has $(\overline{\overline{\Omega}})$ if and only if

$$\forall k \exists l \forall n, \gamma > 0 \exists C \forall r > 0 \quad U_l \subset rU_n + \frac{C}{r^\gamma} U_k,$$

where $U_k = \{x \in E \mid \|x\|_k \leq 1\}$.

Before we can prove Theorem 1.1 we need some preparations. In the following lemma we do for $(\overline{\overline{\Omega}})$ exactly what is done for $(\overline{\Omega})$ in [24, Lemma 29.16].

Lemma 1.2 *The Fréchet space E has property $(\overline{\overline{\Omega}})$ if, and only if, there exists a bounded Banach disk $B \subset E$ having the following property:*

For each $p \in \mathbb{N}$ there exists $q \in \mathbb{N}$ so that for every $\gamma > 0$ there is $D > 0$ with

$$U_q \subset rB + \frac{D}{r^\gamma} U_p \text{ for all } r > 0 \tag{1}$$

If E is a Fréchet-Hilbert space, then B can be chosen as a Hilbert disk.

Proof: Obviously the given condition implies that E has the property $(\overline{\overline{\Omega}})$. To prove the reverse implication we proceed, mutatis mutandis, exactly as in the proof of [24, Lemma 29.16]. We indicate here, for the convenience of the reader, only the necessary changes.

Instead of formula [24, Lemma 29.16, (1)] we choose $q \in \mathbb{N}$ with $q > p$ in accordance with property $(\overline{\overline{\Omega}})$. Then we let $C_1 := 1$ and choose, for each $n \in \mathbb{N}, n \geq 2$, a $C_n \geq \max(n, C_{n-1}^{\frac{n}{n-1}})$ with

$$U_q \subset rU_n + \frac{C_n}{r^{2n}}U_p \text{ for all } r > 0. \quad (2)$$

This replaces formula [24, Lemma 29.16, (1)]. Now we choose $\varepsilon_k := \min(\frac{1}{2}, C_{k+2}^{-1})$ and arrive instead of formula [24, Lemma 29.16, (4)] at

$$U_q \subset rM + \left(\frac{r}{C_{n+1}} + \frac{C_n}{r^{2n}} \right) U_p \text{ for all } r > 0 \text{ and } n \in \mathbb{N}, n > p. \quad (3)$$

For $r \geq C_{p+1}^{\frac{1}{p+1}}$ we choose $n \in \mathbb{N}$ with $n > p$, so that $C_n^{\frac{1}{n}} \leq r \leq C_{n+1}^{\frac{1}{n+1}}$ and obtain

$$U_q \subset rM + \frac{2}{r^n}U_p. \quad (4)$$

This holds for $r \geq C_{p+1}^{\frac{1}{p+1}}$. Replacing 2 by a bigger constant, if necessary, we have it for all $r > 0$. \square

For the next three results we will assume that E has property $(\overline{\overline{\Omega}})$, B is as in Lemma 1.2 and for given $p \in \mathbb{N}$ we have chosen $q \in \mathbb{N}$ so that for every $\gamma > 0$ we have (1).

Corollary 1.3 *There are an increasing function h and a decreasing function g , so that for large r*

$$\begin{aligned} h(r) &= O(r^\varepsilon) \text{ for every } \varepsilon > 0 \\ g(r) &= O\left(\frac{1}{r^m}\right) \text{ for every } m \in \mathbb{N} \end{aligned}$$

and

$$U_q \subset h(r)B + g(r)U_p$$

for every $r > 0$.

Proof: We use Lemma 1.2 and set

$$g_0(r) := \sup_{\|x\|_q \leq 1} \inf_{y \in rB} \|x - y\|_p.$$

Then g_0 is decreasing and has the required asymptotic property and for every decreasing $g_1 > g_0$ with the same asymptotic property we have $U_q \subset rB + g_1(r)U_p$ for every $r > 0$. By a standard construction we find now an increasing h so that h and $g := g_1 \circ h$ have the required asymptotic properties and g is decreasing. \square

For entire functions g_1, \dots, g_m on \mathbb{C}^d and $r_j > 0, j = 1, \dots, m$ we put $U = \{z \in \mathbb{C}^d : \max_j |g_j(z)| < 1\}$. This is an analytic polyhedron. We will assume that it is bounded.

Lemma 1.4 *For every $f \in H(U, E_q)$ there exist $g \in H(\mathbb{C}^d, E_p)$ and $h \in H(U, E_B)$ so that $j_q^p \circ f = g + j^p \circ h$ on U .*

Proof: We may assume that $U \subset \mathbb{D}^d$ where $\mathbb{D}^d = \{z \in \mathbb{C}^d : |z|_\infty < 1\}$ and use the classical method of Oka. For $z \in U$ we put $\varphi(z) = (z, g_1(z), \dots, g_m(z)) \in \mathbb{C}^{d+m} = \mathbb{C}^N$. This defines a biholomorphic map φ from U onto a closed complex submanifold $\varphi(U)$ of \mathbb{D}^N . From the Cartan-Oka theory [18, Th. 7.2.7] and a tensor argument we conclude: for every $f \in H(U, E_q)$ there is $F \in H(\mathbb{D}^N, E_q)$ with $F \circ \varphi = f$.

