On the $GL(V)$–module structure of $K(n)^*(BV)$

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Abstract

We study the question of whether the Morava K-theory of the classifying space of an elementary abelian group V is a permutation module (in either of two distinct senses, defined below) for the automorphism group of V. We use Brauer characters and computer calculations. Our algorithm for finding permutation submodules of modules for p-groups may be of independent interest.

1. *Introduction*

Let p be a prime, let $K(n)^*$ denote the nth Morava K-theory, V an elementary abelian p-group, or equivalently an $\mathbf{F}_p$-vector space, and $GL(V)$ the group of automorphisms of V. Then $GL(V)$ acts naturally on the classifying space BV of V and hence on $h^*(BV)$ for any cohomology theory h. In the case when $h = K(n)$, $K(n)^*(BV)$ is a finitely generated free module over the coefficient ring $K(n)^*$ whose structure is known [**9**], and it is natural to ask what may be said about its structure as a module for the group ring $K(n)^*[GL(V)]$. The Morava K-theory of arbitrary finite groups is not known, and there is no direct construction of Morava K-theory itself. We hope that a better understanding of $K(n)^*(BV)$ may lead to progress with these questions.

For any ring R, and finite group G, we say that an R-free $R[G]$-module M is a permutation module if there is an R-basis for M which is permuted by the action of G. Call such an R-basis a permutation basis for M. If S is a G-set, write $R[S]$ for the permutation module with permutation basis S. If M is a graded module for the graded ring $R[G]$ (where elements of G are given grading zero), we call M a graded permutation module if it is a permutation module with a permutation basis consisting of homogeneous elements.

If M is a graded module for $K(n)^*[G]$, and G_p is a Sylow p-subgroup of G, the following four conditions on M are progressively weaker, in the sense that each is implied by the previous one.

(1) M is a graded permutation module;
(2) M is a permutation module;
(3) M is a direct summand of a permutation module;
(4) as a $K(n)^*[G_p]$-module, M is a graded permutation module.

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The implications (1) $\implies$ (2) $\implies$ (3) are obvious, and hold for M a graded $R[G]$-module for any R. The implication (3) $\implies$ (4) is explained below in Sections 2 and 7.

One might hope for $K(n)^*(BV)$ to satisfy condition (1) for $G = GL(V)$, i.e., for $K(n)^*(BV)$ to be a graded permutation module for the group ring $K(n)^*[GL(V)]$. This would be useful for the following reason: The ordinary cohomology of a group with coefficients in any permutation module is determined by the Eckmann-Shapiro lemma. Hence if $K(n)^*(BV)$ is a graded permutation module for $GL(V)$, and H is a group expressed as an extension with kernel V, the E_2 page of the Atiyah-Hirzebruch spectral sequence converging to $K(n)^*(BH)$ is easily computable. Even condition (4) would be very useful, as it would facilitate the computation of the E_2-page of the Atiyah-Hirzebruch spectral sequence for any p-group H expressed as an extension with kernel V.

The recent work of I. Kriz on Morava K-theory, including his dramatic discovery of a 3-group G such that $K(2)^*(BG)$ is not concentrated in even degrees, has emphasised the importance of studying the $\mathrm{Aut}(H)$-module structure of $K(n)^*(BH)$ [**6**]. For example, Kriz has shown that for any prime p and any cyclic p-subgroup C of $GL(V)$, $K(n)^*(BV)$ is a (graded) permutation module for C. He uses this result to deduce that for p odd and G a split extension with kernel V and quotient C, $K(n)^*(BG)$ is concentrated in even degrees. N. Yagita has another proof of this result [**17**].

It should be noted that if the dimension of V is at least three, there are infinitely many indecomposable graded $K(n)^*[GL(V)]$-modules, of which only finitely many occur as summands of modules satisfying condition (4), which suggests that a 'random' module will not satisfy any of the conditions. On the other hand, work of Hopkins, Kuhn and Ravenel [**4**] shows that for certain generalized cohomology theories h^*, $h^*(BV)$ is a permutation module for $h^*[GL(V)]$. Amongst these h^* are theories closely related to $K(n)^*$, albeit that their coefficient rings are torsion-free and contain an inverse for p.

A result due to Kuhn [**7**] shows that $K(n)^*(BV)$ has the same Brauer character as the permutation module $K(n)^*[\mathrm{Hom}(V, (\mathbf{F}_p)^n)]$ for $GL(V)$. We shall show however that in general $K(n)^*(BV)$ does not have the same Brauer character as a graded permutation module for $K(n)^*[GL(V)]$. Note that Brauer characters give no information whatsoever concerning the structure of $K(n)^*(BV)$ as a module for the Sylow p-subgroup of $GL(V)$. We give an algorithm to determine, for any p-group G, whether an $\mathbf{F}_p[G]$-module is a permutation module, and use this algorithm and computer calculations to determine in some cases whether $K(n)^*(BV)$ satisfies condition (4) above.

The main results of this paper are summarised in the following four statements. Before making them, we fix some notation.

Definition. *Throughout the paper, let p be a prime, $K(n)^*$ the nth Morava K-theory (at the prime p), and let V be a vector space over the field of p elements of dimension d. Let $GL(V)$ act on the right of V, which will have the advantage that the modules we consider will be left modules. Let $U(V)$ be a Sylow p-subgroup of $GL(V)$.*

THEOREM 1.1. *Let p be a prime, let V be a vector space of dimension d over $\mathbf{F}_p$, and let $K(n)^*$ stand for the nth Morava K-theory.*
(a) If p is odd, then $K(n)^*(BV)$ is not a graded permutation module for $GL(V)$.
(b) If $p = 2$ and $n = 1$, then for any d, $K(n)^*(BV)$ is a graded permutation module for $GL(V)$.
(c) For $p = 2$, $n > 1$ and $d \geq 4$, $K(n)^*(BV)$ is not a graded permutation module for $GL(V)$ if d is greater or equal to the smallest prime divisor of n.
(d) For $p = 2$ and $d = 3$, $K(n)^*(BV)$ is not a graded permutation module if n is a multiple of three, or if n is 2, 4, or 5.
(e) For $p = 2$ and $d = 2$, $K(n)^*(BV)$ is a graded permutation module for $GL(V)$ if and only if n is odd.

THEOREM 1.2. *The $K(n)^*[GL(V)]$-modules $K(n)^*(BV)$ and $K(n)^*[\mathrm{Hom}(V, (\mathbf{F}_p)^n)]$ are (ungraded) isomorphic in the following cases:*
(a) For $n = 1$, for any p and d.
(b) For $d = 2$, $p = 2$, and any n.
And are isomorphic as $K(n)^*[SL(V)]$-modules in the case:
(c) $d = 2$, $p = 3$, $n = 1$, 2 or 3.

THEOREM 1.3. *$K(n)^*(BV)$ is not a permutation module for $K(n)^*[U(V)]$ in the following cases:*
(a) $d = 3$, $p = 3$, $n = 2$,
(b) $d = 3$, $p = 5$, $n = 2$.
In the following cases, as well as those implied by Theorems 1.1 and 1.2, $K(n)^*(BV)$ is a graded permutation module for $K(n)^*[U(V)]$:
(c) $d = 3$, $p = 2$, $n = 2$, 3 or 4.

