Morava K-theory of Classifying Spaces

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Preface

The theme of this book is calculating Morava K-theory of classifying spaces, in particular of finite groups. This topic has roots both in homotopy theory and in group cohomology; in fact, given the lack of a proper geometric model for Morava K-theory, many calculations have the flavour of group cohomology with complicated coefficients.

There are many such computations in the literature, and apart from offering some new ones, it was also intended to give a survey of the known results.

The first part of the book, consisting of two chapters, contains the background necessary for the calculations carried out in the later chapters. For most of the theory presented there I do not claim originality. A new feature though is an adaptation of the Rothenberg-Steenrod spectral sequence to central products of groups; this leads to various simplifications of existing work.

In Part 2 the techniques of Part 1 are applied to concrete calculations. The first of its chapters is intended as a survey of results scattered over the literature. Some new proofs are given, but mostly the results are just stated, with one exception: Kriz's celebrated example of a group with odd Morava K-theory is presented with full details. Examples of groups whose Morava K-theory is completely determined by the representation ring of the group are given next. The following chapter concentrates on the prime 2, were new calculations, sometimes computer assisted, are performed. Since it seemed to fit with the rest of the material, an earlier paper on the structure of the Morava K-theory of an elementary abelian group as a module for its automorphism group was also reproduced. The book ends with a few preliminary observations on discrete groups; this area in particular needs further study.

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Introduction

Morava's extraordinary K-theories have been around since the early seventies and have proven their usefulness in homotopy theory. They play a prominent role in much of the recent work in this area of mathematics, as evidenced by the beautiful nilpotence and periodicity theorems of Hopkins, Devinatz, and Smith. From this point of view, the Morava K-theories $K(n)$ can be considered as filtering out certain layers of p-local stable category, the so-called chromatic strata. Adding further to their charm, they tend to be computable, as evidenced e.g. by Ravenel and Wilson in their computation of the Morava K-theory of Eilenberg-Mac Lane spaces, always a good test case. This brings us closer to the main theme of this book: calculating Morava K-theories of classifying spaces.

Although the Morava K-theories are fairly well understood, we do not have good models for the spaces representing them, with exception of $K(1)$, which is closely related to complex K-theory. The (so far elusive) hope is that analysing such natural spaces as BG's might shed some light on their nature.

The only known constructions of the spectra $K(n)$ are purely homotopy theoretic. In particular, we do not have a geometric interpretation akin to the theory of vector bundles for complex K-theory. This considerably complicates matters, including calculations. On the other hand, there is an intriguing connection to number theory. As any complex oriented cohomology theory, $K(n)$ comes equipped with a formal group law. The n-th (integral) Morava K-theory realises the Lubin-Tate formal group of height n, which plays a prominent role in the theory of abelian extensions.

In this work we are mainly concerned with classifying spaces of finite groups. It was Ravenel who realised that the Morava K-theory of a finite group G always has finite rank [R]. Some time later, Kuhn calculated the rank of $K(n)^*(BG)$ when G has an abelian Sylow p-subgroup [Ku1]. Since the mid eighties, a preprint by Hopkins, Kuhn and Ravenel circulated, which proved highly influential, long before the final version appeared [HKR]. The results were startling: Although there is no bundle theory for Morava K-theory, there is a character theory associated to a variant of Morava K-theory, often called Morava E-theory. The algebra $E^*(BG)$ may thus be studied using these 'generalised characters' in much the same way one uses Artin's theorem for the complex representation ring. Furthermore, they calculated the $K(n)$ Euler characteristic, i.e. the difference of the

ranks of the even-dimensional and the odd-dimensional part of $K(n)^*(BG)$, in terms of the subgroup structure of G. Some examples calculated up till then (e.g. [HKR, Hu1, TY2, TY3, T, Y3]) led them to conjecture that the Morava K-theory of a finite group is always concentrated in even degrees. This conjecture in turn prompted many new calculations which seemed to support it, until finally Kriz came up with a counterexample [K]: he showed that the group of unipotent (4×4)-matrices over $\mathbb{F}_3$ has odd second Morava K-theory. (It is perhaps ironic that the counterexample to the conjecture appeared before the paper of its instigators.) This example was later generalised by Kriz and Lee to all odd primes and all $n > 2$ [KL].
After it became clear that the conjecture was dead, the interest turned more to structural questions, but new computations were still carried out ([Sc, Y5, Hu2, St1, St3, B3, St4, GS], to mention a few). However, there are still relatively few groups whose Morava K-theory is known.

This book is mainly about calculating $K(n)^*(BG)$, and not intended to be a comprehensive account of the subject. Much of the material presented here is other people's work. When embarking on this project, our intention was to give a survey of the known computations on the one hand, and add a few more. For example, we consider all 51 groups of order 32 (although for some we used computer calculations, which restricted us to $K(2)$ in these cases).
So the first question to be addressed is: how does one calculate Morava K-theory of classifying spaces? There are in fact many possible approaches: an assortment of spectral sequences, generalised character theory, duality, Chern approximations, transfer methods, and finally computer calculations implementing some of the above methods. Pioneering work on the Serre spectral sequence of fibrations over BC_p was instrumental in Kriz's construction of his counterexample: the question of whether the Morava K-theory of the total space is concentrated in even degrees turns out to depend solely on the structure of the (integral) Morava K-theory of the fibre (assumed to be even degree) as a module for C_p; if this module is a permutation module, then $K(n)^{\mathrm{odd}}(E)$ is zero, otherwise there are classes of odd degree. This condition is something one may check on the computer, once one overcomes the basic difficulty of determining the action. This is not quite as easy as it may sound, the formal group law complicates matters, and furthermore one usually needs to know $K(n)^*(F)$ as an algebra. There are not many groups where such knowledge is available. At the other extreme, one might be tempted to evade the problem of calculating with the formal group law by considering central extensions. However, this is rarely a viable approach, since one needs the (ordinary) cohomology of the quotient as input.
A natural source of complex oriented cohomology of BG's are Chern classes of complex representations of G. This led Strickland to construct his Chern approximations [St4]: consider all irreducible representations of G, introduce formal variables for each Chern class of these representations, and impose all

relations dictated by relations between the representations and all λ-operations. This results in an object of finite rank, and is sometimes computable. In fact, the only complete calculations of algebra structures we are aware of use this method; in these cases, the approximation turns out to be exact.
Bakuradze and Priddy try a different approach: they try to find multiplicative relations between Chern classes using formal properties of the transfer. They come up with a formula linking the transfer of a Chern classes of a representation of (index p) subgroups with Chern classes of the induced representation. Since irreducible representations of p-groups are always induced from subgroups, one obtains relations this way.

Almost all groups considered here are p-groups, mainly for one reason: the classifying space of a finite group G is always a stable summand of the classifying space of a Sylow p subgroup P of G. In a sense, p-groups are the building blocks of finite groups, as evidenced by the many decomposition theorems in the literature.

The reader will notice that we refrain from using the algebro-geometric language of formal schemes. This deprives us of its inherent elegance of exposition as well as some geometric insight. We do so for two reasons: one, since the main emphasis is on calculations, and two, we hope that writing in purely algebraic terms renders the book more accessible.

Organisation of the book

This monograph is divided into two parts. Part 1 collects the prerequisites for the calculations in Part 2. Although we intended this book to be largely self-contained, the first part came out rather condensed, the principal reason being that in many instances, we thought that any attempt on our side to improve on the original exposition would be futile. Chapter I gives a rudimentary account of complex oriented cohomology in general and Morava K-theory in particular. Chapter II describes the methods employed in the subsequent calculations of Part 2. One new feature is that we describe a way to use the Rothenberg-Steenrod spectral sequence for central products.
The heart of the book is clearly the second part. Chapter III surveys many known computations. When we could offer a new approach, we have generally done so, but in many cases we chose not to redo the original calculation, but rather state the results. There is one notable exception to this rule: we felt that without a detailed account of Kriz's counterexample this work would be incomplete. The next chapter gives a few worked examples of Chern approximations, before turning to 2-primary calculations in Chapter V. This chapter contains several new calculations, notably the groups of order 32, many of which are not covered in the existing literature. The vain hope was that by going through the list, we might find a counterexample to the even-dimensionality conjecture at the prime 2: to this date, no such example is known.

Chapter VI is an adaption of [LS]; it studies the question when the Morava K-theory of an elementary abelian group V is a permutation module for (subgroups of) $\mathrm{Aut}(V)$. Chapter VII makes a few obervations on discrete groups.

Sins of omission

We already said at the beginning that this book is far from comprehensive, nor was it intended to be. Consequently, several important topics were left out, of which we now list but a few.
Several versions of Eilenberg-Moore spectral sequences have been constructed by Kriz [K], Bauer (not yet published), Tamaki, and Tanabe [T]; apart from a passing reference to Tanabe's work, we fail to mention any of them. This is mainly since we have not used them in the calculations presented (or do not know how to employ them to get better results).
For the same reason, we give no account of duality theory; we refer the reader to Strickland's paper [St3].
Since we do not use geometric language, we were also unable to include Greenlees's and Strickland's theory of level structures on Morava E-theory of classifying spaces [GS], which refines some results of [HKR].

Finally, a word on notation. When referring to a numbered statement or equation within the same chapter, we omit the number of the chapter.

Part 1

Background

Chapter I

Morava K-theories

In this chapter we very briefly collect a few fundamental facts about Morava K-theories. A more detailed source is Würgler's survey article [Wü2].

1 Complex oriented cohomology

A multiplicative generalised cohomology theory E is called *complex oriented*, if there is a class $x = x^E \in E^2(\mathbb{C}P^\infty)$, called complex orientation, which pulls back to a generator of the free rank one module $E^2(\mathbb{C}P^1)$ under the inclusion $\mathbb{C}P^1 \subset \mathbb{C}P^\infty$.
An essentially formal calculation with the Atiyah-Hirzebruch spectral sequence shows there are isomorphisms

$$E^*(\mathbb{C}P^\infty) = E^*[\![x]\!]$$
$$E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = E^*[\![x_1, x_2]\!] .$$

A similar argument with the Atiyah-Hirzebruch spectral sequence shows

Lemma 1.1 (Adams)**.** *Any theory E whose coefficients are concentrated in even degrees has a complex orientation.* □

Thus there are plenty of examples, such as

- singular cohomology HR with coefficients in a commutative ring R,
- complex K-theory (but not real K-theory KO),
- complex cobordism MU with coefficients $MU_* = \mathbb{Z}[x_1, x_2, \dots]$, $|x_i| = 2i$,
- the Brown Peterson spectrum BP which arises as summand of MU localised at a prime p and has coefficients $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$, $|v_i| = 2(p^i - 1)$,
- the Johnson-Wilson spectra $E(n)$ with $E(n)_* = \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n, v_n^{-1}]$,

among many others.
Complex oriented theories have a theory of Chern classes for complex vector bundles, constructed as usual via the splitting principle: Let $T = S^1 \times \cdots \times S^1$

(m factors) be a maximal torus in $U(m)$, iterating the above argument gives $E^*(BT) = E^*[\![x_1, \ldots, x_m]\!]$. Define c_i as the coefficient of X^{m-i} in $\prod_{i=1}^m (X - x_i)$. Then

$$E^*(BU(m)) \cong E^*[\![c_1, \ldots, c_m]\!] .$$

The Thom isomorphism theorem holds, too; in fact, one could use this as a definition: E is complex oriented if and only if any n-dimensional complex vector bundle $\xi \to X$ has a Thom class $u_\xi \in E^{2n}(X^\xi)$.