We expand F into its power series

$$F(w) = \sum_{\beta \in \mathbb{N}_0^N} a_\beta w^\beta.$$

Then Cauchy's inequalities yield: for every $s > 0$ there is $C_s > 0$ so that

$$\|a_\beta\|_q \leq C_s e^{s|\beta|}, \quad \beta \in \mathbb{N}_0^N.$$

Therefore we can find a sequence $s_j \searrow 0$ and a constant C so that

$$\|a_\beta\|_q \leq C e^{s_j |\beta|}, \quad \beta \in \mathbb{N}_0^N.$$

For given β we set $r = e^{s_j |\beta|}$ and, by Lemma 1.3, find

$$\begin{aligned} u_\beta &\in e^{s_j |\beta|} h(e^{|\beta|}) B \\ v_\beta &\in e^{s_j |\beta|} g(e^{|\beta|}) U_p \end{aligned}$$

so that $j_q^p(a_\beta) = j^p(u_\beta) + v_\beta$ for all $\beta \in \mathbb{N}_0^N$. We set

$$h(z) = \sum_{\beta \in \mathbb{N}_0^N} u_\beta \varphi(z)^\beta, \quad g(z) = \sum_{\beta \in \mathbb{N}_0^N} v_\beta \varphi(z)^\beta.$$

They obviously fulfill the assertion. □

Lemma 1.5 *Let $K \subset \mathbb{R}^d$ be compact and $f \in H(K, E_q)$, then there exist $g \in H(\mathbb{C}^d, E_p)$ and $h \in H(K, E_B)$ so that $j_q^p \circ f = g + j^p \circ h$ in $H(K, E_p)$.*

Proof: This is an immediate consequence of Lemma 1.4 and the fact that K has a neighborhood basis of analytic polyhedra (see [16, Lemma 5.4.1]). □

We are now in the position to prove Theorem 1.1.

Proof: $A(\Omega)$ has a complemented subspace F isomorphic to the 2π -periodic real analytic functions in one dimension. For $d = 1$ this follows from [21], for $d > 1$ we may use any circle in Ω and [4, Lemma 4.4] (more precise: the remark after this Lemma). Since F is isomorphic to $\Lambda'_0(n) = \{\xi \in \mathbb{C}^d : \sup_n |\xi_n| e^{tn} < +\infty \text{ for some } t < 0\}$, the result follows from [30, Theorem 4.2]. Notice that $0 = \text{Proj}^1(F, E) \cong \text{Ext}^1(\Lambda_0(n), E)$.

To prove the converse we may assume that the fundamental system of seminorms is chosen so that with a closed, bounded, absolutely convex set B

$$\forall k, \gamma > 0 \exists C \forall r > 0 : U_{k+1} \subset rB + \frac{C}{r^\gamma} U_k.$$

Let functions $f_k \in H(K_k, E_k)$ be given. We have to find a sequence of functions $F_k \in H(K_k, E_k)$ so that $f_k = F_k - j_{k+1}^k \circ F_{k+1}$ for all k .

First we use Lemma 1.5 to find, for $k = 2, 3, \dots$, functions $g_k \in H(\mathbb{C}^d, E_{k-1})$ and $h_k \in H(K_k, E_B)$ so that $j_k^{k-1} \circ f_k = g_k + j^{k-1} \circ h_k$ in $H(K_k, E_B)$.

Clearly we have $\text{Proj}_k^1 H(\mathbb{C}^d, E_k) = 0$, hence there are $G_k \in H(\mathbb{C}^d, E_k)$ so that $g_{k+1} = G_k - j_{k+1}^k \circ G_{k+1}$ for $k \in \mathbb{N}$.

Since E_B is a Banach space we have $\text{Proj}_k^1 H(K_k, E_B) = \text{Proj}_k^1 A(\Omega, E_B) = 0$. The latter because of the same reason, as $\text{Proj}_k^1 A(\Omega) = 0$ (see e.g. [10, Proposition 1.5]). Hence there are $H_k \in H(K_k, E_B)$ so that $h_k = H_k - H_{k+1}$ for all $k = 2, 3, \dots$

Finally we set $F_k = f_k + g_{k+1} + j_{k+1}^k \circ G_{k+1} + j^k \circ H_{k+1} \in H(K_k, E_k)$ to obtain $F_k - F_{k+1} = f_k$ for all $k \in \mathbb{N}$ and therefore the result. \square

2 Real analytic functions on subvarieties of \mathbb{R}^d

Let $\Omega \subset \mathbb{R}^d$ be open and let X be a *coherent subvariety* of Ω . This means that there is an open pseudoconvex set $\Omega_0 \subset \mathbb{C}^d$ as and a closed complex subvariety V of Ω_0 so that $X = V \cap \mathbb{R}^d$ and for any point $x \in X$ the ideal of X in ${}_d\mathcal{O}_x$ coincides with the ideal of V in ${}_d\mathcal{O}_x$, i.e. every germ of a holomorphic function at x which vanishes on X also vanishes on V .

A function f on X is called real analytic if it can be represented in a neighborhood of any point by a power series. f then extends uniquely into a holomorphic function F defined on a neighborhood of X in V . Since, by the Cartan-Grauert theorem (see [5, 13]), Ω has a neighborhood basis in \mathbb{C}^d of pseudoconvex sets, F can be extended into a holomorphic function defined on a neighborhood of Ω .

