ON UNIVERSALLY STABLE ELEMENTS

Ian J. Leary, Björn Schuster, and Nobuaki Yagita

Abstract. We show that certain subrings of the cohomology of a finite p-group P may be realised as the images of restriction from suitable virtually free groups. We deduce that the cohomology of P is a finite module for any such subring. Examples include the ring of 'universally stable elements' defined by Evens and Priddy, and rings of invariants such as the mod-2 Dickson algebras.

Let P be a finite p-group, and let $\mathcal{C}_u$ be the category whose objects are the subgroups of P, with morphisms all injective group homomorphisms. Let $\mathcal{C}$ be any subcategory of $\mathcal{C}_u$ such that P is an object of $\mathcal{C}$, and such that for any object Q of $\mathcal{C}$, the inclusion of Q in P is a morphism in $\mathcal{C}$. Let $H^*(\,\cdot\,)$ stand for mod-p group cohomology, which may be viewed as a contravariant functor from $\mathcal{C}_u$ to $\mathbf{F}_p$-algebras. We shall study the limit $I(P,\mathcal{C})$ of this functor:

$$I(P,\mathcal{C}) = \lim_{Q\in\mathcal{C}} H^*(Q).$$

Given our assumptions on $\mathcal{C}$, we may identify $I(P,\mathcal{C})$ with a subring of $H^*(P)$. In the final remarks we discuss generalizations of our results in which most of the conditions that we impose upon P, $\mathcal{C}$, and $H^*(\,\cdot\,)$ are weakened.

The classical case of this construction occurs in Cartan-Eilenberg's description of the image of the cohomology of a finite group in the cohomology of its Sylow subgroup as the 'stable elements' [2]. Let G be a finite group with P as its Sylow subgroup, and let $\mathcal{C}_G$ be the subcategory of $\mathcal{C}_u$ containing all the objects, but with morphisms only those homomorphisms Q to Q' induced by conjugation by some element of G. Then the image, $\mathrm{Im}(\mathrm{Res}^G_P)$, of $H^*(G)$ in $H^*(P)$ is equal to $I(P,\mathcal{C}_G)$.

Rings of invariants also arise in this way. If $\mathcal{C}$ is a category whose only object is P, with morphisms a subgroup H of $\mathrm{Aut}(P)$, then $I(P,\mathcal{C})$ is just the subring $H^*(P)^H$ of invariants under the action of H.

Another case considered already, which motivated our work, is the ring of universally stable elements defined by Evens-Priddy in [4]. Let $\mathcal{C}_s$ be the subcategory of $\mathcal{C}_u$ generated by all subcategories of the form $\mathcal{C}_G$ as defined above. Then $I(P,\mathcal{C}_s)$ is the subring $I(P)$ of $H^*(P)$ introduced in [4], consisting of those elements of $H^*(P)$ which are in the image of Res^G_P for every finite group G with Sylow subgroup P.

A fourth case of interest is $I(P, \mathcal{C}_u)$, which might be viewed as the elements of $H^*(P)$ which are 'even more stable' than the elements of $I(P, \mathcal{C}_s)$. It is easy to see that in general $\mathcal{C}_s$ is strictly contained in $\mathcal{C}_u$. For example, the endomorphism monoid $\mathrm{Hom}_{\mathcal{C}_s}(P, P)$ of P is the subgroup of $\mathrm{Aut}(P)$ generated by elements of order coprime to p, whereas $\mathrm{Hom}_{\mathcal{C}_u}(P, P)$ is the whole of $\mathrm{Aut}(P)$. Our main result is the following theorem.

Theorem 1. *Let P be a finite p-group, and let $\mathcal{C}$ be any subcategory of $\mathcal{C}_u = \mathcal{C}_u(P)$ satisfying the conditions stated in the first paragraph. Then there exists a discrete group Γ containing P as a subgroup such that:*

a) $\mathrm{Im}(\mathrm{Res}^\Gamma_P)$ *is equal to* $I(P, \mathcal{C})$*;*
b) $(\mathrm{Ker}(\mathrm{Res}^\Gamma_P))^2$ *is trivial;*
c) Res^Γ_P *induces an isomorphism from* $H^*(\Gamma)/\sqrt{0}$ *to* $I(P, \mathcal{C})/\sqrt{0}$*;*
d) Γ *is virtually free. More precisely,* Γ *has a free normal subgroup of index dividing* $|P|!$.

If Γ' is a free normal subgroup of Γ of finite index, then P maps injectively to the finite group Γ/Γ', so by the Evens-Venkov theorem [5], $H^*(P)$ is a finite module for $H^*(\Gamma/\Gamma')$ and hence *a fortiori* for $H^*(\Gamma)$. Thus one obtains the following corollary.