The image of the complex orientation x under the homomorphism induced by the map $\mu\colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ classifying the tensor product of line bundles (the H-space multiplication for $\mathbb{C}P^\infty$) is thus a formal power series in the two variables x_1, x_2:

$$\mu^*(x) =: F_E(x_1, x_2)$$

When the theory E is understood from the context, we shall drop the subscript E. This power series enjoys the formal properties of a commutative one-dimensional formal group law, i.e.

$$\begin{aligned} F(x,0) = 0 &= F(0,x) && \text{Identity} \\ F(x_1, F(x_2, x_3)) &= F(F(x_1, x_2), x_3) && \text{Associativity} \\ F(x_1, x_2) &= F(x_2, x_1) && \text{Commutativity} \end{aligned}$$

The remarkable fact is that the formal group law for MU is universal in the sense that if F is any formal group law over a ring R, there is a ring homomorphism $MU_* \to R$ mapping the coefficients of the universal formal group law to those of F. For more about formal group laws, see [Ha], [Se]. One also uses the suggestive notation $x_1 +_E x_2$ instead of $F_E(x_1, x_2)$, and $[n]_E(x) = \underbrace{x +_E x +_E \cdots +_E x}_{n}$.

2 Morava K-theories

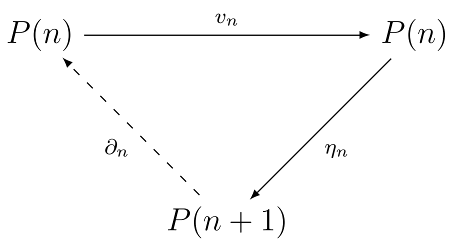
For every prime p and every nonnegative integer n there is a $2(p^n - 1)$ periodic generalised cohomology theory $K(n)^*$, called the n-th Morava K-theory. The representing spectra $K(n)$, or more precisely their connective analogues $k(n)$, can be obtained from the Brown-Peterson spectrum BP using the Sullivan-Baas construction. By this construction one can make a new cohomology theory by killing a regular ideal I in $BP_* \cong \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$ with $\deg(v_i) = 2(p^i - 1)$. Using the ideal $(p, v_1, v_2, \ldots, v_{n-1}, v_{n+1}, v_{n+2}, \ldots)$, one obtains a copy of $k(n)$ with coefficients $\mathbb{F}_p[v_n]$. If p is odd, then there is a unique product map $m_n : k(n) \wedge k(n) \to k(n)$, turning $k(n)$ into a BP module spectrum with respect to the module structure induced by the natural map $BP \to k(n)$ (also called the Thom map). For the prime 2 the situation is a bit more involved, but one can show that there are, up to homotopy, two compatible products, neither of which is commutative; see below. One thus has a map

$$\Sigma^{2(p^n-1)}k(n) \longrightarrow k(n) \tag{2.1}$$

given as the composition

$$S^{2(p^n-1)} \wedge k(n) \xrightarrow{v_n \wedge 1} k(n) \wedge k(n) \xrightarrow{m_n} k(n)\,,$$

choosing either product map when $p = 2$. Here v_n denotes a map representing v_n in homotopy. The homotopy colimit of iterates of this map then is a spectrum which has v_n inverted in its coefficients, and represents periodic Morava K-theory. It is a commutative ring spectrum for odd primes, by construction, whereas for $p = 2$ one has two noncommutative products. Under certain circumstances, though, e.g. if X is a space whose Morava K-theory is entirely concentrated in even degrees, both products on $K(n)^*(X)$ agree and are commutative. In general, the difference between the two products is measured by a higher order Bockstein operation: For all non negative n there are BP-module spectra $P(n)$ with $P(n)_* \cong BP_*/I_n$ where $I_n = (p, v_1, v_2, \ldots v_{n-1})$ is the n-th invariant prime ideal of BP_*, see [JW] for details. The spectra $P(n)$ are related by exact triangles of BP-module maps



One has the following result due to U. Würgler:

Theorem 2.1 ([Wü1]). *Let $n \geq 1$ and $p = 2$. Then there are exactly two products $m_n, \overline{m}_n : P(n) \wedge P(n) \to P(n)$ which make $P(n)$ a BP-algebra spectrum compatible with the given BP-module structure. Both are associative and have a two-sided unit. m_n and $\overline{m}_n$ are related by the formula*

$$\overline{m}_n = m_n \circ T = m_n + v_n m_n (Q_{n-1} \wedge Q_{n-1})\,,$$

where T denotes the twist map. Moreover, $\eta_{n-1} : P(n-1) \to P(n)$ is a map of ring spectra with respect to any admissible product chosen on $P(n-1)$ and $P(n)$. □

Here $Q_{n-1} = \eta_n \circ \partial_n$ is a Bockstein operation of degree $2^n - 1$. There is a map $\lambda_n : P(n) \to K(n)$ which turns $K(n)_*$ into a $P(n)_*$-module. Using the exact functor theorem mod I_n [Y1], one then gets a natural equivalence

$$P(n)_*(X) \otimes_{P(n)_*} K(n)_* \xrightarrow{\ \sim\ } K(n)_*(X)\,.$$

This is the mod I_n version of the Conner-Floyd theorem [CF]. Combined with Theorem 2.1 above one can see that an analogous statement holds for Morava K-theory, with $P(n)$ replaced by $K(n)$. The failure of either one of the two products to be commutative is thus measured by

$$m_n - m_n \circ T = v_n m_n (Q_{n-1} \wedge Q_{n-1}) .$$

In particular, since the degree of Q_{n-1} is odd, if X is a space whose Morava K-theory is concentrated in even dimensions, both products on $K(n)_*(X)$ will agree and be commutative.

The coefficients $K(n)_* = \mathbb{F}_p[v_n, v_n^{-1}]$ are a graded field in the sense that all its graded modules are free. Thus Morava K-theories enjoy Künneth isomorphisms, and there is a linear duality between $K(n)$-homology and $K(n)$-cohomology:

$$K(n)^*(X) \cong \mathrm{Hom}_{K(n)_*}(K(n)_*(X), K(n)_*) .$$

Other variants of Morava K-theory frequently used include an integral version $\widetilde{K}(n)$ with coefficients $\widetilde{K}(n)^* \cong W\mathbb{F}_{p^n}[v_n, v_n^{-1}]$, where $W\mathbb{F}_{p^n}$ is the ring of Witt vectors over $\mathbb{F}_{p^n}$. One can obtain $W\mathbb{F}_{p^n}$ from the p-adics by adjoining a $(p^n - 1)$-st root of unity ζ.
Finally, there is the so-called Morava E-theory, whose coefficients

$$E_{n*} \cong W\mathbb{F}_{p^n}[\![u_1, \ldots, u_n]\!][u, u^{-1}] \quad , \quad |u_i| = 0 \, , \; |u| = 1 \, ,$$

represent the universal deformation of a p-typical formal group of height n.
We shall say a bit more about $\widetilde{K}(n)$ in later sections.

3 The formal group law

Formal group laws over a commutative $\mathbb{F}_p$-algebra R are characterised (up to isomorphism over the separable closure) by a single invariant called the *height*: a formal group law F over such R is of height n if the p-series $[p](x)$ has leading term ax^{p^n} with $a \neq 0$.
As an illustration, we give the following well-known formula for the formal group law for $K(n)$ (with the complex orientation inherited from BP):

Proposition 3.1. *Modulo the ideal generated by $x_1^{p^n}$ and $x_2^{p^n}$, the formal sum $x_1 +_{K(n)} x_2$ for $K(n)$ is*

$$x_1 +_{K(n)} x_2 = x_1 + x_2 - v_n \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} x_1^{ip^{n-1}} x_2^{(p-i)p^{n-1}} .$$

PROOF. First we recall the formal sum for BP^*, Brown-Peterson cohomology [Wi]. Let l be the power series

$$l(x) = \sum_{i \geq 0} m_i x^{p^i},$$

where $m_0 = 1$, but the remaining m_i's are viewed as indeterminates, and let $e(x)$ be the compositional inverse to l, i.e., a power series such that $e(l(x)) = l(e(x)) = x$. The BP^* formal sum is the power series $e(l(x_1)+l(x_2))$. The $K(n)^*$ formal sum may be obtained as follows: Take the BP^* formal sum, replace the indeterminates m_i by indeterminates v_i using the relation

$$v_j = pm_j - \sum_{i=1}^{j-1} m_i v_{j-i}^{p^i},$$

set $v_i = 0$ for $i \neq n$, by which point all the coefficients lie in $\mathbb{Z}_{(p)}$, and take the reduction modulo p. To calculate the $K(n)^*$ formal sum, it is helpful to set $v_i = 0$ for $i \neq n$ as early as possible, and one may as well set $v_n = 1$, since every term in $x_1 +_{K(n)} x_2$ has degree 2. Solving for the m_i's in terms of the v_i's gives

$$\begin{aligned} m_i &= 0 && \text{if } n \text{ does not divide } i, \\ m_{ni} &= 1/p^i \,. \end{aligned}$$

Thus to compute $x_1 +_{K(n)} x_2$, let $e'(x)$ be the compositional inverse to

$$l'(x) = \sum_{i \geq 0} x^{p^{ni}}/p^i \,,$$

and then $x_1 +_{K(n)} x_2$ is the mod-p reduction of $e'(l'(x_1) + l'(x_2))$.
It is easy to see that

$$e'(x) \equiv x - x^{p^n}/p \quad \text{modulo } x^{2p^n},$$

and so

$$x_1 +_{K(n)} x_2 \equiv x_1 + x_2 - (x_1 + x_2)^{p^n}/p \quad \text{modulo } (x_1^{p^n}, x_2^{p^n}, p)\,.$$

The claimed result follows. □

A somewhat more subtle calculation carried out in [BV] gives a few more terms; we shall come back to that where we need to.
For the integral version $\widetilde{K}(n)$, there exists a complex orientation x satisfying

$$[-p]_{\widetilde{K}(n)}(x) = -px + v_n x^{p^n} \,.$$

Recall that $W\mathbb{F}_{p^n} = \mathbb{Z}_p[\zeta_{p^n-1}]$ for a primitive (p^n-1)-th root of unity. For any $a \in \mathbb{F}_{p^n}$ and any power series $\alpha(x)$ with $\alpha(x) \equiv ax \mod x^2$ which commutes with $[-p]_F$, one has $\alpha(x) = [a]_F(x)$, cf. [Se, §3]. If in addition $a^{p^n-1} = 1$, it follows that $[a]_F(x) = ax$ and that all coefficients a_{ij} of the formal group law

$$F(x,y) = \sum a_{ij} x^i y^j$$

are zero unless $i + j \equiv 1 \mod (p^n - 1)$.
Furthermore, if $p > 2$, then

$$[-1]_F\widetilde{K}(n)(x) = -x \quad , \quad [p]_{\widetilde{K}(n)}(x) = px - v_n x^{p^n} .$$

(This is false for $p = 2$.)