Work of Kriz [**6**] shows that for any V, $K(n)^*(BV)$ is a (graded) permutation module for any subgroup of $GL(V)$ of order p. In the cases covered by Theorem 1.3, the group $U(V)$ has order p^3, and for $p > 2$ it contains no element of order p^2. The gap between Theorem 1.3 and a special case of Kriz's result is filled by:

THEOREM 1.4. *Let $d = 3$, let $p = 3$ or 5, and let H be any subgroup of $GL(V)$ of order p^2. Then $K(2)^*(BV)$ is not a permutation module for H.*

Statements 1.1(b) and 1.2(a) are corollaries of Kuhn's description of the mod-p K-theory of BG [**8**]. Our interest in these questions was aroused by [**2**], in which it is shown that in the case $V = (\mathbf{Z}/2)^2$, $K(n)^*(BV)$ is a graded permutation module for $n = 3$ but is not a graded permutation module for $n = 2$, i.e., the cases $n = 2$ and $n = 3$ of 1.1(e).

The remaining sections of the paper are organised as follows. In Section 2 we describe $K(n)^*(BV)$, the action of $GL(V)$, and the process of reduction to questions concerning finite-dimensional $\mathbf{F}_p$-vector spaces. This material is well-known to many topologists, but we hope that its inclusion will make the rest of the paper accessible to the reader who knows nothing about Morava K-theory. We also include some remarks concerning $K(n)^*[\mathrm{Hom}(V, (\mathbf{F}_p)^n)]$. In Section 3 we prove those of our results that require only Brauer character methods. In Section 4 we deduce 1.1(b) and 1.2(a) from Kuhn's work on mod-p K-theory and give a second proof of 1.1(b). Section 5 studies $\mathbf{F}_2[GL_2(\mathbf{F}_2)]$-modules, and contains proofs of 1.1(e) and 1.2(b). In Section 6 we describe how to decompose $\mathbf{F}_3[SL_2(\mathbf{F}_3)]$-modules, and outline the proof of 1.2(c). In Section 7 we describe our algorithm for determining when a module for a p-group is a permutation module, and outline the proofs of the rest of the results we have obtained using computer calculations. The algorithm of Section 7 may be of independent interest. Section 8 contains the tables of computer output relevant to Sections 6 and 7, together with some final remarks.

2. *Preliminaries*

Fix a prime p. The nth Morava K-theory, $K(n)^*$ (which depends on p as well as on the positive integer n), is a generalized cohomology theory whose coefficient ring is the ring $\mathbf{F}_p[v_n, v_n^{-1}]$ of Laurent polynomials in v_n, which has degree $-2(p^n-1)$. All graded modules for this ring are free, which implies that there is a good Künneth theorem for $K(n)^*$. For any graded $K(n)^*$-module M, let $\overline{M}$ be the quotient $M/(1-v_n)M$. Then $\overline{M}$ is an $\mathbf{F}_p$-vector space, naturally graded by the cyclic group $\mathbf{Z}/2(p^n-1)$. If M is a graded $K(n)^*[G]$-module for some finite group G, then $\overline{M}$ is naturally a $\mathbf{Z}/2(p^n-1)$-graded $\mathbf{F}_p[G]$-module, and M is determined up to isomorphism by $\overline{M}$. It is easy to see that M is a (graded) permutation module for $K(n)^*[G]$ if and only if $\overline{M}$ is a ($\mathbf{Z}/2(p^n-1)$-graded) permutation module for $\mathbf{F}_p[G]$.

For C a cyclic group of order p^m, it may be shown [**9**] that the Morava K-theory of BC is a truncated polynomial ring on a generator of degree two:

$$K(n)^*(BC) = K(n)^*[x]/(x^{p^{mn}}).$$

The generator x is a Chern class in the sense that it is the image of a certain element of $K(n)^2(BU(1))$ under the map induced by an inclusion of C in the unitary group $U(1)$. From the Künneth theorem mentioned above it follows that if V is an elementary abelian p-group of rank d, then

$$K(n)^*(BV) = K(n)^*[x_1, \dots, x_d]/(x_1^{p^n}, \dots, x_d^{p^n}),$$

where $x_1, \dots, x_d$ are Chern classes of d 1-dimensional representations of V whose kernels intersect trivially. The Chern class of a representation is natural, and the d representations taken above must generate the representation ring of V. Thus the action of $GL(V)$ on $K(n)^*(BV)$ may be computed from its action on $\mathrm{Hom}(V, U(1))$ together with an expression for the Chern class of a tensor product $\rho \otimes \theta$ of two 1-dimensional representations in terms of the Chern classes of ρ and θ.

For any generalized cohomology theory h^* such that $h^*(BU(1))$ is a power series ring $h^*[[x]]$ (Morava K-theory has this property), Chern classes may be defined, and there is a power series $x +_F y \in h^*[[x, y]]$ expressing the Chern class of a tensor product of line bundles in terms of the two Chern classes. This power series is called the formal group law for h^*, because it satisfies the axioms for a 1-dimensional commutative formal group law over the ring h^*. Since each Chern class in $K(n)^*(BV)$ is nilpotent of class p^n, we need only determine $x +_F y$ modulo (x^{p^n}, y^{p^n}). This is the content of the following proposition, which is well-known, but for which we can find no reference.

PROPOSITION 2.1. *Modulo the ideal generated by x^{p^n} and y^{p^n}, the formal sum $x+_F y$ for $K(n)^*$ is*

$$x+_F y = x + y - v_n\Big(\sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x^{ip^{n-1}} y^{(p-i)p^{n-1}}\Big).$$

Sketch proof. First we recall the formal sum for BP^*, Brown-Peterson cohomology [**15**]. Let l be the power series

$$l(x) = \sum_{i\geq 0} m_i x^{p^i},$$

where $m_0 = 1$, but the remaining m_i's are viewed as indeterminates, and let $e(x)$ be the compositional inverse to l, i.e., a power series such that $e(l(x)) = l(e(x)) = x$. The BP^* formal sum is the power series $e(l(x) + l(y))$. The $K(n)^*$ formal sum may be obtained as follows: Take the BP^* formal sum, replace the indeterminates m_i by indeterminates v_i using the relation

$$v_j = pm_j - \sum_{i=1}^{j-1} m_i v_{j-i}^{p^i},$$

set $v_i = 0$ for $i \neq n$, by which point all the coefficients lie in $\mathbf{Z}_{(p)}$, and take the reduction modulo p. To calculate the $K(n)^*$ formal sum, it is helpful to set $v_i = 0$ for $i \neq n$ as early as possible, and one may as well set $v_n = 1$, since every term in $x+_F y$ has degree 2. Solving for the m_i's in terms of the v_i's gives

$$\begin{aligned} m_i &= 0 \quad \text{if } n \text{ does not divide } i,\\ m_{ni} &= 1/p^i. \end{aligned}$$

Thus to compute $x+_F y$, let $e'(x)$ be the compositional inverse to

$$l'(x) = \sum_{i\geq 0} x^{p^{ni}}/p^i,$$

and then $x+_F y$ is the mod-p reduction of $e'(l'(x) + l'(y))$. It is easy to see that

$$e'(x) \equiv x - x^{p^n}/p \quad \text{modulo } x^{2p^n},$$

and so

$$x+_F y \equiv x + y - (x+y)^{p^n}/p \quad \text{modulo } (x^{p^n}, y^{p^n}, p).$$

The claimed result follows.