Corollary 2. *Let P and $\mathcal{C}$ be as in the statement of Theorem 1. Then $H^*(P)$ is a finite module for its subring $I(P, \mathcal{C})$.*

The case $\mathcal{C} = \mathcal{C}_s$ is Theorem A of [4]. Our result is stronger, since it applies to categories such as $\mathcal{C}_u$ itself, and our proof is more elementary. There is an even shorter proof of Corollary 2 however, which is to deduce it from the following simpler theorem.

Theorem 3. *Let P be a finite p-group, and let G be the symmetric group on a set X bijective with P. Regard P as a subgroup of G via a Cayley embedding (or regular permutation representation). Then* $\mathrm{Im}(\mathrm{Res}^G_P)$ *is contained in* $I(P, \mathcal{C}_u)$.

To deduce Corollary 2 from Theorem 3, note that for any $\mathcal{C}$ as above, one has

$$\mathrm{Im}(\mathrm{Res}^G_P) \subseteq I(P, \mathcal{C}_u) \subseteq I(P, \mathcal{C}) \subseteq H^*(P),$$

and $H^*(P)$ is a finite module for $\mathrm{Im}(\mathrm{Res}^G_P)$ by the Evens-Venkov theorem.
Proof of Theorem 3. We deduce Theorem 3 from the following group-theoretic lemma.

Lemma 4. *Let $Q \leq P \leq G$ be as in the statement of Theorem 3, and let ϕ be any injective homomorphism from Q to P. Then there exists $g \in G$ such that for all $q \in Q$, $\phi(q) = gqg^{-1}$.*

Proof. Fix a bijection between P and the set X permuted by G. This fixes an embedding i_P of P in G. Let i_Q be the induced inclusion of Q in G. Write ${}^{i_P}X$ for X viewed as a

P-set. Thus ${}^{i_P}X$ is a free P-set of rank one. There are two ways to view X as a Q-set, either via i_Q or $i_P \circ \phi$. The Q-sets ${}^{i_Q}X$ and ${}^{i_P\circ\phi}X$ are both free of rank equal to the index, $|P : Q|$, of Q in P. Let g be an isomorphism of Q-sets between ${}^{i_Q}X$ and ${}^{i_P\circ\phi}X$. Then g is an element of G having the required property, because for each $x \in X$ and $q \in Q$, $g \cdot q \cdot x = \phi(q) \cdot g \cdot x$. □

Returning now to the proof of Theorem 3, any morphism in $\mathcal{C}_u$ factors as the composite of an isomorphism followed by an inclusion. Thus it suffices to show that for ϕ as in Lemma 4, Res^G_Q and $\phi^* \circ \mathrm{Res}^G_P$ are equal. Writing c_g for the automorphism of G given by conjugation by g, we have shown that there exists g such that $c_g \circ i_Q = i_P \circ \phi$. But c_g^* is the identity map on $H^*(G)$, and hence $i_Q^* = \phi^* \circ i_P^*$ as required. □

This completes the proofs of all of our statements except for Theorem 1. For the proof of Theorem 1 we recall the following theorem (see for example [3], I.7.4 or IV.1.6).

Theorem 5. *Let Γ be a group that acts simplicially (i.e., without reversing any edges) on a tree with all stabilizer groups of order dividing a fixed integer M. Then there is a homomorphism from Γ to the symmetric group on M letters whose kernel, K, is torsion-free. Since K acts freely, simplicially on the tree, it follows that K is a free group.*

In fact, the short direct proof of Theorem 3 is based on some of the ideas in the proof of Theorem 5 given in [3].

Proof of Theorem 1. The group Γ will be constructed as the fundamental group of a graph of groups (see [3], I.3, [6], I.5, or [1], VII.9 for the definitions and basic theorems). Let $Q_1, \ldots, Q_M$ be the objects of $\mathcal{C}$, and let $\phi_1, \ldots, \phi_N$ be the morphisms of $\mathcal{C}$. Define a function m so that the domain of ϕ_i is $Q_{m(i)}$. Now let Γ be the group generated by the elements of P and new elements $t_1, \ldots, t_N$ subject to all relations that hold in P, together with the relations

$$t_i q t_i^{-1} = \phi_i(q),$$

for all $i \in \{1, \ldots, N\}$ and all $q \in Q_{m(i)}$. Thus Γ is the fundamental group of a graph of groups with one vertex and N edges. The vertex group is of course P and the ith edge group is $Q_{m(i)}$. The two maps from the ith edge group to the vertex group (corresponding to its initial and terminal ends) are the inclusion and ϕ_i.