4 The Atiyah-Hirzebruch spectral sequence

Let $\mathcal{A}$ denote the mod p Steenrod algebra, and Q_n the n-th Milnor primitive, defined recursively by $Q_0 = \beta$ and $Q_n = [P^{p^{n-1}}, Q_{n-1}]$ (respectivley $Q_0 = Sq^1$ and $Q_n = [Sq^{2^n}, Q_{n-1}] = Sq^{\Delta_{n+1}}$ for $p = 2$). Then in the cofibre sequence

$$\cdots \to \Sigma^{2(p^n-1)}k(n) \xrightarrow{v_n} k(n) \xrightarrow{\pi_n} H\mathbb{F}_p \xrightarrow{\overline{Q}_n} \Sigma^{2p^n-1}k(n) \to \cdots$$

the Thom map $\pi_n \colon k(n) \to H\mathbb{F}_p$ (killing v_n) induces an isomorphism

$$H^*(k(n); \mathbb{F}_p) \cong \mathcal{A}/\mathcal{A}Q_n \,,$$

and one has $Q_n = \pi_n \overline{Q}_n$.

One tool for computations is the Atiyah-Hirzebruch spectral sequence

$$E_2 = H^*(X; K(n)^*) \Longrightarrow E^*(X) \,.$$

This is a first and fourth quadrant spectral sequence, and in the case of $K(n)$ with non-zero rows only in vertical degrees divisible by $2(p^n-1)$, the degree of v_n. Thus the first potentially non-zero differential is d_{2p^n-1}.

Lemma 4.1 ([Y2]). $d_{2p^n-1} = v_n \otimes Q_n$.

PROOF. Since $d_{2p^n-1} \otimes v_n^{-1}$ is both a derivation and a stable cohomology operation, it has to be a scalar multiple of Q_n, and checking on lens spaces gives the result, see [Y2]. □

For our purposes though the Atiyah-Hirzebruch spectral sequence is of limited use. Since we want to calculate Morava K-theories of classifying spaces of groups, we would need the mod p cohomology of the group as input — but the problem of calculating the mod p cohomology of p-groups is notoriously hard. In a few simple cases where not only the cohomology is known but also the action of Q_n easy to describe, one can actually do the calculation. We shall illustrate this with examples at $p = 2$ in Chapter III.

Another example where this approach works is A. Yamaguchi's calculation for braid groups [Ym].

Chapter II

Methods of calculation

There are several methods one may employ for calculation: representation theory, transfer methods, generalised character theory, and various spectral sequences. Roughly speaking, one first tries to compute $K(n)^*(BG)$ additively, say by a spectral sequence and/or character theory. To get at the multiplicative structure one then can use character theory again, or transfer methods, or representation theory — usually a combination of all of the above.

1 Complex oriented cohomology of finite abelian groups

Let E be a complex oriented cohomology theory with formal group law $+_E$ and C_m a cyclic group of order m. Then the Gysin sequence for the fibration $S^1 \to BC_m BS^1 = \mathbb{C}P^\infty$ gives

Lemma 1.1. *$E^*(BC_m) \cong E^*[\![x]\!]/([m]_E(x))$ where x is the Euler class of the standard generator of the complex character ring of C_m.* □

In particular,

$$\widetilde{K}(n)^*(BC_p) \cong \widetilde{K}(n)^*[x]/(px - v_n x^{p^n}), \text{ and}$$
$$K(n)^*(BC_{p^k}) \cong K(n)^*[x]/(x^{p^{kn}}).$$

Thus $E^*(BC_m)$ is concentrated in even degrees. Now when E^* is a (graded) field, or a complete local ring, then $E^*(BC_m)$ is a free E^*-module: this follows from the formal Weierstraß preparation theorem, whose proof may be found in [La, Chapter V, §2].

Theorem 1.2. *Let R be a commutative ring and I an ideal of R, such that R is complete in the I-adic topology. Let $f(x)$ be a power series over R such that $f(x) \equiv \varepsilon \cdot x^d \mod (I, x^{d+1})$ where ε is a unit in R. Then $R[\![x]\!]/(f(x))$ is a free R-module with basis $\{x^i \mid 0 \leq i \leq d-1\}$. Furthermore, f factors uniquely as $f(x) = u(x) \cdot w(x)$ where $u(x)$ is a unit and $w(x)$ a monic polynomial of degree d.* □

Thus under the above hypotheses, one has Künneth isomorphisms

$$E^*(BC_m \times X) \cong E^*(BC_m) \otimes_{E^*} E^*(X)$$

for arbitrary spaces X. This determines the E-cohomology of finite abelian groups. More precisely, suppose $A = C_{m_1} \times \cdots \times C_{m_k}$, then

$$E^*(BA) \cong E^*[\![x_1, \ldots, x_k]\!]/([m_1](x_1), \ldots, [m_k](x_k)) .$$

2 Chern approximations

A good source of cohomology classes for classifying spaces BG are complex representations of the group G. Recall from Chapter I that complex oriented cohomology theories come with a theory of Chern classes.
Strickland proposes in [St4] to study Morava K-theory of finite groups by comparing it to an algebra obtained as follows: take all irreducible complex representations ρ of G, assign an indeterminate to every Chern class of such ρ, and divide out by the relations obtained from the product structure of the representation ring and all λ-operations. He describes the resulting object in geometric terms, i.e., the resulting formal scheme over the formal group.
Since we have consistently shunned the geometric language, we shall give a much simplified account of the theory.

2.1 Chern classes of products and exterior powers

Let μ and ρ be complex representations of dimension m and r, respectively. Let $\sigma_i(s)$ and $\sigma_j(t)$ denote the elementary symmetric functions in $s_1, \ldots, s_m$ and $t_1, \ldots, t_r$. Then the coefficient of X^k in

$$\prod_{\substack{1 \le i \le m \\ 1 \le j \le r}} \bigl(1 + X(s_i +_{K(n)} t_j)\bigr)$$

is a polynomial in the $\sigma_i(s)$ and $\sigma_j(t)$, say $P_k(\sigma_1(s), \ldots, \sigma_m(s); \sigma_1(t), \ldots, \sigma_r(t))$. Similarly, the coefficient of X^k in

$$\prod_{i_1 < \cdots < i_q} \bigl(1 + X(s_{i_1} +_{K(n)} s_{i_2} +_{K(n)} \cdots +_{K(n)} s_{i_q})\bigr)$$

is a polynomial L_k in the $\sigma_i(s)$.
These power series determine Chern classes of products and exterior powers:

Proposition 2.1. *(a)* $c_k(\mu \otimes \rho) = P_k\bigl(c_1(\mu), \ldots, c_m(\mu); c_1(\rho), \ldots, c_r(\rho)\bigr)$.

(b) $c_k(\lambda^q \mu) = L_k(c_1(\mu), \ldots, c_m(\mu))$.

Proof. This follows at once from the splitting principle resp. the very construction of the Chern class c_k of an n-dimensional bundle as the coefficient of X^{n-k} in $\prod_{i=1}^{n}(X - t_i)$. □

Next, recall the Adams operations on the represenation ring. Let μ be a representation of dimension m; set $\lambda_t(\mu) = \sum_{i\geq 0} \lambda^i(\mu)t^i$ (where $\lambda^0\mu = 1$), and define

$$\psi_t(\mu) = m - \frac{t}{\lambda_{-t}(\mu)}\frac{d}{dt}\lambda_{-t}(\mu)\,.$$

Then $\psi^l\mu$ is the coefficient of t^l in $\psi_t(\mu)$. There are the well-known formulae linking Adams operations and exterior powers via the Newton polynomials; in particular, $\psi^k(\mu) = \mu^k$ for any line bundle (one-dimensional representation). Hence for a direct sum of line bundles one has

$$c_k(\psi^l(\mu_1 \oplus \cdots \oplus \mu_m)) = c_k(\mu_1^l \oplus \cdots \oplus \mu_m^l) = \sigma_k\big([l](x_1), \ldots, [l](x_m)\big)$$

where $x_i = c_1(\mu_i)$. Thus

Proposition 2.2. *For the $K(n)$ Chern classes one has $c_k(\psi^{p^r}\mu) = c_k(\mu)^{p^{rn}}$.* □

Definition 2.3. *Let $\rho_1, \ldots, \rho_k$ be the distinct non-trivial irreducible complex representations of G. For each ρ_i, choose indeterminates $c_{l,i}$, $1 \leq l \leq \dim(\rho_i)$. Define $C(G; K(n))$ as the quotient of the $K(n)^*$-algebra on the $c_{l,i}$ by the relations imposed by Proposition 2.1.*

As a consequence of Proposition 2.2, one gets the following special case of Corollary 10.3 of [St4]. The proof is a paraphrase of the argument given there.

Corollary 2.4. *For any finite group G, the rank of $C(G; K(n))$ over $K(n)^*$ is finite.*

Proof. It suffices to show that all generators of $C(G; K(n))$ are nilpotent. Let e be the exponent of G and p^r its p-part, i.e., $e = p^r f$ with f coprime to p. Then $\psi^e(\mu) = \dim(\mu)$ for any representation μ of G. Thus for $k \geq 1$, one has $0 = c_k(\psi^e\mu) = c_k(\psi^{p^r}\psi^f\mu) = c_k(\psi^f\mu)^{p^{rn}}$. Now let $c_\bullet$ denote the total Chern class; since we are working modulo p, we find that

$$1 = c_\bullet(\psi^f\mu)^{p^{rn}} = c_\bullet(p^{rn}\psi^f\mu) = c_\bullet(\psi^f(p^{rn}\mu))$$

(using additivity) and thus $c_k(\psi^f(p^{rn}\mu)) = 0$ for all $k \geq 1$. But when f is coprime to p, the series $[f](x)$ is an automorphism of the formal group law; thus $c_k\psi^f = 0$ for all $k > 0$ iff $c_k = 0$ for all $k > 0$. This implies $1 = c_\bullet(p^{rn}\mu) = c_\bullet(\mu)^{p^{rn}}$, whence the claim. □

There is an obvious map

$$\mathrm{ch}_G\colon C(G; K(n)) \longrightarrow K(n)^*(BG)$$

assigning to $c_{k,i}$ the Chern class $c_k(\rho_i)$. In general, this map is neither injective not surjective. We call the Chern approximation of G *exact* if ch_G is an isomorphism.

2.2 The example Σ_3

This is somewhat trivial, but illustrates the method; more examples are to follow at a later stage. The representation ring of Σ_3 is generated by $1, \varepsilon, \delta$ of dimensions $1, 1, 2$, respectively, with relations $\varepsilon^2 = 1$, $\varepsilon\delta = \delta$, and $\delta^2 = 1+\varepsilon+\delta$. Furthermore, $\psi^2\delta = 1-\varepsilon+\delta$, $\psi^3\delta = 1+\varepsilon$, and $\lambda^2\delta = \varepsilon$. Let $x = c_1(\varepsilon)$, $y = c_1(\delta)$, and $z = c_2(\delta)$. First look at $p = 2$: from $\varepsilon^2 = 1$ one immediately concludes $0 = [2](x) = x^{2^n}$. The quickest way to arrive at the remaining relations is using Adams operations: from $2 = \psi^3\psi^2\delta = \psi^3(1-\varepsilon+\delta)$ we see $1 = c_\bullet(1-\varepsilon+\delta)$, where $c_\bullet$ denotes the total Chern class; consequently $1 + x = 1 + y + z$ and thus $y = x$ and $z = 0$ for degree reasons. Thus we recover the obvious isomorphism $K(n)^*(B\Sigma_3) \cong K(n)^*(BC_2)$ for $p = 2$.