For any open $\omega \subset \Omega$ we set

$$G_X(\omega) = \{f \in A(\omega) : f|_X = 0\}.$$

This defines a subsheaf \mathcal{G}_X of the sheaf \mathcal{A} of real analytic functions on Ω . It is the restriction to Ω of the ideal sheaf \mathcal{I}_V of V , which is a coherent subsheaf of ${}_d\mathcal{O}$. This implies that for every compact $K \subset \Omega$ there are finitely many generators $f_1, \dots, f_l \in J_X(\Omega_0) := \Gamma(\Omega_0, \mathcal{I}_X)$ such that the map

$$S : H(K)^l \longrightarrow G_X(K), \quad S(g_1, \dots, g_l) := \sum_{j=1}^l g_j f_j,$$

is surjective (see [18, 7.1.6, 7.2.5]).

Here $G_X(K)$ denotes the sections of \mathcal{G}_X over K and it is clear that

$$G_X(K) = \lim \operatorname{ind}_n J_X^\infty(U_n)$$

where U_n is a basis of complex neighborhoods of K and $J_X^\infty(U_n)$ are the bounded sections on U_n with the sup-norm. So $G_X(K)$ is a nuclear (LB)-space and the map S is open, by the open mapping theorem for (LB)-spaces. Moreover, by [24, 26.26], $G_X(K) \subset H(K)$ as a topological subspace.

Since, by [8, Proposition 1.2], $G_X(\omega)$ as a closed subspace of $A(\omega)$ is an (PLB)-space, we have that $G_X(\omega) = \lim \operatorname{proj}_{n \in \mathbb{N}} G_X(K_n)$ if $K_1 \subset K_2 \subset \dots$ is an exhaustion of $\omega \subseteq \Omega$.

If E is a locally convex space then we set for $\omega \subset \Omega$ open

$$G_X(\omega, E) = \{f \in A(\omega, E) : f|_X = 0\}.$$

This defines a sheaf \mathcal{G}_X^E of E -valued real analytic functions on Ω .

Therefore, by a simple tensor argument, for every Banach space B also the map

$$S \otimes \operatorname{id}_B : H(K, B)^l \longrightarrow G_X(K, B), \quad S \otimes \operatorname{id}_B(g_1, \dots, g_l) := \sum_{j=1}^l g_j f_j, \quad (5)$$

is surjective.

Let us observe also that if $K_1 \subset K_2 \subset \dots$ is an exhaustion of $\omega \subseteq \Omega$, then

$$G_X(\omega, E) = \lim \operatorname{proj}_{n \in \mathbb{N}} G_X(K_n, E_n).$$

In particular, $G_X(\omega, E)$ is a (PLB)-space. We obtain:

Lemma 2.1 *If E has property $(\overline{\Omega})$ then $\operatorname{Proj}^1 G_X(\Omega, E) = 0$.*

Proof: We apply Lemma 1.5 to the functions g_1, g_2, \dots in (5) and proceed as in the proof of Theorem 1.1. \square

Since for every compact $K \subset \Omega$ we have an exact sequence

$$0 \longrightarrow G_X(K) \xrightarrow{i} H(K) \xrightarrow{q} H(X \cap K) \longrightarrow 0$$

where i is the imbedding and q the restriction map, the same holds Banach valued. Therefore we have for every n an exact sequence

$$0 \longrightarrow G_X(K_n, E_n) \xrightarrow{i^{E_n}} H(K_n, E_n) \xrightarrow{q^{E_n}} H(X \cap K_n, E_n) \longrightarrow 0.$$

Since $A(X, E) = \lim \operatorname{proj}_n H(X \cap K_n, E_n)$ we obtain a long exact sequence

$$\begin{aligned} 0 \longrightarrow G_X(\Omega, E) \xrightarrow{i^E} A(\Omega, E) \xrightarrow{q^E} A(X, E) \longrightarrow \operatorname{Proj}^1 G_X(\Omega, E) \longrightarrow \\ \longrightarrow \operatorname{Proj}^1 A(\Omega, E) \longrightarrow \operatorname{Proj}^1 A(X, E) \longrightarrow 0. \end{aligned}$$

Now we are able to extend our main theorem to the case of real analytic varieties.

Theorem 2.2 *If $\dim X \geq 1$ then $\text{Proj}^1 A(X, E) = 0$ if and only if E has property $(\overline{\overline{\Omega}})$. In this case every E -valued real analytic function on X can be extended to Ω .*

Proof: If E has property $(\overline{\overline{\Omega}})$ then the fourth and fifth term in the long exact sequence above vanish by Theorem 1.1 and Lemma 2.1, respectively. Therefore every E -valued real analytic function on X can be extended to Ω and $\text{Proj}^1 A(X, E) = 0$.

We have to show necessity of property $(\overline{\overline{\Omega}})$. Let $d_0 = \dim X$. We choose $x_0 \in X$ so that X is a manifold of dimension at least d_0 in a neighborhood of x_0 . For $d_0 \geq 2$ let S be a real analytic diffeomorphic copy of the unit circle in this neighborhood, then S is a real analytic submanifold of \mathbb{R}^d . By [31] there is an extension operator to the whole of \mathbb{R}^d , hence to X , and therefore $\text{Proj}^1 A(S, E) = 0$ which implies that X has $(\overline{\overline{\Omega}})$. For $d_0 = 1$ we may assume that $x_0 = 0$ and that $\mathbb{R} := \{(t, 0, \dots, 0) : t \in \mathbb{R}\}$ is a tangent to X in 0 . Then there is a neighborhood U of 0 in X and $\varepsilon > 0$ so that for $\pi_1 : x \mapsto x_1$ the map $\varphi = \pi_1|_U$ is a real analytic diffeomorphism from U onto $(-\varepsilon, \varepsilon)$. We choose $r < \varepsilon$ and find a projection P_0 from $A(-\varepsilon, \varepsilon)$ onto the space \mathcal{A}_r of r -periodic real analytic functions on \mathbb{R} . Then $P : f \mapsto (P_0(f \circ \varphi^{-1})) \circ \pi_1|_X$ is a projection in $A(X)$ onto a subspace isomorphic to $\mathcal{A}_r \cong H(S^1)$. Now we proceed as previously. \square