Using the reduction $M \mapsto \overline{M}$ as at the start of this section and Proposition 2.1, the study of the graded $K(n)^*[GL(V)]$-module structure of $K(n)^*(BV)$ reduces to the study of the $\mathbf{Z}/(p^n-1)$-graded $\mathbf{F}_p[GL_d(\mathbf{F}_p)]$-module $K^*_{n,d}$ defined below.

As an $\mathbf{F}_p$-algebra,

$$K^*_{n,d} \cong \mathbf{F}_p[x_1, \dots , x_d]/(x_1^{p^n}, \dots , x_d^{p^n}).$$

Each x_i has degree 1, and the $GL_d(\mathbf{F}_p)$-action is compatible with the product. The action of the matrix $(a_{ij}) \in GL_d(\mathbf{F}_p)$ is given by

$$x_j \mapsto e'\big(\sum_i l'(x_i)a_{ij}\big),$$

where e' and l' are as in the proof of Proposition 2.1. (Recall that for any V we take $GL(V)$ to act on the right of V, and hence obtain a left $K(n)^*[GL(V)]$-module structure on $K(n)^*(BV)$.)

Note that we have halved the original degrees because $K(n)^*(BV)$ is concentrated in even degrees. Until recently it was an open problem whether a similar statement holds for arbitrary finite groups, although some cases had been verified [**9**,**5**,**13**,**14**,**11**]. Kriz has recently announced that this is not the case [**6**].

If we are only interested in Brauer characters, or equivalently composition factors, then a further simplification may be made, see [**7**]. Let $L^*_{n,d}$ denote the algebra of polynomial functions on $(\mathbf{F}_p)^d$, modulo the ideal of p^nth powers of elements of positive degree. Grade $L^*_{n,d}$ by $\mathbf{Z}/(p^n - 1)$, and let $GL_d(\mathbf{F}_p)$ act on $L^*_{n,d}$ by its natural action on the polynomial functions. Thus $L^*_{n,d}$ is a truncated polynomial algebra $\mathbf{F}_p[x_1, \dots , x_d]/(x_1^{p^n}, \dots , x_d^{p^n})$, cyclically graded, and having the standard action of $GL_d(\mathbf{F}_p)$.

Lemma 2.2. *$K^*_{n,d}$ has a series of (graded) submodules such that the direct sum of the corresponding quotients is isomorphic to $L^*_{n,d}$. In particular, $K^*_{n,d}$ and $L^*_{n,d}$ have the same composition factors (as graded modules).*

Proof. For each degree k, take the basis consisting of monomials of length congruent to k modulo $p^n - 1$, and arrange them in blocks with respect to length. For any g, the matrix of its action on $L^k_{n,d}$ with respect to this basis consists of square blocks along the diagonal, whereas the corresponding matrix for the action on $K^k_{n,d}$ has some extra entries below the blocks.

The permutation module $K(n)^*[\mathrm{Hom}(V, (\mathbf{F}_p)^n)]$ occurs in the statement of Theorem 1.2, so we complete this preliminary section with some remarks concerning this module. If ϕ is a homomorphism from V to $\mathbf{F}_p^n$, then $g \in GL(V)$ acts by composition, i.e.,

$$g\phi(v) = \phi(vg).$$

Since we view $GL(V)$ as acting on the right of V, this makes $\mathrm{Hom}(V, (\mathbf{F}_p)^n)$ into a left $GL(V)$-set. The $GL(V)$-orbits in $\mathrm{Hom}(V, (\mathbf{F}_p)^n)$ may be described as follows. For W a subspace of V, let $H(W) \le GL(V)$ be

$$H(W) = \{g \in GL(V) : vg - v \in W \ \forall v \in V\}.$$

For example, $H(\{0\}) = \{1\}$, and $H(V) = GL(V)$. For $0 \leq i \leq \dim(V)$, let H_i be $H(W_i)$ for some W_i of dimension i. Thus H_i is defined only up to conjugacy, but this suffices to determine the isomorphism type of the $GL(V)$-set $GL(V)/H_i$. Now let ϕ be an element of $\mathrm{Hom}(V, (\mathbf{F}_p)^n)$. The stabilizer of ϕ in $GL(V)$ is the subgroup $H(\ker(\phi))$, and the orbit of ϕ consists of all ϕ' such that $\mathrm{Im}(\phi') = \mathrm{Im}(\phi)$. It follows that as $GL(V)$-sets,

$$\mathrm{Hom}(V, (\mathbf{F}_p)^n) \cong \coprod_{0 \leq i \leq \dim(V)} m(n,i) \cdot G/H_i,$$

where $m(n,i)$ is the number of subspaces of $(\mathbf{F}_p)^n$ of dimension i. Thus to decompose the module $\mathbf{F}_p[\mathrm{Hom}(V, (\mathbf{F}_p)^n)]$, it suffices to decompose each $\mathbf{F}_p[GL(V)/H_i]$.

3. *Brauer characters*

In this section we shall prove most of the negative results of Theorem 1.1. Firstly, we describe how to compute the values of the modular characters afforded by the modules $L^k_{n,d}$. As a general reference, see [**3**], in particular §17. Fix an embedding of the multiplicative group of the algebraic closure of $\mathbf{F}_p$ in the group of roots of 1 in $\mathbf{C}$. Let g be a p-regular element of $GL(V)$, i.e., an element whose order is coprime to p, and let $\lambda_1, \lambda_2, \ldots, \lambda_d$ denote the images in $\mathbf{C}$ of the eigenvalues of its action on V^*. Then the Brauer character of g is

$$\chi_{V^*}(g) = \lambda_1 + \lambda_2 + \cdots + \lambda_d\,.$$

Two $\mathbf{F}_p[G]$-modules have the same Brauer character if and only if they have the same composition factors. To compute the character of $L^k_{n,d}$ we proceed as follows: an argument similar to the one used to prove Molien's theorem (see e.g. [**3**], p. 329) shows that the character of a truncated polynomial algebra has a generating function

$$f_g(t) = \prod_{i=1}^{d} \Big(\frac{1-(\lambda_i t)^{p^n}}{1-\lambda_i t} \Big)\,.$$

Then the character of $L^*_{n,d}$ evaluated at g is simply $f_g(1)$, whereas for each degree k (recall that we are grading cyclically) one has

$$\chi_{L^k_{n,d}}(g) = \frac{1}{p^n-1} \sum_{\tau} \tau^{-k} f_g(\tau)\,, \tag{3.1}$$

where the sum ranges over all (p^n-1)-st roots of unity—to see this, recall that the sum, over all mth roots of unity λ, of λ^k is equal to zero if m does not divide k, and equal to m if m does divide k.

Proof of 1.1(a). Let D be the subgroup of diagonal matrices in $GL_d(\mathbf{F}_p)$, so that D is isomorphic to a direct product of d cyclic groups of order $p-1$. In $K^*_{n,d}$, each monomial in $x_1, \ldots, x_d$ is an eigenvector for D, and the monomials fixed by D are those in which the exponent of each x_i is divisible by $p-1$. Hence if $p-1$ does not divide k, then $K^k_{n,d}$ cannot be a permutation module for D because it contains no D-fixed point.