At the prime 3, $\varepsilon^2 = 1$ gives $[2](x) = 0$, hence $x = 0$. This implies $c_k(\psi^2\delta) = c_k(\delta)$ for all k. From $\psi^3\delta = 1 + \varepsilon$ we then learn $y^{3^n} = z^{3^n} = 0$. Now use the splitting principle, i.e., write $\delta = \lambda_1 + \lambda_2$ and set $t_i = c_1(\lambda_i)$. Then

$$c_1(\psi^2\delta) = c_1(\lambda_1^2 + \lambda_2^2) = [2](t_1) + [2](t_2) = (-t_1 +_F [3](t_1)) + (-t_2 +_F [3](t_2))\,.$$

Since we may calculate modulo y^{3^n} and z^{3^n}, we conclude $y = c_1(\psi^2\delta) = -y$, hence $y = 0$. A similar calculation gives

$$c_2(\lambda_1^2 + \lambda_2^2) \equiv c_2(\lambda_1 + \lambda_2) - c_2(\lambda_1 + \lambda_2)^{(3^n+1)/2} \mod (c_1, c_2^{3^n})\,,$$

hence $z^{(3^n+1)/2} = 0$. In summary:

Proposition 2.5. *Let $p = 3$ and z denote the Euler class of the two-dimensional irreducible representation of Σ_3. Then $K(n)^*(B\Sigma_3) \cong K(n)^*[z]/z^{(3^n+1)/2}$.* □

3 Transfer

When H is a finite index subgroup of G, the restriction map $BH \to BG$ is a finite covering, and for such maps one has a stable transfer map $BG_+ \to BH_+$ inducing the transfer homomorphism

$$\mathrm{Tr}_H^G\colon E^*(BH) \longrightarrow E^*(BG)$$

for any cohomology theory E. Being induced by a geometric map, the transfer homomorphism is natural.

Naturality is at the base of all formal properties enjoyed by the transfer, of which we mention but the two of most interest to us: Frobenius reciprocity and the double coset formula.

Both are consequences of a simple observation: let H and K be subgroups of the group G, which we assume to be finite (finite index would suffice). The set $G/H \times G/K$ is a G-set with diagonal G-action, and as such decomposes as

$$G/H \times G/K = \coprod_{KgH} G/(K \cap H^g)$$

where g runs over a set of double coset representatives of $K\backslash G/H$. Since the functor $EG \times_G -$ preserves pullbacks up to homotopy, this decomposition yields a pullback diagram

$$\begin{array}{ccc} \coprod_{KgH} BK \cap H^g & \xrightarrow{c_g \circ \mathrm{Res}} & BH \\ \downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle \mathrm{Res}} \\ BK & \xrightarrow[\mathrm{Res}]{} & BG \end{array}$$

Here c_g denotes the map induced by conjugation with g.
Applying a cohomology functor E to the diagram yields, using naturality of the transfer, the double coset formula

$$\mathrm{Res}^G_K \mathrm{Tr}^G_H = \sum_{KgH} \mathrm{Tr}^K_{K\cap H^g} \mathrm{Res}^{H^g}_{K\cap H^g} c_g \, .$$

Frobenius reciprocity is the special case

$$\begin{array}{ccc} BH & \longrightarrow & BG \\ \downarrow & & \downarrow{\scriptstyle \Delta} \\ B(G \times H) & \longrightarrow & BG \times BG \end{array}$$

giving

$$\mathrm{Tr}^G_H(\mathrm{Res}^G_H(x) \cdot y) = x \cdot \mathrm{Tr}^G_H(y) \, ,$$

i.e., the transfer is a map of $E^*(BG)$-modules.
Another special case occurs for $H, K \leq G$, when $H \times K$ and ΔG (the diagonal subgroup) are considered as subgroups of $G \times G$. Then the double coset decomposition of $G \times G$ as left $H \times K$-set and right ΔG-set gives rise to the pullback diagram

$$\begin{array}{ccc} \coprod_{g_i} BG_i & \longrightarrow & BG \\ \downarrow & & \downarrow{\scriptstyle \Delta} \\ BH \times K & \longrightarrow & BG \times BG \end{array} \tag{3.1}$$

with G_i of the form $\Delta G \cap (H \times K)^{g_i}$ where again g_i runs over a set of double coset representatives of $\Delta G\backslash(G \times G)/(H \times K)$.

An influential but now disproved conjecture of Hopkins, Kuhn, and Ravenel claimed that the Morava K-theory of classifying spaces was concentrated in even degrees. This statement is equivalent to Morava E-theory being concentrated in even degrees and torsion-free. A slightly sharper conjecture by the above authors claimed that finite groups are 'good' in the sense of the following definition (taken from [HKR]):

Definition 3.1. *Let G be a finite group.*

(a) *An element $x \in K(n)^*(BG)$ is called good if there exists a subgroup $H \leq G$ and a complex representation ρ of H such that $x = \mathrm{Tr}_H^G(e(\rho))$.*

(b) *G is called good if $K(n)^*(BG)$ has a basis consisting of good elements.*

It is observed in [HKR, Proposition 7.2] that additive generation by transferred Euler classes is equivalent to multiplicative generation by such classes. This follows from diagram (3.1) above.

Later on, Bakuradze and Priddy [BP1, BP2] studied the multiplicative structure of $K(n)^*(BG)$ using the transfer: Suppose one knew that $K(n)^*(BG)$ was 'good' in the above sense. One might then ask which of the relations between the generating transfered Euler classes are consequences of formal properties of the transfer.
In particular, they came up with a formula expressing transfers of Chern classes of a representation of an index p subgroup in terms of Chern classes of the induced representation on the Euler class of the generator of the representation ring of the quotient C_p.
Their result can be summarised as follows (for details we refer to the original sources):

Theorem 3.2 (Bakuradze-Priddy [BP2])**.** *Let G be a finite group and H a normal subgroup of index p. Let ε be a generator of the complex representation ring of C_p, and z its Euler class. Finally, let η be a complex representation of H, and $\rho = \mathrm{Ind}_H^G(\eta)$. Then there is an identity*

$$c_k(\rho) - \mathrm{Tr}_H^G(\omega_k(\eta)) = A_k^{(n)}(z^{p-1}), c_p(\rho), \ldots, c_{np}(\rho))$$

where $A_k^{(n)}$ is an explicitly given polynomial, and $\omega_k(\eta)$ is a certain polynomial in the Chern classes of η (defined by way of universal example).

4 The Serre spectral sequence

The Serre spectral sequence associated to a group extension is probably the first tool coming to mind when attempting calculations. Suppose given a group extension

$$1 \to N \longrightarrow G \longrightarrow Q \to 1\,,$$

then one has the Serre spectral sequence

$$E_2 = H^*(BQ; K(n)^*(BN)) \Longrightarrow K(n)^*(BG)\,.$$

Here $H^*(BQ; K(n)^*(BN))$ is the ordinary cohomology of Q with coefficients in the $\mathbb{F}_p[Q]$-module $K(n)^*(BN)$, the action of Q being induced by conjugation in

G as usual. This module structure can be quite messy, even in the simplest cases: suppose N is an elementary abelian group V of rank r with a linear action of Q. Then $K(n)^*(BV) \cong K(n)^*[x_1, \ldots, x_r]/(x_i^{p^n})$, where the x_i are the Euler classes of a set of generators for the complex representation ring of V. Thus the action can be calculated in terms of tensor products of line bundles, and this involves the formal group law. Such actions were considered in [LS], in particular with regard to the question whether such modules are permutation modules (the relevance of which shall be explained later). In Chapter VI we reproduce some of this material.
Two "extreme" cases thus come to mind:

(1) extensions with trivial action, such as central extensions, and

(2) extensions with quotient C_p.

Case (1) has the drawback that the quotient is usually a large (p-)group, whose mod p cohomology poses a real problem, whereas in (2) we have to determine the C_p-module $K(n)^*(BN)$, which typically means intricate calculations involving the formal group law. Nevertheless both strategies are useful in special cases. A non-trivial example for (1) is the extraspecial group of order 32 (see Chapter V), the wreath product theorem III.2.1 is an easy calculation using (2). The Serre spectral sequence for extensions of type (2) was also used by Ravenel to give an inductive proof of finite generation [R]. We shall give a slightly different argument below.

In [K], Igor Kriz proved a beautiful theorem about the Serre spectral sequence associated to fibrations over BC_p. This theorem is one of the few practical tools for calculation; Kriz used it to great effect to supply the first counterexample to the even degree conjecture. His result gives a useful criterion to decide whether a group G has even Morava K-theory. For odd primes p, it may be simply stated as follows: suppose H is a normal subgroup of index p in G. The quotient $G/H \cong C_p$ acts on H and thus on $K(n)^*(BH)$. Suppose $K(n)^{\mathrm{odd}}(BH) = 0$, then the same is true for G if and only if $K(n)^*(BH)$ is a permutation module for G/H. For $p = 2$ this is trivially false (all $\mathbb{F}_2[C_2]$-modules are permutation modules), but the statement is true if one replaces mod p Morava K-theory with the integral variant $\widetilde{K}(n)$.

The Serre spectral sequence for bundles over BC_p. We begin by recalling the mod p cohomology of the cyclic group $C_p = \langle t \mid t^p \rangle$. In the group ring $\mathbb{Z}[C_p]$ there are two special elements, namely $D := t - 1$ and $N := 1 + t + t^2 + \cdots + t^{p-1}$. Then

$$\cdots \longrightarrow \mathbb{Z}[C_p] \xrightarrow{D} \mathbb{Z}[C_p] \xrightarrow{N} \mathbb{Z}[C_p] \xrightarrow{D} \mathbb{Z}[C_p] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

is a $\mathbb{Z}[C_p]$-free resolution of $\mathbb{Z}$. Let M be a $\mathbb{Z}[C_p]$-module, then one has

$$H^i(C_p; M) = \begin{cases} M^{C_p} = \ker(D) & i = 0; \\ \ker D / \operatorname{Im} N & i \equiv 0\ (2)\,,\ i > 0; \\ \ker N / \operatorname{Im} D & i \equiv 1\ (2)\,. \end{cases}$$

Theorem 4.1. *Let* $F \xrightarrow{\pi} E \xrightarrow{f} BC_p$ *be a fibration with* $\widetilde{K}(n)^*(F)$ *finitely generated over* $\widetilde{K}(n)^*$.

(a) The Serre spectral sequence

$$E_2^{s,t} = H^s(BC_p; \widetilde{K}(n)^*(F)) \Longrightarrow \widetilde{K}(n)^*(E) \tag{4.1}$$

has only finitely many differentials.