For $\dim X = 0$, which means that X is a discrete subset of Ω , we have $A(X, E) = E^X$ hence $\text{Proj}^1 A(X, E) = 0$. Moreover, due to a tensor argument, q^E is surjective for any Fréchet space E , since for every $f \in A(X, E)$ we find even $F \in H(\Omega, E)$ with $F|_X = f$. So this is a special case, interpolation is always possible.

Even for $\dim X \geq 1$ we cannot expect to conclude much about E from the surjectivity of q^E . There are varieties X so that there is a continuous extension operator $A(X) \rightarrow A(\Omega)$. A characterization in the compact case is given in [31]. Any compact real analytic submanifold has this property. A concrete extension operator for the case of the unit circle in \mathbb{R}^2 is given in [4]. If X has this property then clearly q^E is surjective for any Fréchet space E . However $\text{Proj}^1 \mathcal{G}_X(X, E)$ need not to be zero.

3 Examples

Before we go on in the theory we want to study some consequences of Theorems 1.1 and 2.2 for solving equations with real analytic parameters. We will use the following Lemma.

Lemma 3.1 *If $0 \rightarrow E \rightarrow F \xrightarrow{\varphi} G \rightarrow 0$ is an exact sequence of Fréchet spaces then we obtain an exact sequence*

$$\begin{aligned} 0 \rightarrow A(\Omega, E) \rightarrow A(\Omega, F) \xrightarrow{\varphi^*} A(\Omega, G) \rightarrow \text{Proj}^1 A(\Omega, E) \rightarrow \\ \rightarrow \text{Proj}^1 A(\Omega, F) \rightarrow \text{Proj}^1 A(\Omega, G) \rightarrow 0. \end{aligned}$$

Proof: We may assume that for every n we have an exact sequence $0 \rightarrow E_n \rightarrow F_n \rightarrow G_n \rightarrow 0$ of local Banach spaces which yields an exact sequence

$$0 \rightarrow A(K_n, E_n) \rightarrow A(K_n, F_n) \rightarrow A(K_n, G_n) \rightarrow 0.$$

This proves the claim. \square

We derive immediately the following corollary. We use the notation of Lemma 3.1.

Corollary 3.2 *Let F have property $(\overline{\overline{\Omega}})$. Then φ_* is surjective if and only if E has property $(\overline{\overline{\Omega}})$.*

To obtain examples where this can be applied, we observe that any *quojection* has property $(\overline{\overline{\Omega}})$. A Fréchet space is called a quojection if it is the projective limit of Banach spaces with surjective linking maps. Typical examples are the spaces $C^m(\Omega)$, $L_{loc}^p(\Omega)$, $H_{loc}^s(\Omega)$, $\mathcal{B}_{p,k}^{loc}(\Omega)$ in the sense of Hörmander [15, Definition 2.2.1]. Here $m \in \mathbb{N}_0$, $1 \leq p \leq \infty$, $s \in \mathbb{R}$, k a temperate weight function as defined in [15, Definition 2.1.1] and Ω an open subset of \mathbb{R}^d . That for a quojection $\text{Proj}^1 A(\Omega, E) = 0$ has already been observed in [2].

If Z denotes one of these spaces then $A(T, Z(\Omega))$ where Ω is an open subset of \mathbb{R}^d and T an open subset or a coherent subvariety of \mathbb{R}^d with $\dim T \geq 1$ (e.g. a submanifold of \mathbb{R}^d), is the set of all functions $f(x, t)$ so that for every $t_0 \in T$ and every seminorm $\| \cdot \|$ on $Z(\Omega)$ there is a neighborhood U_1 of t_0 and on this neighborhood an expansion $f(x, t) = \sum_{\alpha} c_{\alpha}(x)(t - t_0)^{\alpha}$ so that $\sum_{\alpha} \|c_{\alpha}\| |t - t_0|^{\alpha} < \infty$. We may also consider the elements of $A(T, Z(\Omega)) = Z(\Omega) \hat{\otimes}_{\pi} A(T) =: Z(\Omega, T)$ as functions in $Z(\Omega)$ depending on a real analytic parameter t running through T .

Those spaces appear, for instance, in the following situation: let $P(D)$ be a linear partial differential operator with constant coefficients. We use the notation of [15, Chapter II]. Then, by [15, Theorem 2.3.4] and [15, Theorem 3.5.5], we get for $P(D)$ -convex Ω an exact sequence

$$0 \longrightarrow \mathcal{N}_{p, \tilde{P}k}(\Omega) \longrightarrow \mathcal{B}_{p, \tilde{P}k}^{loc}(\Omega) \xrightarrow{P(D)} \mathcal{B}_{p, k}^{loc}(\Omega) \longrightarrow 0 \quad (6)$$

where $\mathcal{N}_{p, \tilde{P}k}(\Omega)$ is the space of solutions of the homogenous equation.