Proof of 1.1(c). As already said above, this is done by computing the character values on certain 2-regular elements of $GL(V)$. We shall first look at the case where d equals a prime divisor q of the fixed number n. Consider an element, g_q say, of $GL_q(\mathbf{F}_2)$ which permutes the 2^q-1 nontrivial elements of $(\mathbf{F}_2)^q$ cyclically. (To see that there is always such an element consider the action of the multiplicative group of $\mathbf{F}_{2^q}$ on the additive group of $\mathbf{F}_{2^q}$.) The set of eigenvalues of g_q contains a primitive (2^q-1)st root of unity, and is closed under the action of the Galois group $\mathrm{Gal}(\mathbf{F}_{2^q}/\mathbf{F}_2)$. Hence the Brauer lifts of the eigenvalues of g_q are λ, $\lambda^2, \dots, \lambda^{2^{q-1}}$ for some primitive (2^q-1)-st root of unity $\lambda \in \mathbf{C}$. Consequently, the generating function for the character afforded by $L^*_{n,q}$ is given by

$$f_{g_q}(t) = \prod_{i=0}^{q-1} \Big(\frac{1-(\lambda^{2^i}t)^{2^n}}{1-\lambda^{2^i}t} \Big).$$

If τ is a (2^n-1)-st root of unity, one gets

$$f_{g_q}(\tau) = \begin{cases} 2^n & \text{if } \tau \in \{\lambda^{-2^i}, \quad i=0,1,\dots q-1\} \\ 1 & \text{otherwise.} \end{cases}$$

Thus evaluating the formula (3.1) for the character afforded by $L^k_{n,q}$ yields

$$\chi_{L^k_{n,q}}(g_q) = \frac{1}{2^n-1} \sum_{\tau \neq \lambda^{-2^i}} \tau^k + \frac{2^n}{2^n-1} \sum_{i=0}^{q-1} \lambda^{2^i k}$$

which is equal to

$$\begin{cases} q+1 & \text{for } k=0 \\ \displaystyle\sum_{i=0}^{q-1} \lambda^{2^i k} & \text{for } k \neq 0. \end{cases}$$

Specializing to the case $k=1$, this sum is never zero, since the powers λ^i for i coprime to 2^q-1 form a $\mathbf{Q}$-basis for $\mathbf{Q}[\lambda]$. For $q>2$ the sum is not a rational, because it is not fixed by the whole Galois group $\mathrm{Gal}(\mathbf{Q}[\lambda]/\mathbf{Q})$. In the case $q=2$, one obtains -1 (the sum of the two primitive third roots of unity). Since permutation modules have positive integer character values, this shows that $K(n)^*(BV_q)$ is not a graded $GL_q(\mathbf{F}_2)$-permutation module. To proceed with vector spaces of dimension bigger than q we consider the cases $q>2$ and $q=2$ separately. In the first case we use the following lemma to conclude that the character still takes non-integer values on certain elements of $GL(V)$. Let g be an (arbitrary) 2-regular element of $GL_d(\mathbf{F}_2)$ and I_r the $r \times r$ identity matrix. If we denote by $g \times I_r$ the element of $GL_{d+r}(\mathbf{F}_2)$ which acts like g on the first d generators of $L^*_{n,d+r}$ and trivially on the last r, one has

LEMMA 3.2. $\chi_{L^k_{n,d+r}}(g\times I_r)=\chi_{L^k_{n,d}}(g)+\big((2^{nr}-1)/(2^n-1)\big)\chi_L(g)$.

Proof. The generating function for $g\times I_r$ is obtained from the one for g as the product with r factors $(1+t+t^2+\ldots+t^{2^n-1})$, thus

$$\begin{aligned}\chi_{L^k_{n,d+r}}(g\times I_r)&=\frac{1}{2^n-1}\sum_{\tau}\tau^{-k}f_{g\times I_r}(\tau)\\&=\frac{1}{2^n-1}\sum_{\tau\neq 1}\tau^{-k}f_g(\tau)\Big(\frac{1-\tau^{2^n}}{1-\tau}\Big)^r+\frac{2^{nr}}{2^n-1}f_g(1)\\&=\chi_{L^k_{n,d}}(g)+\big(\frac{2^{nr}-1}{2^n-1}\big)f_g(1)\,.\end{aligned}$$

Thus $g_q\times I_r$ will do the trick when V has rank $q+r$. This fails for $q=2$, whence we choose the element g' which consists of $d/2$ copies of $g_2=\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$ arranged along the diagonal if d is even, and add an extra diagonal entry 1 if d is odd. Then a computation similar to the one carried out in the previous lemma shows that for $k\not\equiv 0 \bmod 3$,

$$\chi_{L^k_{n,d}}(g')=\begin{cases}-\dfrac{2^{nd/2}-1}{2^n-1} & \text{if } d \text{ is even}\\ -\dfrac{2^{n(d-1)/2}-1}{2^n-1}+1 & \text{if } d \text{ is odd}\end{cases}$$

For $d>3$ these numbers are negative.

The other parts of Theorem 1.1 that may be proved using Brauer characters are some cases of 1.1(d) and both implications of 1.1(e). The details are similar to the above proof so we shall not give them. In the case when V has rank 3, evaluation of the Brauer character of an element of $GL(V)$ of order 7 shows that $L^1_{n,3}$ is not a $GL(V)$-permutation module if 3 divides n. Similarly, when V has rank 2, the Brauer character of an element of $GL(V)$ of order 3 on $L^1_{n,2}$ is negative if n is even. When V has rank 2 and n is odd, it may be shown that for each k, any $GL(V)$-module having the same Brauer character as $L^k_{n,2}$ is a permutation module. This shows that for n odd, $K^k_{n,2}$ is a permutation module, but does not specify which one. In Section 5 we shall describe the isomorphism type of $K^k_{n,2}$ and $L^k_{n,2}$ for all n and k, giving an alternative proof of 1.1(e).

4. *On* $K(1)$

Here we describe how those parts of Theorems 1.1 and 1.2 that concern $K(1)^*$ (i.e., 1.1(b) and 1.2(a)) follow from Kuhn's description of the mod-p K-theory of finite groups [**8**]. An 'elementary' proof, working directly with the description of $K^*_{1,d}$ in the previous section, would be more in keeping with the rest of the paper. We give such a proof in the case $p=2$.

Proof of 1.1(b) and 1.2(a). First, note that for $p=2$, v_1 has degree -2, so that the 'cyclically graded' modules $K^*_{1,d}$ are in fact concentrated in a single degree. Hence

1.2(a) implies 1.1(b). To prove 1.2(a) recall [**15**] that the spectrum representing mod-p K-theory splits as a wedge of one copy of each of the 0th, 2nd, ... , $(2p-4)$th suspensions of the spectrum representing $K(1)^*$. Since $K(1)^*(BV)$ is concentrated in even degrees it follows that $K^*_{1,d}$ is naturally isomorphic to $K^0(BV;\mathbf{F}_p)$. In [**8**] it is shown that for any p-group G, $K^0(BG;\mathbf{F}_p)$ is naturally isomorphic to $\mathbf{F}_p\otimes R(G)$, where $R(G)$ is the (complex) representation ring of G. The case $G=V$ gives 1.2(a), because as a $GL(V)$-module, $\mathbf{F}_p\otimes R(V)$ is isomorphic to $\mathbf{F}_p[\mathrm{Hom}(V,\mathbf{F}_p)]$.