(b) $\widetilde{K}(n)^*(E)$ *is a finitely generated* $\widetilde{K}(n)^*$*-module.*

Proof. Let $y \in H^2(BC_p; \mathbb{Z})$ be a generator for the integral cohomology of C_p. Then $A := \widetilde{K}(n)^* \otimes \mathbb{Z}[y]$ is the $E_2 = E_\infty$ page of the Atiyah-Hirzebruch spectral sequence for BC_p. The fibration $F \to E \to BC_p$ maps to the fibration $* \to BC_p \to BC_p$, turning $E_r^{*,*}$ into an A-module. The class y is a permanent cycle since it is represented by $f^*(x)$, the image under f^* of the generator of $\widetilde{K}(n)^*(BC_p) = \widetilde{K}(n)^*[x]/(px - v_n x^{p^n})$. By assumption, $\widetilde{K}(n)^*(F)$ is finitely generated. Now $y\colon E_2^{i,*} \to E_2^{i+2,*}$ is an isomorphism for $i > 0$, whence $E_r^{*,*}$ is a finitely generated A-module, since A is noetherian. Thus the chain of the A-submodules of boundaries

$$0 = B_2^{*,*} \subset B_3^{*,*} \subset \cdots \subset B_r^{*,*} \subset \ldots B_\infty^{*,*} \subset E_\infty^{*,*}$$

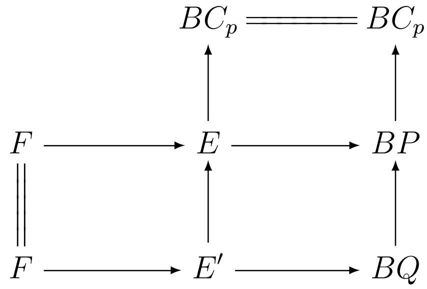
becomes stationary. This shows (a). Furthermore, since $E_2^{*,*}$ is finitely generated over A, so is $E_\infty^{*,*}$. Standard commutative algebra implies that $\widetilde{K}(n)^*(E)$ is finitely generated over $\widetilde{K}(n)^*(BC_p)$ and thus over $\widetilde{K}(n)^*$. □

As immediate consequence we obtain Ravenel's finite generation theorem for classifying spaces of finite groups [R]:

Corollary 4.2 (Ravenel)**.** *Let G be a finite group. Then* $\widetilde{K}(n)^*(BG)$ *is a finitely generated* $\widetilde{K}(n)^*$*-module.*

Proof. Let P be a Sylow p-subgroup of G. Since p-locally, BG is a stable summand of BP, it is enough to prove this for p-groups. Since any p-group has a normal subgroup of index p, the corollary follows from the theorem by induction. □

Ravenel's original formulation was that $K(n)^*(BG)$ has finite rank over $K(n)^*$. Part (b) of the theorem above easily generalises to fibrations over classifying spaces of finite p-groups, and even further. Let P be a finite p-group and $F \to E \to BP$ a fibration. Let Q be a normal index p subgroup of P; consider the diagram of fibrations



Then $\widetilde{K}(n)^*(E)$ is finitely generated if $\widetilde{K}(n)^*(E')$ is, by the theorem, and we can work by induction on the order of P. Thus

Corollary 4.3. *Let P be a finite p-group and $F \to E \to BP$ a fibration. If $\widetilde{K}(n)^*(F)$ is finitely generated over $\widetilde{K}(n)^*$, the so is $\widetilde{K}(n)^*(E)$.* □

Returning to fibrations over BC_p, let E and F be as in Theorem 4.1. Then $\pi\colon F \to E$ is a p-fold covering, with t, say, generating the group of deck transformations. The transfer map for this covering induces a homomorphism

$$\mathrm{Tr}\colon \widetilde{K}(n)^*(F) \to \widetilde{K}(n)^*(E)$$

of $\widetilde{K}(n)^*(E)$-modules, since $\mathrm{Tr}(\pi^*(x)\cdot y) = x \cdot \mathrm{Tr}(y)$ by Frobenius reciprocity. In particular, one has $\mathrm{Tr}(\pi^*(x)) = x \cdot \mathrm{Tr}(1)$. As the transfer commutes with deck transformations, so does Tr. Quillen [Q2] gives a formula for Tr(1):

Lemma 4.4 (Quillen)**.** $\mathrm{Tr}(1) = p - v_n x^{p^n-1} \in \widetilde{K}(n)^*(E)$.

Proof. By Frobenius reciprocity, it suffices to consider $E = BC_p$ and $F = *$. Then since $\pi^*(x) = 0$, one computes

$$0 = \mathrm{Tr}(\pi^*(x)\cdot 1) = x \cdot \mathrm{Tr}(1)\,.$$

On the other hand, $\mathrm{Tr}(1) \equiv p$ modulo higher skeletal filtration, whence

$$\mathrm{Tr}(1) = \frac{[p](x)}{x}\,.$$

More precisely, $x\,\mathrm{Tr}(1) = 0$ implies that there is a power series $f(x)$ with

$$\mathrm{Tr}(1) = f(x)\frac{[p](x)}{x} = f(0)\frac{[p](x)}{x}$$

where the last equality holds since multiplication by x annihilates the expression. Now the two conditions $\mathrm{Tr}(1) \equiv p \bmod x$ and $[p](x) \equiv px \bmod x^2$ imply $f(0) = 1$. □

Since in ordinary cohomology, the transfer is given on chain level as the action by the sum of all deck transformations,, one has a commutative diagram

$$\begin{array}{ccccccc} \widetilde{K}(n)^*(F) & \xrightarrow{\mathrm{Tr}} & \widetilde{K}(n)^*(E) & \twoheadrightarrow & E_\infty^{0,*} & \hookrightarrow & E_2^{0,*} \\ {\scriptstyle N}\downarrow & & & & & & \Vert \\ \widetilde{K}(n)^*(F) & & \longleftarrow & & & \hookleftarrow & H^0(BC_p; \widetilde{K}(n)^*(F)) \end{array}$$

i.e., in the spectral sequence (4.1) the transfer represents the norm map. In particular, the elements in the image of the norm are permanent cycles.

Lemma 4.5. *Suppose* $K(n)^{odd}(E) = 0$. *Let* $x \in \widetilde{K}(n)^*(F)$ *be an element with* $N(x) = 0$. *Then* $\mathrm{Tr}(x) = 0$.

Proof. If $N(x) = 0$, then x represents an element $[x] \in H^1(BC_p; \widetilde{K}(n)^*(F))$. By assumption on E, this class is p-torsion, hence $px = (1-t)y$ for a suitable y. Since the transfer commutes with deck transformations, one obtains

$$0 = \mathrm{Tr}(y - ty) = \mathrm{Tr}(px) = p\,\mathrm{Tr}(x)\,,$$

but $\widetilde{K}(n)^*(E)$ is p-torsion free. □

Since $\mathrm{Im}(N)$ injects into $E_r^{0,*}$ for any r, the Serre spectral sequence has a quotient spectral sequence

$$\bar{E}_r^{p,q} = \begin{cases} E_r^{p,q}/\,\mathrm{Im}(N) & \text{for } p = 0, \\ E_r^{p,q} & \text{for } p > 0, \end{cases} \tag{4.2}$$

This spectral sequence is multiplicative since $\mathrm{Im}(\mathrm{Tr})$ and $\mathrm{Im}(N)$ are ideals by Frobenius reciprocity. If furthermore $K(n)^{\mathrm{odd}}(E) = 0$, Lemma 4.5 implies that it converges to $\widetilde{K}(n)^*(E)/\,\mathrm{Im}(\mathrm{Tr})$.

Remark. The E_2-page of this quotient spectral sequence is just the cohomology of C_p made periodic, i.e., Tate cohomology $\hat{H}^*(C_p; \widetilde{K}(n)^*(F))$.

Kriz's theorem. In the rest of this section we shall discuss

Theorem 4.6 (Kriz [K]). *Let* $F \to E \to BC_p$ *be a fibration with* $K(n)^*(F)$ *finitely generated and* $K(n)^{odd}(E) = 0$. *Then* $H^1(BC_p; \widetilde{K}(n)^{2*}(F)) = 0$.

We break down the proof into a series of lemmas; the arguments closely follow the original in [K].
Below $y \in H^2(BC_p)$ denotes a generator of integral cohomology (as in the proof of Theorem 4.1), $[x]$ shall always denote an E_∞-representative of a class $x \in \widetilde{K}(n)^*(E)$.

Lemma 4.7. *Let $x \in \widetilde{K}(n)^*(E)$ be of filtration s but not $s+1$ (such that $[x] \neq 0$ in $\bar{E}_\infty^{s,*}$). Let $z = px - \mathrm{Tr}(\pi^*(x))$.*

(a) If $y^{p^n-1}[x]$ is non-zero in $\bar{E}_\infty$, then $[z] = y^{p^n-1}[x]$.

(b) If $y^{p^n-1}[x] = 0$, then z has filtration at least $s + |v_n|$.

Proof. This follows from $\mathrm{Tr}(\pi^*(x)) = x \cdot \mathrm{Tr}(1) = x(p - y^{p^n-1})$. □

Lemma 4.8. *The map $\bar{E}_r^{i,*} \xrightarrow{y} \bar{E}_r^{i+2,*}$ given by multiplication by y is*

(i) onto, and

(ii) injective for $i \geq r$.

Proof. Induction on r. For $r = 2$ multiplication by y is an isomorphism (periodicity). Suppose the claim holds for r. Let $\alpha \in \bar{E}_r$ be a cycle for d_r and $\bar{\alpha}$ its image in $\bar{E}_{r+1}$. Then $\alpha = y\beta$ for some β with $yd_r(\beta) = 0$. Since β must have higher filtration, part (ii) of the inductive hypothesis implies $d_r(\beta) = 0$, whence $\bar{\alpha} = y\bar{\beta}$; this shows (i). For (ii), suppose $i \geq r+1$ and $\bar{\alpha} \in \bar{E}_{r+1}^{i,*}$ is an element annihilated by y. Then $y\alpha = d_r(\beta)$ for some $\beta \in \bar{E}_r^{i+2-r,*}$. Since $i + 2 - r > 1$, there is a γ with $\beta = y\gamma$, hence $y\alpha = d_r(\beta) = yd_r(\gamma)$. Now α has filtration larger than r, and multiplication by y is injective in this range by (ii). Thus $\alpha = d_r(\gamma)$. □

In other words, $\bar{E}_r$ is generated as $\widetilde{K}(n)^*[y]$-module in degrees 0 and 1, with relations in filtrations at most r.

Lemma 4.9. *Let $2 \leq s < r$, and $\alpha \in \bar{E}_s^{2i+1,*}$ with $d_s(\alpha) \neq 0$. Then there exists k, $1 \leq k \leq r-1$, and a non-zero class $\beta \in \bar{E}_r^{k,k+2*}$ with $y\beta = 0$ in $\bar{E}_r$.*

Proof. Induction on r. The statement is trivial for $r = 2$, so assume the lemma for r, and let $2 \leq s < r+1$. Suppose first that $s < r$. By induction hypothesis, there exists a non-zero element $\beta' \in \bar{E}_r^{k,k+2*}$ with $y\beta' = 0$, hence $yd_r(\beta') = 0$, and $d_r(\beta') = 0$ by Lemma 4.8. Then $\beta = \bar{\beta}' \neq 0$ since β' cannot be a boundary for degree reasons, and $y\beta = 0$.
If $s = r$ then dividing by y as often as possible (cf. Lemma 4.7) we may assume $\alpha \in \bar{E}_s^{1,2*}$ with $d_r(\alpha) \neq 0$. Then $d_r(\alpha) = y\beta$ for some $\beta \in \bar{E}_r^{r-1,2*}$ with $yd_r(\beta) = d_rd_r(\alpha) = 0$. Since $d_r(\beta)$ is in filtration $2r-1 > r$, Lamma 4.8 (ii) implies $d_r(\beta) = 0$. Thus $\bar{\beta} \in \bar{E}_{r+1}$ is defined, non=zero since it is in filtration $r-1$, and $y\bar{\beta} = 0$. □

The lemma says that a nontrivial differential on a class of odd filtration produces y-torsion. The idea of proof for the theorem consists in the observation that y-torsion ultimately produces p-torsion.