We obtain the following

Proposition 3.3 *For every $g = g(x, t) \in \mathcal{B}_{p, k}^{loc}(\Omega, T)$ there is $f \in \mathcal{B}_{p, \tilde{P}k}^{loc}(\Omega, T)$ so that $P(D_x)f(x, t) = g(x, t)$ for all $t \in T$ if and only if $\mathcal{N}_{p, \tilde{P}k}(\Omega)$ has property $(\overline{\overline{\Omega}})$. If $P(D)$ is elliptic and Ω convex this is never fulfilled, except for $d = 1$.*

Proof: The first part is an immediate consequence of Corollary 3.2, the second follows from the fact that $\mathcal{N}_{p, \tilde{P}k}(\Omega) = \{f \in A(\Omega) : P(D)f = 0\}$ and the latter space is a Fréchet subspace of $A(\Omega)$. If it has property $(\overline{\overline{\Omega}})$ then it is finite dimensional (see the proof of [9, Theorem 3.7]), which means that $d = 1$. \square

Other operators for which we know that $\mathcal{N}_{p, \tilde{P}k}(\Omega)$ does not have property $(\overline{\overline{\Omega}})$ are the hypoelliptic operators which appear in Langenbruch [20], among which is the heat equation.

In precisely the same way as in Proposition 3.3 we can show that an elliptic operator $P(D)$ on $A(\mathbb{R}^d)$ acting only on $1 < n < d$ variables is never surjective. We use there

that $\text{Proj}^1 A(\mathbb{R}^d) = 0$. This is a well known example of De Giorgi and Cattabriga [7] for nonsurjectivity.

4 Consequences of the main theorem

There are other natural, related but not equivalent ways of considering the space $A(\Omega, E)$ as a projective limit: $A(\Omega, E) = \lim \text{proj}_n A(\Omega, E_n)$, where E_n runs through a fundamental system of local Banach spaces or $A(\Omega, E) = \lim \text{proj}_n A(K_n, E)$ where the K_n are a compact exhaustion of Ω .

Proposition 4.1 *For every Fréchet space E we have $\text{Proj}^1 A(\Omega, E) = \text{Proj}_n^1 A(\Omega, E_n)$.*

Proof: We consider the canonical resolution (see [24, p. 317]) of E

$$0 \longrightarrow E \xrightarrow{i} \prod_n E_n \xrightarrow{\sigma} \prod_n E_n \longrightarrow 0.$$

where $\sigma(x_n)_n = (i_{n+1}^n x_{n+1} - x_n)_n$ and i_{n+1}^n is the canonical linking map. We apply Lemma 3.1 and observe that due to

$$\text{Proj}^1 A(\Omega, \prod_n E_n) = \prod_n \text{Proj}^1 A(\Omega, E_n) = 0$$

we have

$$\text{Proj}^1 A(\Omega, E) = \prod_n E_n / \text{im}(\sigma) = \text{Proj}_n^1 A(\Omega, E_n),$$

the latter by definition. □

In the following we denote by \mathcal{G} either \mathcal{G}_X for some variety or \mathcal{A} , i.e. X may be empty. We set for the Fréchet space E and every compact $K \in \mathbb{C}^d$

$$G(K, E) = \lim \text{proj}_k G(K, E_k).$$

Then we have

$$G(\Omega, E) = \lim \text{proj}_n G(K_n, E).$$

We investigate this case in an analogous manner. We consider the canonical resolution

$$0 \longrightarrow G(\Omega) \xrightarrow{j} \prod_n G(K_n) \xrightarrow{\sigma} \prod_n G(K_n) \longrightarrow 0$$

belonging to the projective limit we want to investigate. Here $j(f) = (j^n f)_n$ and $\sigma(f)_n = (j_{n+1}^n f_{n+1} - f_n)_n$, where j^n and j_{n+1}^n are the natural restriction maps. This yields for every p an exact sequence

$$0 \longrightarrow G(K_p, E_p) \xrightarrow{J^p} \prod_{n=1}^p G(K_n, E_p) \xrightarrow{\Sigma^p} Z_p \longrightarrow 0$$

where $J^p f = (j_p^n f)_{n=1}^p$ and $\Sigma^p(f)_{n=1}^p = (j_{n+1}^n f_{n+1} - f_n)_{n=1}^{p-1}$ and we have set $Z_p = \text{im } \Sigma^p \subset \prod_{n=1}^{p-1} G(K_n, E_p)$.

It is easily seen that the projective spectra $(Z_p)_p$ and $(\prod_{n=1}^p G(K_n, E_p))_p$ are equivalent. Therefore we obtain an exact sequence

$$\begin{aligned} 0 \longrightarrow G(\Omega, E) &\xrightarrow{J} \prod_n G(K_n, E) \xrightarrow{\Sigma} \prod_n G(K_n, E) \longrightarrow \text{Proj}^1 G(\Omega, E) \xrightarrow{J_1} \\ &\xrightarrow{J_1} \prod_n \text{Proj}^1 G(K_n, E) \xrightarrow{\Sigma_1} \prod_n \text{Proj}^1 G(K_n, E) \longrightarrow 0. \end{aligned}$$

This case is essentially different from the previous one as the last two terms in the long exact sequence will not vanish in general. In fact they will vanish if and only if $\text{Proj}^1 G(K_n, E) = 0$ for all n , i.e. if E has property $(\overline{\overline{\Omega}})$, and in this case also $\text{Proj}^1 G(\Omega, E) = 0$. So the proof of the necessity of condition $(\overline{\overline{\Omega}})$ fails in this case. Even more: necessity cannot be expected. For E with property (DN) we can find for any sequence $f_n \in G(K_n, E)$, $n \in \mathbb{N}$ a bounded Banach disc $B \subset E$ so that $f_n \in G(K_n, E_B)$ for all $n \in \mathbb{N}$ (cf. [3]). This implies $\text{Proj}_n^1 G(K_n, E) = 0$ since the proof is reduced to the Banach space case. However a Fréchet space belongs to the classes $(\overline{\overline{\Omega}})$ and (DN) if and only if it is finite dimensional. So the following result has only trivial intersection with the result [3, Theorem 3].