Alternative proof, $p=2$. In this case, $K^*_{1,d}$ is isomorphic to an exterior algebra $\Lambda[x_1,\dots,x_d]=\mathbf{F}_2[x_1,\dots,x_d]/(x_i^2)$. The monomial 1 generates a trivial $GL(V)$-summand. Let H be the subgroup of $GL(V)$ fixing x_1. Then H is the subgroup of $GL(V)$ stabilizing some hyperplane W and inducing the identity map on the quotient V/W. There is a $GL(V)$-set isomorphism

$$\mathrm{Hom}(V,\mathbf{F}_2)\cong GL(V)/GL(V)\amalg GL(V)/H,$$

so it will suffice to show that the submodule M generated by x_1 contains each monomial in $\Lambda[x_1,\dots,x_d]$ of strictly positive length. The permutation matrices permute the monomials of any given length transitively. Assume that M contains all the monomials of length i (this holds for $i=1$), and let $g\in GL(V)$ be such that

$$gx_1=x_1,\dots,gx_{i-1}=x_{i-1},\quad gx_i=x_i{+_F}x_{i+1}.$$

Then

$$g(x_1\dots x_i)+x_1\dots x_i+x_1\dots x_{i-1}x_{i+1}=x_1\dots x_ix_{i+1}\in M,$$

so M contains all monomials of length $i+1$.

It should be possible to give an ‘elementary’ proof of 1.2(a) for $p>2$ by considering the element $x_1+x_1^2+\cdots+x_1^{p-1}$, but we have not done so.

5. *When V has order four*

Here we shall prove Theorems 1.1(e) and 1.2(b), which concern $K(n)^*(BV)$ for V of dimension two over $\mathbf{F}_2$. We determine the structure of $L^k_{n,2}$ as a $GL_2(\mathbf{F}_2)$-module, and deduce that $K^k_{n,2}$ and $L^k_{n,2}$ are isomorphic. Note that it is also possible to prove 1.1(e) using the methods of Section 3 without determining the isomorphism type of $K^k_{n,2}$. Throughout this section, let $V=(\mathbf{F}_2)^2$.

There are three isomorphism types of indecomposable $\mathbf{F}_2[GL(V)]$-modules: the 1-dimensional trivial module T; the natural module V which is both simple and projective (and is the Steinberg module for $GL(V)$); and a module N expressible as a non-split extension of T by T, which is the projective cover of T. Each module is self-dual. There are four conjugacy classes of subgroups of $GL(V)$. The transitive permutation modules are the following four modules:

$$T,\qquad N,\qquad T\oplus V,\qquad N\oplus 2V.$$

Let $S^*[V^*]$ stand for the algebra of polynomial functions on V as a graded $GL(V)$-module.

Proposition 5.1. *The generating functions P_T, P_N and P_V for the number of each indecomposable $GL(V)$-summand of $S^*[V^*]$ are the following power series:*

$$P_T(t) = \frac{1}{1-t^2}, \qquad P_N(t) = \frac{t^3}{(1-t^2)(1-t^3)}, \qquad P_V(t) = \frac{t}{(1-t)(1-t^3)}.$$

Proof. Recall that the ring of invariants $S^*[V^*]^{GL(V)}$ is a free polynomial ring on two generators of degrees two and three (see [**16**]). The Poincaré series for $S^*[V^*]$, and the ring of invariants, together with the generating function for the Brauer character of an element of $GL(V)$ of order three give the three equations below, whose solution is as claimed.

$$\begin{aligned} P_T + 2P_N + 2P_V &= \frac{1}{(1-t)^2} \\ P_T + P_N &= \frac{1}{(1-t^2)(1-t^3)} \\ P_T + 2P_N - P_V &= \frac{1-t}{1-t^3} \end{aligned}$$

Proposition 5.2. *Let k be an element of $\mathbf{Z}/(2^n-1)$. The direct sum decomposition for the module $L^k_{n,2}$ is:*

$$\begin{aligned} 2T \oplus (2^n-2)/6\,N \oplus (2^n+1)/3\,V &\quad \textit{for } n \textit{ odd, } k=0, \\ T \oplus (2^n-2)/6\,N \oplus (2^n+1)/3\,V &\quad \textit{for } n \textit{ odd, } k\neq 0, \\ 2T \oplus (2^n+2)/6\,N \oplus (2^n-1)/3\,V &\quad \textit{for } n \textit{ even, } k=0, \\ T \oplus (2^n+2)/6\,N \oplus (2^n-1)/3\,V &\quad \textit{for } n \textit{ even, } k\neq 0,\ k\equiv 0 \textit{ mod } 3, \\ T \oplus (2^n-4)/6\,N \oplus (2^n+2)/3\,V &\quad \textit{for } n \textit{ even, } k\not\equiv 0 \textit{ mod } 3. \end{aligned}$$

Proof. Let $\tilde{L}^*$ be the truncated symmetric algebra $L^*_{n,2}$, but graded over the integers rather than over the integers modulo 2^n-1. Then $\tilde{L}^k = \{0\}$ for $k > 2(2^n-1)$, and for $k = 2(2^n-1)$, $\tilde{L}^k$ is isomorphic to T, generated by $x_1^{2^n-1}x_2^{2^n-1}$. For $0 < k < 2^n-1$, viewing k as either an integer or an integer modulo 2^n-1 as appropriate, $L^k_{n,2}$ is isomorphic to $\tilde{L}^k \oplus \tilde{L}^{2^n-1+k}$, while $L^0_{n,2}$ is isomorphic to $\tilde{L}^0 \oplus \tilde{L}^{2^n-1} \oplus \tilde{L}^{2(2^n-1)} \cong 2T \oplus \tilde{L}^{2^n-1}$. For $0 \le k \le 2^n-1$, $\tilde{L}^k$ is isomorphic to $S^k[V^*]$. The product structure on $\tilde{L}^*$ gives a duality pairing

$$\tilde{L}^k \times \tilde{L}^{2(2^n-1)-k} \to \tilde{L}^{2(2^n-1)} \cong T,$$

and since all $GL(V)$-modules are self-dual it follows that for $2^n \le k \le 2(2^n-1)$, $\tilde{L}^k \cong S^{2(2^n-1)-k}[V^*]$. The claimed description of $L^k_{n,2}$ follows from Proposition 5.1.

Corollary 5.3. *For each n and each $k \in \mathbf{Z}/(2^n-1)$, $K^k_{n,2}$ and $L^k_{n,2}$ are isomorphic.*

Proof. For $k \neq 0$ $K^k_{n,2}$ has odd dimension, so must contain at least one direct summand isomorphic to T. It is easy to see that 1 generates a summand of $K^0_{n,2}$, and the same dimension argument applied to a complement of this summand shows that $K^0_{n,2}$ contains at least two summands isomorphic to T. On the other hand, V and N are projective, and (Lemma 2.2) $K^k_{n,2}$ has a filtration such that the sum of the factors is isomorphic to $L^k_{n,2}$. Hence $K^k_{n,2}$ has at least as many N summands and V summands as $L^k_{n,2}$. This accounts for all the summands of $K^k_{n,2}$.