Proof of the theorem. Suppose $\mu \in H^1(C_p; \widetilde{K}(n)^{2t}(F_+))$ is a non-zero class. Then there is an s with $\mu \in \bar{E}_s^{1,2t}$ and $d_s\mu \neq 0$. Let $r > s$ with $\bar{E}_r^{*,*} = \bar{E}_\infty^{*,*}$. By Lemma 4.9 there is a non-zero $\alpha \in \bar{E}_r^{k,k+2*}$ with $y\alpha = 0$. Let α represent $x \in \widetilde{K}(n)^*(E)$, then by Lemma 4.8 (ii) the class $z_0 = px - \mathrm{Tr}(\pi^*(x))$ has filtration $k_1 > k + 2(p^n - 1)$. Let $[z_0] = y^{p^n-1}\gamma_1 \in \bar{E}_r^{k_1,*}$ where $\gamma_1 \in \bar{E}_r^{k_1-2(p^n-1),*}$. Since $\bar{E}_r = \bar{E}_\infty$, γ_1 is represented by a class $c_1 \in \widetilde{K}(n)^*(E)$. Now let

$$z_1 = pc_1 - \mathrm{Tr}(\pi^*(c_1))\,.$$

By Lemma 4.8, z_1 also represents by $y^{p^n-1}\gamma_1$; thus

$$z_0 - z_1 = p(a - c_1) - \mathrm{Tr}(\pi^+(a - c_1))$$

has filtration k_2 strictly larger than k_1. Iterating this procedure, one inductively finds classes $c_i \in \widetilde{K}(n)^*(E)$ of increasing filtration such that

$$p \cdot (x - \textstyle\sum c_i) - \mathrm{Tr}(\pi^*(x - \textstyle\sum c_i)) = 0\,.$$

(The infinite sum of the c_i's converges since $\widetilde{K}(n)^*(E)$ is complete with respect to the skeletal filtration.) Now $z := x - \sum c_i$ has positive filtration whereas the image of the transfer sits in filtration zero, implying that pz has to be zero. On the other hand, z is not zero since $\sum c_i$ has higher filtration than x, by construction. Thus we have produced a p-torsion class in $\widetilde{K}(n)^*(E)$, contradicting the even-dimensionality of $K(n)^*(E)$. □

The following result is stated in [K] without a reference; it follows easily from the classification of indecomposables.

Proposition 4.10. *Let $R = \mathbb{Z}_{(p)}$ or $R = \mathbb{Z}_p$ and M be a finitely generated R-free $R[C_p]$-module. Then the following are equivalent:*

(a) *M is a permutation module;*

(b) *$H^1(C_p; M) = 0$.*

Proof. There are three isomorphism classes of indecomposable $R[C_p]$-lattices, namely the trivial module T, the regular module F, and the reduced regular module, $\tilde{F}$, say; see [CR, p. 690 and §34B]. For the first two, H^1 clearly vanishes. Furthermore, $H^2(C_p; T) \cong \mathbb{Z}/p$, and $H^i(C_p; F) = 0$ for $i > 0$. The long exact sequence in cohomology associated to the short exact sequence $0 \to T \to F \to \tilde{F} \to 0$ gives $H^1(C_p; \tilde{F}) \cong \mathbb{Z}/p$. □

Remark. For p odd, it furthermore transpires that a torsion free $\mathbb{Z}_p[C_p]$ module is a permutation module if and only if its mod p reduction is a permutation module. (This is clearly false for $p = 2$.)

5 Central products and the Rothenberg-Steenrod spectral sequence

Let p be a prime and denote by k the field of p elements. Suppose G is a p-group which is expressible as the central product of two subgroups P and Q, ,*i.e.*, there is a central subgroup Z of G contained in $P \cap Q$ and a central extension

$$1 \to Z \xrightarrow{\Delta} P \times Q \longrightarrow G \to 1$$

where Δ is the diagonal inclusion of Z. We shall also write $P \times_Z Q$ or just $P \circ Q$ to denote this central product.
$K(n)^*(BZ)$ is a $K(n)^*$-coalgebra by virtue of the group product, and $K(n)^*(BP)$ and $K(n)^*(BQ)$ become comodules over $K(n)^*(BZ)$ via the multiplication maps

$$\mu_1\colon P \times Z \to P \quad \text{and} \quad \mu_2\colon Z \times Q \to Q\,.$$

By definition,

$$P \times Z \times Q \underset{1\times\mu_2}{\overset{\mu_1\times 1}{\rightrightarrows}} P \times Q \xrightarrow{\quad\mu\quad} G$$

is a coequalizer in the category of groups. Applying $K(n)^*$ to this coequalizer yields a diagram

$$\begin{array}{c} K(n)^*(BG) \\ \big\downarrow{\scriptstyle \mu^*} \\ K(n)^*(BP) \otimes K(n)^*(BQ) \\ {\scriptstyle 1\otimes\mu_2^*}\big\downarrow\;\big\downarrow{\scriptstyle \mu_1^*\otimes 1} \\ K(n)^*(BP) \otimes K(n)^*(BZ) \otimes K(n)^*(BQ)\,. \end{array}$$

This certainly will fail to be an equalizer, since the inflation map is generally not injective. Note though that by definition, the equalizer of the above parallel arrows is the cotensor product $K(n)^*(BP)\square_{K(n)^*(BZ)}K(n)^*(BQ)$. Recall that if Γ is a coalgebra, (M_1, ψ_2) a right Γ-comodule, and (M_2, ψ_2) a left Γ-comodule, their cotensor product $M_1 \square_\Gamma M_2$ is by definition the kernel of the map

$$M_1 \otimes M_1 \xrightarrow{\psi_1\otimes 1 - 1\otimes\psi_2} M_1 \otimes \Gamma \otimes M_2\,.$$

By construction, the image of inflation is contained in the cotensor product, and it is tempting to ask when it is all of it.
We may regard BZ as a topological group acting on $B(P \times Q)$ with orbit space BG. The Rothenberg-Steenrod spectral sequence of this principal fibration is then a spectral sequence of algebras

$$E_2 = \mathrm{Cotor}_{K(n)^*(BZ)}(K(n)^*(BP), K(n)^*(BQ)) \Longrightarrow K(n)^*(BG)\,.$$

This spectral sequence is concentrated in the right half plane; the zero column is just the cotensor product

$$K(n)^*(BP)\Box_{K(n)^*(BZ)}K(n)^*(BQ))\,.$$

For details see [P], or [V, McC].
Write Γ for $K(n)^*(BZ)$ and ψ_Γ for the comultiplication of Γ. Since we are working over a field, all modules are flat (Γ, in particular) and we may calculate Cotor_Γ using resolutions by extended comodules, *i.e.*, comodules of the form $(\Gamma\otimes V,\psi_\Gamma\otimes 1)$ for some $K(n)^*$-module V. (Unadorned tensor products shall always be over $K(n)^*$.) Now let M be a left Γ-comodule and

$$M \xrightarrow{\;i\;} \Gamma\otimes V_0 \xrightarrow{\;\partial_0\;} \Gamma\otimes V_1 \xrightarrow{\;\partial_1\;} \Gamma\otimes V_2 \longrightarrow \cdots$$

be a resolution of M by extended comodules. Then for any right Γ-comodule M',

$$\mathrm{Cotor}^i_\Gamma(M',M) = H^i(M'\Box_\Gamma(\Gamma\otimes V_\bullet))\,.$$

Using the isomorphism $M\Box_\Gamma(\Gamma\otimes V)\overset{\cong}{\to} M\otimes V$ given by $m\otimes r\otimes v\mapsto \varepsilon(r)m\otimes v$ (the reverse isomorphism being given by $m\otimes v\mapsto \psi_M(m)\otimes v$), the above complex for calculating Cotor simplifies to $M'\otimes V_\bullet$.
Sometimes it is easier to dualise everything and work with $K(n)$ homology, although that might only be due to being more used to modules rather than comodules. In order to do that, one needs to know the Morava K-homology of abelian groups as an algebra. Obviously, it is enough to know $K(n)_*(BC)$ for C a cyclic group of prime power order. For example, it is easy to caclulate $K(n)_*(BC_p)$: Recall $K(n)^*(BC_p)\cong K(n)^*[x]/(x^{p^n})$ where x is the Euler class of a generator of the (oridinary) character ring. The coproduct being given by the formal group law, one obtains from Lemma I.3.1:

Theorem 5.1 ([RW], Theorem 5.7)**.** *Let $\xi_{(i)}=\xi_{p^i}$ be dual to x^{p^i}. Then*

$$K(n)_*(BC_{p^m})\cong K(n)_*[\xi_{(i)}\,,\ 0\le i<nm]/\big(\xi^p_{(n+i-1)}-v_n^{p^i}\xi_{(i)}\big)$$

where $\xi_{(i)}=0$ for $i<0$. □

For example, if $n>1$ one has

$$K(n)_*(BC_p)\cong K(n)_*[\xi_p,\xi_{p^2},\dots,\xi_{p^{n-1}}]/(\xi_p^p,\xi_{p^2}^p,\dots,\xi^p_{p^{n-2}},\xi^{p^2}_{p^{n-1}})\,.$$

Unfortunately, we have not been able to use this spectral sequence for effective calculation (yet). Later we shall present an example showing that the spectral sequence is highly non-collapsing even in simple cases. At this point we only offer the following essentially trivial example:

Central products with abelian groups. Let H be a p-group and $\langle h \rangle$ a central element of order p. Let G be the central product of H with $C_{p^m} = \langle t \rangle$, $m \geq 2$, obtained by identifying $t^{p^{m-1}}$ with h. To apply the Rothenberg-Steenrod spectral sequence, we need to know $K(n)^*(BC_{p^m})$ as a $K(n)^*(BC_p)$-comodule, but this is very simple: again from Lemma I.3.1 one obtains

Lemma 5.2. *$K(n)^*(BC_{p^m}) \cong K(n)^*[z]/(z^{p^{mn}})$ is an extended $K(n)^*(BC_p)$-comodule. More precisely,*

$$K(n)^*(BC_{p^m}) \cong K(n)^*(BC_p) \otimes K(n)^*\{a_i \mid 0 \leq i < p^{(m-1)n}\}$$

as comodules, where $a_i = z^{ip^n}$

Proof. It is immediate from Lemma I.3.1 that the a_i are comodule primitives. □

Remark. By contrast, the analogous statment for ordinary cohomology is false.

Corollary 5.3. *The Rothenberg-Steenrod spectral sequence*

$$\mathrm{Cotor}_{K(n)^*(BC_p)}(K(n)^*(BH), K(n)^*(BC_{p^m})) \Longrightarrow K(n)^*(BG)$$

collapses. □

Thus $K(n)^*(BG)$ is a module of rank $p^{(m-1)n}$ over $K(n)^*(BH)$. In particular, if H has even Morava K-theory, so does G.

Remark. This last corollary also follows easily from Theorem 4.6, working by induction on m: one can build up G by successive extensions of the form $H' \to G' \to C_p$ where the quotient C_p acts trivially on H'.

6 A brief introduction to generalised characters

Generalised character theory was invented by Hopkins, Kuhn, and Ravenel in their seminal paper [HKR]. For suitable complex oriented theories E (in a sense to be made precise below) it relates the E-cohomology of the classifying space BG of a finite group G to the ring of "generalised class functions" with values in a suitably chosen E^*-algebra L, i.e., functions on the set $C_{n,p}(G)$ of commuting n-tuples of elements of p-power order in G which are constant on conjugacy classes of n-tuples.

The account we shall give below is rather terse: we do not think we can improve on the exposition in the original source, which we highly recommend to the reader. In Chapter VII we shall present a generalisation to certain classes of discrete groups. This is not really new, all the necessary ingredients are in [HKR].