Proposition 4.2 *If E has property $(\overline{\overline{\Omega}})$ then $\text{Proj}_n^1 G(K_n, E) = 0$.*

Proof: In this case $\text{Proj}^1 G(\Omega, E) = 0$ and we conclude like in the proof of Proposition 4.1. \square

It should be finally remarked that the results on $\text{Proj}_n^1 G(K_n, E) = 0$ remain unsatisfactory and are probably far from a characterization. This is part of the difficult problem of studying projective spectra of (PDF)-spaces and of evaluating characterizations for the vanishing of Proj^1 in this case (see [11], [19]).

5 Application to the functor Ext

We are now ready to prove our second main result. We resume the notation $\mathcal{G} = \mathcal{G}_X$ where X is a coherent real analytic variety or $\mathcal{G} = \mathcal{A}$, i.e. $X = \emptyset$. We need some more notation.

Let in this section E be a complete (LB)-space. That means that $E = \bigcup_n E_n$, $E_n \subset E_{n+1}$ for all n , where E_n is a Banach space with the norm $\|\cdot\|_n^*$ and the imbeddings are continuous. We may assume $\|\cdot\|_n^* \geq \|\cdot\|_{n+1}^*$. E carries the finest locally convex topology making all imbeddings $E_n \hookrightarrow E$ continuous. We call it an (LS)-space if the imbeddings $E_n \hookrightarrow E_{n+1}$ are compact.

We will use the following preliminary lemma.

Lemma 5.1 *Let E be a barrelled (DF)-space so that E_b^l has property $(\overline{\overline{\Omega}})$. Let K be a compact subset of \mathbb{C}^d and let for some $l \in \mathbb{N}$*

$$H(K)^l \longrightarrow Y \xrightarrow{q} E \longrightarrow 0$$

be an exact sequence of locally convex spaces, so that for every bounded set $B \subset E$ there is a bounded set $\tilde{B} \subset Y$ so that $q(\tilde{B}) \supset B$, then q has a continuous linear right inverse.

Proof: We set $Z = \ker q$, $i : Z \hookrightarrow Y$ the identical injection. By assumption the dual sequence

$$0 \longrightarrow E'_b \xrightarrow{q'} Y'_b \xrightarrow{i'} Z'_b \longrightarrow 0 \quad (7)$$

is algebraically exact, Z'_b and E'_b are Fréchet spaces, q' is an isomorphic imbedding, i' is continuous and surjective. Moreover Z , as a quotient of $H(K)^l$, is barrelled. Therefore for every bounded, hence equicontinuous set $M \subset Z'_b$ there is an equicontinuous, hence bounded set $\tilde{M} \subset Y'_b$ so that $i'(\tilde{M}) \supset M$. Since Z'_b is a Fréchet space, this implies that i' is open. Therefore the sequence (7) is (topologically) exact, which implies that also Y'_b is a Fréchet space (see [26, Corollary 2.4.4]).

$H(K)_b^l$ is nuclear and has property (DN) (for the definition see e.g. [24, p. 368]). To see this we apply [28, Satz 5.1] to three open neighborhoods $K \subset \Omega_1 \subset \subset \Omega_2 \subset \subset \Omega_3 \subset \subset \mathbb{C}^d$ of K . Therefore $Z'_b \subset (H(K)_b^l)'$ has the same properties. By an easy modification of [30, Theorem 5.1] the sequence (7) splits. Hence there is a right inverse $\lambda \in L(Z'_b, Y'_b)$ for i' .

The space Z , as a quotient of $H(K)^l$, is reflexive. Therefore the map $\lambda' \in L((Y'_b)'_b, Z)$ is a left inverse for i'' . Since E and Z are barrelled, also Y is barrelled (see [27, Theorem 2.6, (a)] or [26, Proposition 4.2.3]). Therefore $Y \subset (Y'_b)'_b$ as a topological subspace and $L := \lambda'|_Y$ is a left inverse for i which implies that q has a right inverse. \square

We are now ready to prove our second main result. We say $q : Y \rightarrow E$ *lifts bounded sets with closure* if for every bounded set $B \subset E$ there is a bounded set $\tilde{B} \subset Y$ so that $q(\tilde{B}) \supset B$. For a thorough discussion of this see [1].

Proposition 5.2 *Let E be a barrelled (DF) -space and \mathcal{G} a coherent sheaf of real analytic functions on Ω . If E'_b has property $(\overline{\Omega})$ then every exact sequence of locally convex spaces*

$$0 \longrightarrow G(\Omega) \xrightarrow{i} Y \xrightarrow{q} E \longrightarrow 0, \quad (8)$$

where q lifts bounded sets with closure, splits.