The proof of 1.1(e) follows easily from the given description of $K^*_{n,2}$. For 1.2(b), recall from the end of Section 2 that

$$\begin{aligned}\mathrm{Hom}(V,(\mathbf{F}_2)^n) = {} & GL(V)/GL(V) \\ & \amalg\ (2^n-1)\cdot GL(V)/H_1 \\ & \amalg\ (2^n-1)(2^n-2)/6\cdot GL(V)/\{1\},\end{aligned}$$

where H_1 is a subgroup of $GL(V)$ of order two, and that

$$\mathbf{F}_2[GL(V)/H_1] \cong T \oplus V, \qquad \mathbf{F}_2[GL(V)/\{1\}] \cong N \oplus 2V.$$

The argument used in the proof of Corollary 5.3 shows that for any p, n, and k such that $L^k_{n,2}$ contains at most one non-projective summand, $K^k_{n,2} \cong L^k_{n,2}$. For odd primes this does not always occur however. If $0 < k < p^n - 1$ then $L^k_{n,2}$ splits as a direct sum of submodules of dimensions $k+1$ and $p^n - k - 2$ coming from the standard $\mathbf{Z}$-grading on the truncated polynomial algebra. If k is not congruent to either -1 or -2 modulo p, the dimensions of these summands are not divisible by p, and hence $L^k_{n,2}$ contains at least two non-projective indecomposable summands. The calculations described in the next section show that for $p = 3$, $n = 2, 3$ and $0 < k < 3^n - 1$, the module $K^k_{n,2}$ has exactly one non-projective indecomposable summand. It follows that for $p = 3$, $K^k_{n,2}$ and $L^k_{n,2}$ are not necessarily isomorphic.

6. *When V has order nine*

Our results concerning the $SL_2(\mathbf{F}_3)$-module structure of $K^*_{n,2}$ in the case when $p = 3$ were obtained by computer. We wrote a Maple program to generate matrices representing the action of a pair of generators for $SL_2(\mathbf{F}_3)$ on $K^k_{n,2}$. These matrices were fed to a GAP [**10**] program which, given a matrix representation of $SL_2(\mathbf{F}_3)$, outputs a list of its indecomposable summands. In fact the output from the Maple program needed a little editing before being read into the GAP program. This was done by a third program, although it could equally have been done by hand.

Using standard techniques of representation theory [**1**,**3**], the following facts may be verified. For $V = (\mathbf{F}_3)^2$, there are three simple $\mathbf{F}_3[SL(V)]$-modules: the trivial module T, the natural module V, and a simple projective module $P = S^2(V)$ of dimension three. There are three blocks. The blocks containing T and V each contain three indecomposable modules, each of which is uniserial. This data may be summarised as follows:

$$\begin{array}{lll} \text{block of } T = I_1: & T \rightarrowtail I_2 \twoheadrightarrow T, & T \rightarrowtail I_3 \twoheadrightarrow I_2, \\ \text{block of } V = I_4: & V \rightarrowtail I_5 \twoheadrightarrow V, & V \rightarrowtail I_6 \twoheadrightarrow I_5, \\ \text{block of } P = I_7: & \text{contains no other indecomposables.} & \end{array}$$

Letting τ stand for the element of order two in $SL(V)$ and σ for the sum of the six elements of $SL(V)$ of order four, the block idempotents are

$$b_T = 2 + 2\tau + 2\sigma, \qquad b_V = 2 + \tau, \qquad b_P = \sigma.$$

The modules in any single block are distinguishable by their restrictions to a cyclic subgroup of $SL(V)$ of order three. Thus if α is an element of $SL(V)$ of order three, and M is an $SL(V)$-module, the direct summands of M are determined by the ranks of the elements of $\mathrm{End}(M)$ representing the actions of the following seven elements of $\mathbf{F}_3[SL(V)]$:

$$b_T, \quad (1-\alpha)b_T, \quad (1-\alpha)^2 b_T, \quad b_V, \quad (1-\alpha)b_V, \quad (1-\alpha)^2 b_V, \quad b_P.$$

More precisely, if the seven ranks are $r_1, \dots, r_7$, and n_i stands for the number of factors of M isomorphic to I_i, then

$$n_1 = r_1 - 2r_2 + r_3, \quad n_2 = r_2 - 2r_3, \quad n_3 = r_3, \quad n_4 = r_5 - 2r_6,$$
$$n_5 = r_4 - 2r_5 + r_6, \quad n_6 = (2r_5 - r_4)/2, \quad n_7 = r_7/3.$$

Our GAP program reads in matrices representing the action on M of a certain pair of generators for $SL(V)$, and calculates $n_1, \dots, n_7$ by first finding $r_1, \dots, r_7$ as above.

Recall from the end of Section 2 that there is an isomorphism of (right) $GL(V)$-sets:

$$\begin{aligned} \mathrm{Hom}(V, (\mathbf{F}_3)^n) = & GL(V)/GL(V) \\ & \amalg\, (3^n - 1)/2 \cdot GL(V)/H_1 \\ & \amalg\, (3^n - 1)(3^n - 3)/48 \cdot GL(V)/\{1\}, \end{aligned}$$

where H_1 is the subgroup stabilizing a line L in V and acting trivially on V/L. As $SL(V)$-modules, it may be checked that

$$\begin{aligned} \mathbf{F}_3[GL(V)/GL(V)] &\cong I_1, \\ \mathbf{F}_3[GL(V)/L(V)] &\cong I_1 \oplus I_5 \oplus I_7, \\ \mathbf{F}_3[GL(V)/\{1\}] &\cong 2I_3 \oplus 4I_6 \oplus 6I_7. \end{aligned}$$

From this information together with the results given in Table 8.1 it is easy to check the claim of Theorem 1.2(c).

There are fourteen indecomposable $GL(V)$-modules in four blocks, two of which contain a single simple projective module. The six indecomposables in the block containing V are comparatively hard to distinguish, which is the reason why we considered only $SL(V)$.

7. *Permutation modules for p-groups*

In this section we shall describe the computer programs used in the proofs of Theorem 1.3, Theorem 1.4, and the cases $n = 2$, 4, and 5 of 1.1(d). In Sections 5 and 6 our programs made use of the fact that there were only finitely many indecomposable modules. If G is a group whose Sylow p-subgroup is not cyclic, then $\mathbf{F}_p[G]$ has infinitely many indecomposable modules, so the same sort of methods cannot work. Here we shall describe an algorithm which may be used to determine, for any p-group G, whether an $\mathbf{F}_p[G]$-module is a permutation module, and if so to decompose it. (For a precise statement, see Proposition 7.1 below.) As before, we use a Maple program to generate matrices representing the action of a Sylow p-subgroup of $GL_d(\mathbf{F}_p)$ on $K^k_{n,d}$ for various p, k, n, and d, and we use a GAP program working with our algorithm to decompose these modules.