The group Γ as defined above has the following properties (see any of the references listed above): P is a subgroup of Γ; for each i, the homomorphism $\phi_i\colon Q_{m(i)} \to P$ is inner in Γ (i.e., is induced by conjugation by the element t_i); Γ acts simplicially on a tree T, with one orbit of vertices and N orbits of edges, with P being a vertex stabilizer and $Q_{m(i)}$ being the stabilizer of some edge in the ith orbit. The quotient T/Γ is the graph used in defining Γ.

Recall from [1], VII.7–VII.9 that for any Γ-CW-complex X, there is a spectral sequence, with $E_1^{p,q} = \bigoplus_\sigma H^q(\Gamma_\sigma)$, where the sum is over a set of orbit representatives of p-cells in X. For coefficients in a ring with trivial Γ-action (such as the field of p elements), this is a spectral sequence of rings. When X is acyclic the spectral sequence converges to a filtration of $H^{p+q}(\Gamma)$. We apply this spectral sequence in the case when $X = T$. In this case

$$E_1^{0,q} \cong H^q(P), \quad E_1^{1,q} \cong \bigoplus_{i=1}^{N} H^q(Q_{m(i)}),$$

and $E_1^{p,q} = 0$ for $p > 1$. Under this isomorphism the differential $d_1 : E_1^{0,q} \to E_1^{1,q}$ has ith coordinate $\mathrm{Res}^P_{Q_{m(i)}} - \phi_i^*$, and so $E_2^{0,*}$ is isomorphic to $I(P, \mathcal{C})$. The fact that $E_2^{p,q} = 0$ for $p > 1$ implies that the spectral sequence collapses at the E_2-page. The edge homomorphism from $E_\infty^{0,*}$ to $H^*(P)$ may be identified with Res^Γ_P (consider the map of spectral sequences induced by the inclusion of the vertex set of the tree in the whole tree, viewed as a map of Γ-spaces), and so a) is proved. For b), note that since $E_2^{p,q} = 0$ for $p > 1$, elements of $E_2^{1,*}$ uniquely determine elements of $H^*(\Gamma)$, and the product of any two such elements is zero in $H^*(\Gamma)$. Since $\mathrm{Ker}(\mathrm{Res}^\Gamma_P)$ may be identified with $E_2^{1,*}$, b) follows, and c) follows immediately from b). Finally, d) follows from Theorem 5 stated above. □

Remarks. 1) There are alternatives to using the equivariant cohomology spectral sequence in the proof of Theorem 1, but following a suggestion of the referee we decided to explain just one method in detail in the proof. Since the spectral sequence has only two non-zero rows it is essentially just a long exact sequence. This long exact sequence may be obtained by applying $H^*(\Gamma; \cdot\,)$ to the augmented chain complex for the tree T, modulo an application of the Eckmann-Shapiro lemma. We felt, however, that the ring structure of $H^*(\Gamma)$ is more easily understood in terms of the spectral sequence.

2) We believe that $I(P, \mathcal{C}_u)$ has some advantages over $I(P, \mathcal{C}_s)$. Both of these rings enjoy the finiteness property stated in Corollary 2. To compute $I(P, \mathcal{C}_s)$ one needs to know something about the p-local structure of all groups with Sylow subgroup P, whereas $I(P, \mathcal{C}_u)$ requires only knowledge of P.

3) On the other hand, $I(P, \mathcal{C}_u)$ does not retain much information concerning P. Let $W(P)$ be the variety of all ring homomorphisms from $I(P, \mathcal{C}_u)$ to an algebraically closed field k of characteristic p. Then $W(P)$ is determined up to homeomorphism by the p-rank of P: If P has p-rank n, then $W(P)$ is homeomorphic to $k^n/GL_n(\mathbf{F}_p)$, and if E is an elementary abelian subgroup of P of rank n, then the induced map from $W(E)$ to $W(P)$ is an homeomorphism. These assertions concerning $W(P)$ follow easily from Quillen's

theorem describing the variety of homomorphisms from $H^*(P)$ to k (see for example [5], chap. 9). Note that this the only place where we use Quillen's theorem.

4) The definitions and theorems that we state remain valid if P is any finite group. We restrict to the case when P is a p-group only because this is the case occurring naturally in the work of Cartan-Eilenberg and Evens-Priddy.

5) The reader may have noticed that Theorems 1 and 3 work perfectly well for cohomology with coefficients in any ring R (viewed as a trivial P-module). Corollary 2 is valid for cohomology with coefficients in any ring R for which the Evens-Venkov theorem holds (see [5], 7.4 for a general statement).