Proof: We choose some exhaustion $(K_n)_{n \in \mathbb{N}}$ and apply the standard push-out construction (see e.g. [32, Definition 5.1.2]) to the exact sequence (8) and the canonical map $j^n : G(\Omega) \rightarrow G(K_n)$. Due to the functorial properties of the push-out we get for every n a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G(K_n) & \xrightarrow{i_n} & Y_n & \xrightarrow{q_n} & E & \longrightarrow & 0 \\ & & \uparrow j_{n+1}^n & & \uparrow h_{n+1}^n & & \uparrow \text{id} & & \\ 0 & \longrightarrow & G(K_{n+1}) & \xrightarrow{i_{n+1}} & Y_{n+1} & \xrightarrow{q_{n+1}} & E & \longrightarrow & 0 \\ & & \uparrow j^{n+1} & & \uparrow h^{n+1} & & \uparrow \text{id} & & \\ 0 & \longrightarrow & G(\Omega) & \xrightarrow{i} & Y & \xrightarrow{q} & E & \longrightarrow & 0. \end{array}$$

Since every bounded set in E is in the closure of the image under q of a bounded set in Y , the same is true for every q_n . For every n the space $G(K_n)$ is a quotient of $H(K_n)^l$ for some $l \in \mathbb{N}$. Therefore, by Lemma 5.1 for every n the map q_n has a right inverse $R_n \in L(E_n, Y_n)$.

For every n the map $R_n - h_{n+1}^n \circ R_{n+1}$ gives rise to a map $A_n \in L(E, G(K_n))$ which, due to [14], can be represented by a kernel $f_n \in G(K_n, E'_b)$, this means $A_n(x) = \langle f_n(\cdot), x \rangle$. By Corollary 4.2 we find $g_n \in G(K_n, E'_b)$ so that $f_n = g_n - g_{n+1}$ in $G(K_n, E'_b)$ for all $n \in \mathbb{N}$. We define B_n by $x \mapsto \langle g_n(z), x \rangle$, then $B_n \in L(E, G(K_n))$ and $A_n x = B_n x - B_{n+1} x$ for all $x \in E$ in $G(K_n)$.

Finally we obtain $R_n - i_n \circ B_n = j_{n+1}^n \circ (R_{n+1} - i_{n+1} \circ B_{n+1})$ so these maps define a map $R \in L(E, \tilde{Y})$ where $\tilde{Y} = \lim \text{proj}_n Y_n$.

It is immediately seen that we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & G(\Omega) & \xrightarrow{\tilde{i}} & \tilde{Y} & \xrightarrow{\tilde{q}} & E \\ & & \uparrow \text{id} & & \uparrow h & & \uparrow \text{id} \\ 0 & \longrightarrow & G(\Omega) & \xrightarrow{i} & Y & \xrightarrow{q} & E \longrightarrow 0. \end{array}$$

We conclude that \tilde{q} is surjective, that h is bijective and that R is a continuous linear right inverse for \tilde{q} . Let \tilde{L} be the corresponding left inverse for \tilde{i} . We set $L = \tilde{L} \circ h \in L(Y, G(\Omega))$ and obtain $L \circ i = \tilde{L} \circ h \circ i = \tilde{L} \circ \tilde{i} = \text{id}$. Therefore i has a left inverse. \square

Remark 5.3 *If E is an (LB)-space, $E = \lim \text{ind}_k E_k$, where E_k is a Banach space, B_k its unit ball, and if for every k there is a bounded set $\tilde{B}_k \subset Y$ so that $q(\tilde{B}_k) \supset B_k$, then q lifts bounded sets with closure. This is true, because the sets \tilde{B}_k are a fundamental system of bounded sets in E .*

We use this remark in the following lemma.

Lemma 5.4 *If in the exact sequence (8) the space Y is a (PLB)-space and E an (LB)-space, then q lifts bounded sets with closure.*

Proof: Let $Y = \lim \text{proj}_n Y_n$ where the Y_n are (LB)-spaces, Then there is m so that q factorizes through Y_n for all $n \geq m$, i.e. we have $q = q_n \circ h_n$ where $q_n \in L(Y_n, E)$ and $h_n : Y \rightarrow Y_n$ is the canonical map. We may assume that $m = 1$.

We set up the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & Z_n & \xrightarrow{i_n} & Y_n & \xrightarrow{q_n} & Y_n/Z_n \longrightarrow 0 \\ & & \uparrow h^n & & \uparrow h^n & & \uparrow \hat{h}_n \\ 0 & \longrightarrow & G(\Omega) & \xrightarrow{i} & Y & \xrightarrow{q} & E \longrightarrow 0 \end{array}$$

where Z_n is the closure of $h_n(A(\Omega))$ in Y_n .

We get for every n exact sequences

$$\begin{array}{ccccccc}
0 & \longrightarrow & L(\ell_1(I), Z_n) & \xrightarrow{i_n^*} & L(\ell_1(I), Y_n) & \xrightarrow{q_n^*} & L(\ell_1(I), Y_n/Z_n) & \longrightarrow & 0 \\
& & \uparrow h_{n+1}^{n*} & & \uparrow h_{n+1}^{n*} & & \uparrow \hat{h}_{n+1}^{n*} & & \\
0 & \longrightarrow & L(\ell_1(I), Z_{n+1}) & \xrightarrow{i_{n+1}^*} & L(\ell_1(I), Y_{n+1}) & \xrightarrow{q_{n+1}^*} & L(\ell_1(I), Y_{n+1}/Z_{n+1}) & \longrightarrow & 0.
\end{array}$$

Here we use the notation $\psi^*(\varphi) := \psi \circ \varphi$. The surjectivity of q_n^* is due to a simple application of Grothendieck's factorization theorem [24, 24.33]. Since

$$\begin{aligned}
\text{Proj}_n^1 L(\ell_1(I), Z_n) &= \text{Proj}_n^1 L(\ell_1(I), G(K_n)) \\
&= \text{Proj}_n^1 G(K_n, \ell_\infty(I)) = \text{Proj}^1 G(\Omega, \ell_\infty(I)) = 0
\end{aligned}$$

the map $q^* : L(\ell_1(I), Y) \rightarrow L(\ell_1(I), E)$ is surjective.