Our algorithm relies on the following fact [**3**]: For G a p-group, any transitive permutation module for $\mathbf{F}_p[G]$ has a unique minimal submodule, which is the trivial module generated by the sum of the elements of a permutation basis. This implies that any transitive permutation module is indecomposable. Note that the Krull-Schmidt theorem and the indecomposability of transitive permutation modules together imply that if a graded $\mathbf{F}_p[G]$-module is a permutation module, then it is also a graded permutation module.

Proposition 7.1. *Let $G_1, \ldots, G_n$ be subgroups of a p-group G, where the order of G_{i+1} is at least the order of G_i, and let M be a (finitely generated) $\mathbf{F}_p[G]$-module. Let $m_1, \ldots, m_n$ be the integers whose calculation is described below. Then M contains a submodule M', where*

$$M' \cong m_1\mathbf{F}_p[G/G_1] \oplus \cdots \oplus m_n\mathbf{F}_p[G/G_n],$$

and M' has maximal dimension among all submodules of M isomorphic to a direct sum of copies of the $\mathbf{F}_p[G/G_i]$.

To compute m_i, proceed as follows. Let M_0 be the zero submodule of M. If M_{i-1} has been defined, let

$$M_i = M_{i-1} + \mathrm{Im}\Big(\big(\sum_{g\in G/G_i} g\big) : M^{G_i} \to M\Big),$$

where the sum ranges over a transversal to G_i in G, M^{G_i} denotes the G_i-fixed points of M, and the sum is an element of $\mathbf{F}_p[G]$ viewed as an element of $\mathrm{End}(M)$. Now define

$$m_i = \dim M_i - \dim M_{i-1}.$$

Without loss of generality, it may be assumed that no two of $G_1, \ldots, G_n$ are conjugate. The dimension of M' is equal to the sum $\sum_i m_i|G : G_i|$. If $G_1, \ldots, G_n$ contains a representative of each conjugacy class of subgroups of G, then $\dim M' = \dim M$ if and only if M is a permutation module.

Proof. First, recall that the socle, $\mathrm{Soc}(N)$, of a module N is the smallest submodule of N containing every minimal submodule. The following statement is easy to prove, and will be useful below. If L is a submodule of M, and $f : N \to M$ is a module homomorphism, then f is injective if and only if its restriction to $\mathrm{Soc}(N)$ is injective. If f is injective, then $\mathrm{Soc}(f(N)) = f(\mathrm{Soc}(N))$, and the sum $L + f(N)$ in M is direct if and only if the sum $\mathrm{Soc}(L) + f(\mathrm{Soc}(N))$ is direct.

Module homomorphisms from $\mathbf{F}_p[G/G_i]$ to M are naturally bijective with elements of M^{G_i}, where the element x corresponds to the homomorphism θ_x sending $1 \cdot G_i$ to x. The socle of $\mathbf{F}_p[G/G_i]$ is a trivial submodule generated by $\sum_{g \in G/G_i} g \cdot G_i$, so its image under θ_x is generated by $\sum_{g \in G/G_i} g \cdot x$. It follows that any submodule of M isomorphic to a direct sum of copies of the modules $\mathbf{F}_p[G/G_1], \dots, \mathbf{F}_p[G/G_i]$ has socle contained in M_i, and in particular consists of at most $\dim M_i$ summands. This shows that any submodule of M isomorphic to a direct sum of $\mathbf{F}_p[G/G_i]$'s has dimension less than or equal to $\sum_i m_i |G : G_i|$, but it remains to exhibit a submodule M' having this dimension.

Define M'_0 to be the zero submodule of M, and assume that for some j with $1 \le j \le n$ we have constructed a submodule M'_{j-1} of M with

$$M'_{j-1} \cong m_1 \mathbf{F}_p[G/G_1] \oplus \cdots \oplus m_{j-1} \mathbf{F}_p[G/G_{j-1}].$$

Let $x_1, \dots, x_{m_j} \in M^{G_j}$ be such that the images $\sum_{g \in G/G_j} g \cdot x_i$ form a basis for a complement to M_{j-1} in M_j. Taking $L = M_{j-1}$, $N = m_j \mathbf{F}_p[G/G_j]$, and $f : N \to M$ the map sending the elements $(0, \dots, 1 \cdot G_j, \dots, 0)$ to the x_i's, the statements in the first paragraph of the proof show that f is injective, and that M'_j defined as the submodule of M spanned by M_{j-1} and the x_i's is isomorphic to $M'_{j-1} \oplus m_j \mathbf{F}_p[G/G_j]$. Now M' may be taken to be M'_n.

The data in Tables 8.2–8.6 of the next section were obtained using the algorithm described above.

8. *Tables and final remarks*

For each p, n, and d, let $\tilde{K}^*_{n,d}$ be the direct summand of $K^*_{n,d}$ corresponding to the reduced Morava K-theory $\tilde{K}(n)^*(B(\mathbf{F}_p)^d)$. Thus $\tilde{K}^k_{n,d} = K^k_{n,d}$ for $k \neq 0$, and $K^0_{n,d} = \tilde{K}^0_{n,d} \oplus T$, where T is the trivial $\mathbf{F}_p[GL(V)]$-submodule of dimension one spanned by the monomial 1. The $\mathbf{F}_p$-dimension of $\tilde{K}^k_{n,d}$ is $(p^{nd} - 1)/(p^n - 1)$. Table 8.1 describes the $SL_2(\mathbf{F}_3)$-module structure of $\tilde{K}^k_{n,2}$ (for $p = 3$) in terms of the indecomposable modules $I_1, \dots, I_7$ as described in Section 6.

Let V have dimension $d = 3$ over $\mathbf{F}_p$. Let l be a line in V, and let π be a plane in V containing l. The group $GL(V)$ acts on the set of all such pairs, and the stabilizer of the pair (l, π) contains a unique Sylow p-subgroup $U(V)$ of $GL(V)$ (and is in fact equal to the normalizer of $U(V)$). Let C be a generator for the centre of $U(V)$, which is cyclic of order p. Let A be a non-central element of $U(V)$ stabilizing every line in π, and let B be a non-central element of $U(V)$ stabilizing every plane containing l. Then A and B generate $U(V)$, and after replacing C by a power if necessary, the commutator of A and B is equal to C. If we identify V with $(\mathbf{F}_p)^3$, and take $U(V)$

Table 8.1: The $SL_2(\mathbf{F}_3)$-summands of $\tilde{K}^k_{n,2}$.

n	k	I_1	I_2	I_3	I_4	I_5	I_6	I_7
1	0	1	0	0	0	0	0	1
1	1	0	0	0	0	1	0	0
2	0,4	1	0	1	0	0	0	2
2	1,3,5,7	0	0	0	0	1	1	0
2	2,6	1	0	0	0	0	0	3
3	even	1	0	2	0	0	0	7
3	odd	0	0	0	0	1	4	0

to be the upper triangular matrices, then we may take

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(Recall that V is to be viewed as the space of row vectors with a right $GL(V)$-action.)