6.1 Detection by abelian subgroups

Let G be a finite group. Before describing the characters, we recall another remarkable theorem from [HKR] which says that under favourable circumstances, the E-cohomology of BG is detected on abelian subgroups. This will e.g. be the case when we can prove that $\widetilde{K}(n)^*(BG)$ is torsion free; thus our emphasis on trying to prove $K(n)^{\mathrm{odd}}(BG) = 0$ in as many cases as possible.
Denote by $\mathcal{A}(G)$ the category with objects the abelian subgroups of G and morphism sets $\mathcal{A}(G)(A, B) = \mathrm{Map}^G(G/B, G/A)$. If Y is a G-space, then a morphism $G/B \to G/A$ induces maps $G \times_N Y \to G \times_A Y$ and $Y^A \to Y^B$. Applying a complex oriented cohomology functor E^* yields a compatible family of maps $E^*(EG \times_G Y) \to E^*(EG \times_A Y)$, thus a homomorphism to the inverse limit over the category $\mathcal{A}(G)$, and furthermore a map from the limit to the end $\int_{A\in\mathcal{A}(G)} E^*(BA \times Y^A)$. Then the following theorem is proved in [HKR]:

Theorem 6.1 ([HKR], Theorem A)**.** *Let E be a complex oriented cohomology theory. Then for any finite group G and any finite G-CW-complex Y, the natural maps*

$$E^*(EG \times_G Y) \to \lim_{A\in\mathcal{A}(G)} E^*(EG \times_A Y) \to \int_{A\in\mathcal{A}(G)} E^*(BA \times Y^A)$$

become isomorphisms upon inverting the order of G. □

Taking Y to be a point immediately gives

Corollary 6.2 ([HKR])**.** *If $E^*(BG)$ is torsion free, the natural map*

$$E^*(BG) \longrightarrow \prod_{A\in\mathcal{A}(G)} E^*(BA)$$

is injective. □

In particular, integral Morava K-theory of BG is detected on (maximal) abelian subgroups if it is torsion free. We shall make repeated use of this fact.

6.2 Characters

Suppose E is a complex oriented ring spectrum satisfying

(i) the coefficients E^* are a complete local ring with maximal ideal $\mathfrak{m}$ and residue characteristic p;

(ii) p is not a zero divisor in E^*;

(iii) the formal group law over the mod $\mathfrak{m}$ reduction has height n.

Furthermore, let $\mathcal{O}$ be the ring of integers in a complete algebraically closed local field L, and suppose we have a local homomorphism $E^* \to \mathcal{O}$. Consider the set $C_{n,p}(G)$ of n-tuples of commuting elements of p-power order in G. Such an n-tuple corresponds to a group homomorphism $g\colon \mathbb{Z}_p^n \to G$. Since G is finite, there is a minimal $i \in \mathbb{N}$ such that g factors through the quotient homomorphism $\mathbb{Z}_p^n \to (\mathbb{Z}/p^i)^n$; the resulting map $(\mathbb{Z}/p^i)^n \to G$ shall again be called g. For E satisfying the above conditions we have Künneth isomorphisms

$$E^*(B(\mathbb{Z}/p^i)^n) \cong E^*[\![x_1, \dots, x_n]\!]/([p^i](x_j)).$$

The generalised character associated to g is the map

$$\chi(g)\colon E^*(BG) \longrightarrow L$$

given as the composition

$$E^*(BG) \xrightarrow{g^*} E^*(B(\mathbb{Z}/p^i)^n) \cong E^*[\![x_1, \dots, x_n]\!]/([p^i](x_j)) \xrightarrow{\varphi_j} L\,,$$

where the maps φ_j form a compatible family of E^*-algebra morphisms sending the generators x_j to generators of the p^i-torsion subgroup of the maximal ideal of the ring of intergers in L. This torsion subgroup is known to be (abstractly) isomorphic as abelian group to $(\mathbb{Z}/p^i)^n$, by Lubin-Tate theory. The above maps assemble to the character homomorphism

$$\chi_G\colon E^*(BG) \longrightarrow \mathrm{Map}_G(\mathrm{Hom}(\mathbb{Z}_p^n, G), L) =: \mathrm{Cl}_{n,p}(G)$$

associating to a class $x \in E^*(BG)$ the map $\chi_x\colon \mathrm{Hom}(\mathbb{Z}_p^n, G) = C_{n,p}(G) \to L$ defined by $\chi_x(g) = \chi(g)(x)$; this is clearly invariant under the action of G by conjugation.

More generally, if Y is a finite G-CW-complex, consider the set

$$\mathrm{Fix}_{n,p}(G, Y) := \coprod_{\alpha \in \mathrm{Hom}(\mathbb{Z}_p^n, G)} Y^{\mathrm{Im}(\alpha)}\,;$$

this is a G-space via the action of G by conjugation. Now set

$$\mathrm{Cl}_{n,p}(G, Y; L) := L \otimes_{E^*} E^*(\mathrm{Fix}_{n,p}(G, Y), L))^G\,.$$

This object comes equipped with an action of $\mathrm{Aut}(\mathbb{Z}_p^n)$. Since $E^*(B(\mathbb{Z}/p^i)^n)$ is a free E^*-module, one has Künneth isomorphisms

$$E^*(B(\mathbb{Z}/p^i)^n \times Z) \cong E^*(B(\mathbb{Z}/p^i)^n) \otimes_{E^*} E^*(Z)$$

for all spaces Z. Thus the inclusion of fixed point sets together with the maps φ_j above combine to give the character map

$$E^*(EG \times_G Y) \longrightarrow \mathrm{Cl}_{n,p}(G, Y; L)^{\mathrm{Aut}(\mathbb{Z}_p^n)}\,.$$

Theorem 6.3 ([HKR], Theorem C)**.** *The generalised character map $\chi^G_{n,p}$ induces isomorphisms*

$$\chi^G_{n,p}\colon L\otimes_{E^*} E^*(EG\times_G Y)\to \mathrm{Cl}_{n,p}(G,Y;L)$$

and

$$\chi^G_{n,p}\colon p^{-1}E^*(EG\times_G Y)\to \mathrm{Cl}_{n,p}(G,Y;L)^{\mathrm{Aut}(\mathbb{Z}_p^n)}\,.$$

□

Remark. The account given here seems to depend on certain choices made for the maps φ_j. This can be avoided by properly saying what L should be; we have not done so in order to keep the exposition short and simple.

In many of our examples, G will have exponent p, whence it suffices to consider $i=1$. In that case we may work over an appropriate extension of the p-adic rationals, where the equation $p-x^{p^n-1}=0$ has a solution: the p-torsion subgroup of the height n Lubin-Tate formal group law consists of 0 and the solutions to this equation. If π is a uniformising element, then all other solutions are of the form $\zeta\pi$ for a (p^n-1)-st root of unity ζ, and calculations become very tractable.

6.3 The Euler characteristic formula

Finally, we record the formula for the Morava K-theory Euler characteristic. Here *Euler characteristic* means the difference between the ranks of the even and odd degree parts of $K(n)^*(BG)$:

$$\chi_{n,p}(G):=\mathrm{rank}_{K(n)^*}K(n)^{\mathrm{ev}}(BG)-\mathrm{rank}_{K(n)^*}K(n)^{\mathrm{odd}}(BG)\,;$$

by a theorem due to D. Ravenel, the rank of $K(n)^*(BG)$ over $K(n)^*$ is always finite if G is a finite group, and sometimes too when G is discrete; see Section 4.

Theorem 6.4 ([HKR], Theorem D)**.**

$$\chi_{n,p}(G)=\sum_{A<G}\frac{|A|}{|G|}\mu_{\mathcal{A}(G)}(A)\chi_{n,p}(A) \tag{6.1}$$

where the sum is over all abelian subgroups $A<G$ and $\mu_{\mathcal{A}(G)}$ is a Möbius function defined recursively by

$$\sum_{A<A'}\mu_{\mathcal{A}(G)}(A')=1 \tag{6.2}$$

where the sum is over all abelian subgroups $A'<G$ containing A.

In particular, $\mu_{\mathcal{A}(G)}(A)=1$ when A is maximal. It is easy to see that one only has to consider subgroups arising as intersections of maximal ones. Furthermore, one clearly has $\chi_{n,p}(A)=|A_{(p)}|^n$ where $A_{(p)}$ denotes the p-part of the abelian group A.

Part 2

Calculating Morava K-theory of classifying spaces

Chapter III

Calculations at any prime

There are not that many groups whose Morava K-theory is known. The purpose of this chapter is to give an overview of existing results, many of which are documented in the literature. We shall however in some cases indicate different proofs, and add a few more groups.

Apart from natural curiosity, one of the driving factors for such calculations was to either prove or disprove the so-called 'even-dimensionality conjecture', i.e., the claim that for any finite group G, $K(n)^{\mathrm{odd}}(BG)$ is always zero. With Kriz's construction of a counterexample [K] this search has come to an end; we shall give a brief description of his example in Section 6. Nevertheless, there are some families of groups for which the conjecture holds, and only in these cases complete calculations do exist.

Generally speaking, many successful calculations follow the same pattern: first try to calculate $K(n)^*(BG)$ additively, by means of a spectral sequence for a suitable extension. If it should turn out that $K(n)^*(BG)$ is 'good', meaning generated by (tranfers of) Chern classes, one can try to find the multiplicative structure using restriction to subgroups, Chern approximations, and formal properties of the transfer.

The computations are generally presented in increasing order of complication. We start in Section 1 with some calculations using only the Atiyah-Hirzebruch spectral sequence. This is really very inefficient, and one normally would not contemplate such an approach, but it does work in some simple cases. Next comes the wreath product theorem, due to Hunton and Hopkins-Kuhn-Ravenel. We present it here as an easy application of Theorem II.4.6. We also include a certain generalisation, due to Hunton, Leary and the author (unpublished).

We continue with a survey of results already in the literature: Yagita's theorem on groups of p-rank 2 (Section 3), Tanabe's theorem on linear groups away from the defining characteristic, the calculation of elementary-abelian by cyclic groups, due independently to Kriz and Yagita (Section 5).

We chose to redo some of the calculations, although they are mostly well-documented. Other known computations include 2-groups with a cyclic maximal subgroup [Sc]. We defer those to the next chapter, which deals exclusively with 2-primary cases.

In the last section we give an account of Kriz's counterexample.

1 Warm up: Calculations with the AHSS

The Atiyah-Hirzebruch spectral sequence is not really suitable for doing calculations, inasmuch it requires knowledge of the mod p cohomology of the group in question. The purpose of this section is to demonstrate that such computations can succeed for simple enough groups.

1.1 Abelian groups

If G is an abelian group, the calculation is essentially trivial: we can deduce the behaviour of the spectral sequence from the answer, obtained from the Gysin sequence. Using the Künneth theorem, we may immediately reduce to the case of a cyclic group C_{p^r} of order p^r. For odd primes or $r > 1$, its mod p cohomology is known to be $H^*(BC_{p^r}; \mathbb{F}_p) \cong \Lambda(x) \otimes \mathbb{F}_p[y]$ where $|x| = 1$, and $y = \beta_r(x)$ is the r-th Bockstein of x; for $p = 2$ and $r = 1$ identify y with x^2. Then y is the Euler class of a generator of the complex character ring of C_{p^r}.

Lemma 1.1. *The Atiyah-Hirzebruch spectral sequence for BC_p has only one differential, namely*

$$d_{2\cdot p^{nr}-1} x = v_n^{k_{n,r}} y^{p^{nr}}$$

with $k_{n,r} = (p^{nr} - 1)/(p^n - 1)$.