Let now $E = \lim \text{ind}_k E_k$ where E_k is a Banach space and B_k its unit ball. We choose an index set I and a map $\varphi : \ell_1(I) \rightarrow E_k$ which maps the unit ball of $\ell_1(I)$ onto B_k . By the previous we find $\psi \in L(\ell_1(I), Y)$ so that $q \circ \psi = \varphi$. We set $\tilde{B}_k = \psi(U)$ where U is the unit ball of $\ell_1(I)$ and get $q(\tilde{B}_k) = B_k$. \square

If, therefore, E is an (LB)-space and Y a (PLB)-space then the sequence (8) splits if E'_b has property $(\overline{\overline{\Omega}})$, without further assumptions on q or E . We express this in the following proposition.

Proposition 5.5 *Whenever E is an (LB)-space and its dual E'_b has property $(\overline{\overline{\Omega}})$, then $\text{Ext}_{PLB}^1(E, G(\Omega)) = 0$.*

Theorem 5.6 *Let E be an (LB)-space. Then $\text{Ext}_{PLB}^1(E, A(\Omega)) = 0$ if and only if E'_b has property $(\overline{\overline{\Omega}})$.*

Proof: It remains to show necessity of $(\overline{\overline{\Omega}})$. If $d = 1$ then either $\mathcal{G} = \mathcal{A}$ or X is a discrete set in Ω , hence there is $g \in H(\Omega_0)$ so that $G(\Omega) = g \cdot A(\Omega)$. Therefore in both cases $G(\Omega) \cong A(\Omega)$ which has a complemented subspace isomorphic to $H(S^1)$. So we have $0 = \text{Ext}^1(E, H(S^1)) \cong \text{Proj}^1 L(E_n, H(S^1)) \cong \text{Proj}^1 E'_n \hat{\otimes}_\pi H(S^1) \cong \text{Proj}^1 L(H(S^1)'_b, E'_n)$. This implies, in particular, that $\text{Proj}^1 E'_n = 0$. Therefore

$$\text{Ext}^1(H(S^1)'_b, E'_b) \cong \text{Proj}^1 L(H(S^1)'_b, E'_n) = 0.$$

By [30, Theorem 4.2] then E'_b has property $(\overline{\overline{\Omega}})$.

If $d \geq 2$ then we chose $x = (x_1, \dots, x_d) \in \Omega \setminus X$, hence there is a small circle $S = \{(\xi, x_2, \dots, x_d) : |\xi - x_1| = \varepsilon\} \subset \Omega$ with $S \cap X = \emptyset$. Clearly the restriction map $G(\Omega) \rightarrow A(S)$ is surjective. So we may choose a function $g \in G(X)$ so that $g|_S = 1$. We choose a continuous linear extension operator $R_0 : A(S) \rightarrow A(\mathbb{R}^d)$ (see [4, Remark 4.6]) and put $R(f) = g \cdot R_0(f)$ for $f \in A(S)$. This defines a continuous linear extension operator $R : A(S) \rightarrow G(\Omega)$. Hence $A(S) \cong H(S^1)$ is complemented in $G(\Omega)$ which implies $\text{Ext}_{PLB}^1(E, H(S^1)) = 0$. Therefore, as before, E'_b has property $(\overline{\overline{\Omega}})$. \square

Let now X be a compact coherent subvariety of Ω . We have an exact sequence

$$0 \longrightarrow G_X(\Omega) \longrightarrow A(\Omega) \xrightarrow{q} A(X) \longrightarrow 0$$

where q is the restriction map. Guided by the case of a complex subvariety one could hope to solve the problem when there is a right inverse for q , i.e. a continuous linear extension map $A(X) \rightarrow A(\Omega)$, by looking for cases where $\text{Ext}^1(A(X), G_X(\Omega)) = 0$. This attempt however fails.

Proposition 5.7 *If Ω is connected then $\text{Ext}^1(A(X), G_X(\Omega)) = 0$ if and only if X is a finite set.*

Proof: $A(X)'_b$ is nuclear and has property $(\underline{\text{DN}})$ (see the proof of Lemma 5.1). If it has also property $(\overline{\Omega})$ then, by [29], it is a nuclear Banach space, hence finite dimensional. This means that the set X is finite. \square

A complete solution of the above mentioned problem is given in [31].

Let us close with some final remarks. While for nuclear Fréchet spaces E and F there are well known necessary and sufficient conditions for $\text{Ext}^1(E, F) = 0$ (see [12], [30], [32]) which can be transferred by dualization also to the case of (LB)-spaces, the case of (PLB)-spaces is much more difficult (see [19]). Only special cases are really known: $\text{Ext}^1(\mathcal{D}', \mathcal{D}') = 0$ [8], conditions for certain (PLB) power series spaces [19] and, related to the present case, in [6] it was shown for a Fréchet space E that $\text{Ext}^1(A(\Omega), E) = 0$ if and only if E has property $(\overline{\Omega})$. The present result carries on this work and again shows the important role of property $(\overline{\Omega})$ for the study of spaces of real analytic functions.

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