6) The easiest way to relax the restrictions on the category $\mathcal{C}$ is to consider arbitrary finite categories with objects finite groups and morphisms injective group homomorphisms (it is unhelpful to view the groups as subgroups of a single group if the inclusion maps are not in the category). Define $I(\mathcal{C})$ to be the limit over this category and for any group Γ, define $\mathcal{D}(\Gamma)$ to be the category of finite subgroups of Γ, with morphisms inclusions and conjugation by elements of Γ. Then one obtains

Theorem $1'$. *Let $\mathcal{C}$ be a connected finite category of finite groups and injective homomorphisms. Then there exists a discrete group Γ and a natural transformation from $\mathcal{C}$ to $\mathcal{D}(\Gamma)$ such that Γ and the induced map from $H^*(\Gamma)$ to $I(\mathcal{C})$ satisfy properties a) to d) of Theorem 1.*

Recall that a category is said to be connected if the equivalence relation on objects generated by 'there is a morphism from Q to Q'' has exactly one class. Note that there cannot be a direct analogue of Theorem 1 unless the category $\mathcal{C}$ is connected, since the degree zero part of $I(\mathcal{C})$ is an $\mathbf{F}_p$ vector space of dimension the number of components of $\mathcal{C}$, whereas $H^0(\Gamma) \cong \mathbf{F}_p$. The proof of Theorem $1'$ is very similar to the proof of Theorem 1, except that one creates a graph of groups with one vertex for every object of $\mathcal{C}$. The restriction to connected categories is not serious, since given any category $\mathcal{C}$ as above, one may make a connected category $\mathcal{C}^+$ by adding a trivial group to $\mathcal{C}$ as an initial object (i.e., add one new object, a trivial group, and one morphism from this object to every other object). The natural map from $I(\mathcal{C}^+)$ to $I(\mathcal{C})$ is an isomorphism, except in degree zero.

The analogue of Corollary 2 in this generality, for which $\mathcal{C}$ need not be assumed to be connected, is:

Corollary $2'$. *Let $\mathcal{C}$ be a finite category of finite groups and injective homomorphisms. Then $\prod_{Q\in\mathcal{C}} H^*(Q)$ is a finite module for $I(\mathcal{C})$.*

7) The following instance of Theorem 1 seems worthy of special note. Let P be an elementary abelian 2-group of rank n, let $\mathcal{C}$ be the category whose only object is P and whose morphisms are the group $GL(n, \mathbf{F}_2)$. Then $H^*(B\Gamma)$ is a ring whose radical is invariant under the action of the Steenrod algebra, and $H^*(B\Gamma)/\sqrt{0}$ is isomorphic to the Dickson algebra $D_n = \mathbf{F}_2[x_1, \ldots, x_n]^{GL(n,\mathbf{F}_2)}$. On the other hand it is known that for $n \geq 6$, D_n itself cannot be the cohomology of any space [7].

Acknowledgements. This work was done when the authors met at the Centre de Recerca Matemàtica. The authors gratefully acknowledge the hospitality and support of the CRM. The first- and second-named authors were supported by E.C. Leibniz Fellowships, at the Max-Planck-Institut für Mathematik and at the CRM respectively.

References.

[1] K. S. Brown, Cohomology of Groups, Graduate Texts in Mathematics **87**, Springer-Verlag, Heidelberg (1982).

[2] H. Cartan and S. Eilenberg, Homological algebra, Princeton Univ. Press, Princeton (1955).

[3] W. Dicks and M. J. Dunwoody, Groups acting on graphs, Cambridge Studies in Advanced Mathematics **17**, Cambridge Univ. Press, Cambridge (1989).

[4] L. Evens and S. Priddy, The ring of universally stable elements, Quart. J. Math. Oxford (2) **40** (1989), 399–407.

[5] L. Evens, Cohomology of groups, Oxford Mathematical Monographs, Clarendon Press, New York (1991).

[6] J.-P. Serre, Trees, Springer-Verlag, Heidelberg (1980).

[7] L. Smith and R. M. Switzer, Realizability and non-realizability of the Dickson algebras as cohomology rings, Proc. Amer. Math. Soc. **89** (1983), 303–313.

I. J. Leary, Max-Planck-Institut für Mathematik, 53225 Bonn, Germany.
From Jan. 1996: Faculty of Math. Studies, Univ. of Southampton, Southampton SO17 1BJ, England.

B. Schuster, CRM, Institut d'Estudis Catalans, E-08193 Bellaterra (Barcelona), Spain.
From Jun. 1996: Fachbereich 7 Mathematik, Bergische Universität Wuppertal, Gaußstr. 20, 42097 Wuppertal, Germany.

N. Yagita, Faculty of Education, Ibaraki University, Mito, Ibaraki, Japan.