Table 8.2: The D_8-summands of $\tilde{K}^k_{n,3}$.

n	k	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
1	0	0	1	0	0	0	0	1	1
2	0	0	2	2	1	0	0	0	1
2	1,2	1	1	1	0	0	1	1	1
3	0	4	4	4	2	0	0	0	1
3	1,6	6	1	3	0	0	3	1	1
3	2,5	5	3	3	1	0	1	1	1
3	3,4	5	2	4	1	0	2	0	1
4	0	24	8	8	4	0	0	0	1
4	1,14	28	1	7	0	0	7	1	1
4	2,13	25	7	7	3	0	1	1	1
4	3,12	27	2	8	1	0	6	0	1
4	4,11	25	6	8	3	0	2	0	1
4	5,10	27	3	7	1	0	5	1	1
4	6,9	26	5	7	2	0	3	1	1
4	7,8	26	4	8	2	0	4	0	1
5	0	112	16	16	8	0	0	0	1

Table 8.3: A maximal $\mathrm{Syl}_3(GL_3(\mathbf{F}_3))$*-permutation submodule of* $\tilde{K}^k_{n,3}$.

n	k	dim. M'	P_1	P_2	P_5	P_6	P_7	P_{10}	P_{11}
1	0,1	13	0	1	0	0	0	1	1
2	0	91	1	3	3	1	0	0	1
2	1	65	1	1	2	0	3	0	2
2	2	71	1	2	2	0	1	1	2
2	3	73	1	2	2	0	1	2	1
2	4	57	1	1	1	0	2	1	3
2	5	65	1	2	1	0	1	2	2
2	6	67	1	2	1	0	2	2	1
2	7	73	1	2	2	0	2	1	1

Table 8.4: A maximal $\mathrm{Syl}_5(GL_3(\mathbf{F}_5))$*-permutation submodule of* $\tilde{K}^k_{n,3}$.

n	k	dim. M'	P_1	P_2	P_7	P_8	P_9	P_{14}	P_{15}
1	0–3	31	0	1	0	0	0	1	1
2	0	651	3	5	5	1	0	0	1
2	1	527	3	1	4	0	5	0	2
2	2	447	2	3	4	0	3	1	2
2	3	467	2	4	4	0	2	1	2
2	4	587	3	4	4	0	1	1	2
2	5	591	3	4	4	0	1	2	1

For $p = 2$ the group $U(V)$ has 8 conjugacy classes of subgroups, which we list in the following order:

$$\{1\}, \langle A\rangle, \langle B\rangle, \langle C\rangle, \langle AB\rangle, \langle A, C\rangle, \langle B, C\rangle, U(V).$$

Let $P_1, \dots, P_8$ be the corresponding transitive permutation modules, so that P_1 is the free module and P_8 is the trivial module. Similarly, for $p > 2$, $U(V)$ has $2p + 5$ conjugacy classes of subgroups, which we list as:

$$\{1\}, \langle A\rangle, \langle AB\rangle, \dots, \langle AB^{p-1}\rangle, \langle B\rangle, \langle C\rangle,$$
$$\langle A, C\rangle, \langle AB, C\rangle, \dots, \langle AB^{p-1}, C\rangle, \langle B, C\rangle, U(V).$$

Again we let $P_1, \dots, P_{2p+5}$ be the corresponding transitive permutation modules.

Tables 8.2, 8.3, and 8.4 describe maximal $U(V)$-permutation submodules M' of $\tilde{K}^k_{n,3}$ in the cases $p = 2$, 3, and 5 respectively. These submodules were found using the algorithm of Proposition 7.1, with the conjugacy classes of subgroups of $U(V)$ listed in the order given above. The permutation modules omitted from Tables 8.3 and 8.4 never arose as summands of any such M'. In Table 8.2 the dimension of M' is omitted since in these cases M' was always the whole of $\tilde{K}^k_{n,3}$, with dimension $2^{2n}+2^n+1$. The dimensions of $\tilde{K}^k_{1,3}$ and $\tilde{K}^k_{2,3}$ are 13 and 91 respectively for $p=3$, and 31 and 651 respectively for $p=5$. Table 8.4 in the case $n=2$ is incomplete in the sense that not all values of k have been considered. This is because each row required over 24 hours' computing time.

Note that for fixed p and n, the modules $K^k_{n,3}$ tend not to be isomorphic to each other, except when $n=1$, and isomorphic in pairs when $p=2$. For each p, the ring structure gives rise to a duality

$$K^k_{n,3} \times K^{N-k}_{n,3} \to \mathbf{F}_p \subseteq K^N_{n,3},$$

where $N = p^n - 1$. This explains the observed fact that whenever $K^k_{n,3}$ is a permutation module, then $K^{N-k}_{n,3} \cong K^k_{n,3}$.

We find it intriguing that in the case $p=2$ we have been unable to find pairs (n,k) such that $K^k_{n,3}$ is not a $U(V)$-permutation module. Note also that for each n and p considered, $K^0_{n,3}$ is a $U(V)$-permutation module, although it is easy to show that usually $K^0_{n,3}$ cannot be a $GL(V)$-permutation module by comparing the information in the tables with the information given by Brauer characters. (This technique may be used to prove the cases $n=2$, 4 and 5 of Theorem 1.1(d), which we leave as an exercise.)

Finally, in Tables 8.5 and 8.6 we give just enough information to prove Theorem 1.4, in the cases $p=3$ and $p=5$ respectively. That is, for each subgroup H of $U(V)$ of order p^2, we give the dimension of a maximal H-permutation submodule M'' of $K^1_{2,3}$. The dimension of $K^1_{2,3}$ is 91 for $p=3$ and 651 for $p=5$. Only one of the subgroups $\langle AB, C\rangle, \dots, \langle AB^{p-1}, C\rangle$ is listed in these tables, because these subgroups are all conjugate in $GL(V)$ and so give rise to M'''s of the same dimension. The programs were run separately for each of these groups however, as a check.

Table 8.5: A maximal H-permutation submodule of $K^1_{2,3}$ $(p=3)$.

Subgroup H	dim. M''
$\langle A, C\rangle$	69
$\langle AB, C\rangle$	84
$\langle B, C\rangle$	87

Kriz's example of a 3-group G such that $K(2)^*(BG)$ is not concentrated in even degrees is the Sylow 3-subgroup of $GL_4(\mathbf{F}_3)$ [**6**]. This group is expressible as the split extension with kernel $(\mathbf{F}_3)^3$ and quotient the Sylow 3-subgroup of $GL_3(\mathbf{F}_3)$, with the natural action. There may be a connection between our result that $K^*_{2,3}$ is not a permutation module for the Sylow 3-subgroup of $GL_3(\mathbf{F}_3)$ and the fact that $K(2)^*(BG)$ is not entirely even. If so, then Theorem 1.4 suggests if H is a split

Table 8.6: A maximal H-permutation submodule of $K^1_{2,3}$ ($p = 5$).

Subgroup H	dim. M''
$\langle A, C\rangle$	535
$\langle AB, C\rangle$	628
$\langle B, C\rangle$	643

extension with kernel $(\mathbf{F}_3)^3$ and quotient a subgroup of $GL_3(\mathbf{F}_3)$ of order nine, then possibly $K(2)^*(BH)$ is not entirely even. Such H include the extraspecial group of order 3^5 and exponent 3.

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