Proof. This is dictated by the relation $[p^r](y) = v_n^{k_{n,r}} y^{p^{nr}} = 0$ which arises from the Gysin sequence. □

For $r = 1$, this differential is none but $v_n Q_n$.

1.2 Dihedral groups at $p = 2$

We start with the dihedral group D_8 of order 8. This group has a presentation

$$\langle g_1, g_2 \mid g_i^2 = [g_1, g_2]^2 = 1 \rangle ,$$

we sometimes write c for the central element $[g_1, g_2]$.
We recall the representation theory of D_8: the representation ring $R(D_8)$ is generated by two one-dimensional representations γ_1, γ_2 and the standard real two-dimensional representation Δ defined as follows:

$$\gamma_i(g_j) = (-1)^{\delta_{ij}} , \quad \Delta(g_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \; \Delta(g_2) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} .$$

By abuse of notation, we shall use the same symbols to denote the complexified representations.
The mod 2 cohomology of D_8 is given by

$$H^*(D_8; \mathbb{F}_2) = \mathbb{F}_2[x_1, x_2, w_2]/(x_1 x_2) \qquad (1.1)$$

where $x_i = w_1(\gamma_i)$ are the Euler classes of γ_i and w_2 is the Euler class of Δ. Thus the E_2-page of the AHSS is

$$E_2 = H^*(D_8; K(n)^*) \cong \mathbb{F}_2[x_1, x_2, w_2]/(x_1 x_2) \otimes K(n)^* \,. \tag{1.2}$$

The first differential is $d_{2^{n+1}-1} = v_n \otimes Q_n$; so in the first step we have to calculate the Q_n-homology of $H^*(D_8; \mathbb{F}_2)$.
By instability (as a module over the Steenrod algebra), we have $Q_n x_i = x_i^{2^{n+1}}$, whereas $Q_n w_2$ may be computed using the Wu formula. Note that $w_1(\Delta) = x_1 + x_2$, we shall often write just w_1 to denote this class. Then $Sq^1(w_2) = w_1 w_2$ and $Sq^2 w_2 = w_2^2$ are the only non-trival Steenrod squares on w_2. Let p be the polynomial

$$p(u, v) := \sum_{j=0}^{n} u^{2^{n+1}-2^{j+1}+1} v^{2^j} \,.$$

By induction, one readily verifies $Q_n w_2 = p(w_1, w_2)$.
Next, since $H^*(D_8; \mathbb{F}_2)$ is detected on elementary abelian subgroups, no odd degree class in degree less than $\deg(Q_n) + 1 = 2^{n+1}$ can be a cycle for Q_n: this follows immediately from the AHSS for C_2. We proceed to calculate all cycles and boundaries.
Obviously, $Q_n w_2$ is a cycle, and so is the class $\zeta = p(x_1, w_2)$: since $x_1 w_1 = x_1^2$, we have

$$\begin{aligned} Q_n(\zeta) &= x_1^{2^{n+1}-2} w_2 + x_1^{2^{n+1}-1} Q_n(w_2) + \sum_{i=1}^{n} x_1^{2^{n+2}-2^{i+1}} w_2^{2^i} \\ &= x_1^{2^{n+1}-2} w_2 + x_1^{2^{n+1}-1} w_1^{2^{n+1}-1} w_2 \\ &\quad + \sum_{i=1}^{n} (x_1^{2^{n+1}-1} w_1^{2^{n+1}-2^{i+1}+1} + x_1^{2^{n+2}-2^{i+1}}) w_2^{2^i} \\ &= 0 \,. \end{aligned}$$

The cohomology of D_8 has an additive basis

$$\{x_1^i w_2^j \mid i, j \geq 0\} \cup \{x_2^k w_2^l \mid k > 0,\ l \geq 0\} \,.$$

Suppose y is an element of degree $2d + 1 > 2^{n+1}$. Then y may be written as $y_1 + y_2$ where $y_1 = \sum_{i=0}^{d} \lambda_i x_1^{2d+1-2i} w_2^i$ and $y_2 = \sum_{j=0}^{d} \mu_j x_2^{2d+1-2j} w_2^j$. We claim that if y_1 is in the kernel of Q_n, then there is a polynomial q in x_1^2 and w_2^2 such that $y_1 = \zeta \cdot q$. We have

$$\begin{aligned} Q_n y_1 &= \sum_{i=0}^{d} \lambda_i \left(x_1^{2d-2i+2^{n+1}} w_2^i + i \sum_{j=0}^{n} x_1^{2d+1-2i+2^{n+1}-2^{j+1}+2} w_2^{2^j+i-1} \right) \\ &= \sum_{k=0}^{[\frac{d}{2}]} \lambda_{2k} x_1^{2d-4k+2^{n+1}} w_2^{2k} + \sum_{k=0}^{[\frac{d-1}{2}]} \lambda_{2k+1} \sum_{j=0}^{n} x_1^{2d-4k+2^{n+1}-2^{j+1}} w_2^{2^j+2k} \end{aligned}$$

This is equal to zero if and only if the relations $\lambda_0 = 0$ and $\lambda_{2k} = \sum_{j=1}^{\nu_2(2k)} \lambda_{2k-2^j+1}$ (for $0 < k \le d$) hold. This leads to

$$\begin{aligned} y_1 &= \sum_{k=0}^{[\frac{d}{2}]} \lambda_{2k} x_1^{2d+1-4k} w_2^{2k} + \sum_{k=0}^{[\frac{d-1}{2}]} \lambda_{2k+1} x_1^{2d-4k-1} w_2^{2k+1} \\ &= \sum_{k=1}^{[\frac{d}{2}]} \left(\sum_{j=1}^{\nu_2(2k)} \lambda_{2k-2^j+1} \right) x_1^{2d+1-4k} w_2^{2k} + \sum_{k=0}^{[\frac{d-1}{2}]} \lambda_{2k+1} x_1^{2d-4k-1} w_2^{2k+1} \\ &= \zeta \cdot \sum_{k=0}^{[\frac{d-1}{2}]} \lambda_{2k+1} x_1^{2d-4k-2^{n+1}} w_2^{2k} . \end{aligned}$$

Similarly, if $Q_n y_2 = 0$ then y_2 is a product of $p(x_2, w_2) = \zeta + Q_n w_2$ with a polynomial in x_2^2 and w_2^2.
Next, the even degree cycles have to be squares: suppose

$$y = \sum_{i=0}^{d} (\lambda_i x_1^{2d-2i} + \mu_i x_2^{2d-2i}) w_2^i .$$

Then λ_{2i+1} is the coefficient of $x_1^{2d-4i+2^{n+1}-3} w_2^{2i+1}$ (respectively μ_{2i+1} the coefficient of $x_2^{2d-4i+2^{n+1}-3} w_2^{2i+1}$) in $Q_n y$:

$$Q_n \left(\sum_{i=0}^{d} \lambda_i x_1^{2d-2i} w_2^i \right) = \sum_{j=0}^{[\frac{d-1}{2}]} \lambda_{2j+1} x_1^{2d-4j-2} w_2^{2j} \left(x_1^{2^{n+1}-1} w_2 + f(x_1, w_2^2) \right) ,$$

where f is a poynomial in x_1 and w_2^2; similarly for x_2. Thus the cycles for Q_n are given by (note that $x_1^2 \zeta = x_1^2 Q_n w_2$ and $x_2^2 \zeta = 0$)

$$Z = \mathbb{F}_2[x_1^2, x_2^2, w_2^2]/(x_1^2 x_2^2)\{1, Q_n w_2\} \oplus \mathbb{F}_2[w_2^2]\{\zeta\} . \tag{1.3}$$

Now to the boundaries: Clearly the image of Q_n in odd degrees is given by

$$\mathbb{F}_2[x_1^2, x_2^2, w_2^2]/(x_1^2 x_2^2)\{Q_n w_2\} .$$

In even degrees, the classes $x_i^{2k+2^{n+1}} w_2^{2m} = Q_n(x_i^{2k+1} w_2^{2m})$ are obvious boundaries ($i = 1, 2$). Furthermore, one easily sees that for $k + m \ge 2^n$, all classes of the form $x_i^{2k+2} w_2^{2m+2^n}$ ($i = 1, 2$) are boundaries, too. For the homology, one arrives at

$$\mathbb{F}_2[x_1^2, x_2^2, w_2^2]/(x_1^2 x_2^2, x_1^{2^{n+1}}, x_2^{2^{n+1}}, w_2^{2^n}) \oplus \mathbb{F}_2[w_2^2]\{w_2^{2^n}, \zeta\} . \tag{1.4}$$

This is not yet finite, whence there has to be another differential. Restriction to the cyclic supgroup of order 4 says that ζ will eventually support a differential $d_{2\cdot 4^n-1}$; the only possibility is

$$d_{2\cdot 4^n-1}\zeta = v_n^{2^n+1} w_2^{4^n+2^n} . \tag{1.5}$$

Summing up:

Lemma 1.2. *(a) The $E_{2^{n+1}}$-page of the Atiyah-Hirzebruch spectral sequence for $K(n)^*(BD_8)$ is isomorphic to*

$$K(n)^*[x_1^2, x_2^2]/(x_1^2x_2^2, x_1^{2^{n+1}}, x_2^{2^{n+1}}) \otimes K(n)^*[w_2^2]/(w_2^{2^n}) \;\oplus\; K(n)^*[w_2^2]\{w_2^{2^n}, \zeta\}$$

where z is the class in degree $2^{n+1}+1$ obtained by replacing w_1 by x_1 in the expression for $Q_n(w_2)$.

(b) There is one more differential, namely $d_{2\cdot 4^n-1}(\zeta) = v_n^{2^n+1} w_2^{4^n+2^n}$. □

Let y_1, y_2, c_2 denote the Euler classes of $\gamma_2, \gamma_2, \Delta$, regarded as complex representations. Then y_j is a representative for $x_j^2 \in E_\infty$, and c_2 represents w_2^2. The E_∞-page is thus additively isomorphic to

$$K(n)^*[y_1, y_2, c_2]/(y_1y_2, y_1^{2^n}, y_2^{2^n}, c_2^{2^{n-1}}) \oplus K(n)^*[c_2]/(c_2^{2^{2n-1}})\{c_2^{2^{n-1}}\}\,.$$

There are extension problems to check:

(i) $y_1^{2^n}$, $y_2^{2^n}$, $c_2^{2^{2n-1}+2^{n-1}}$;

(ii) y_1y_2 and $y_ic_2^{2^{n-1}}$, $i = 1, 2$.

We have $y_j^{2^n} = [2](e(\gamma_i)) = e(\gamma_j^2) = e(1) = 0$. Also, $c_2^{2^{2n-1}+2^{n-1}} = 0$ since the spectral sequence is concentrated in horizontal degrees less than the degree of this class.
The classes in (ii) are in degree 4, and fortunately, $K(n)^4(BD_8)$ is detected on subgroups, as we shall now see (the other degrees however are not).
The maximal subgroups of D_8 are all abelian, namely

$$W_1 := \langle a_1, c\rangle\,, \quad W_2 := \langle a_2, c\rangle\,, \quad C := \langle a_1a_2\rangle$$

with W_j isomorphic to $C_2 \times C_2$ and $C \cong C_4$. Denote by η the character of W_j given by $\eta(c) = -1$ and $\eta(a_j) = 1$, and define $\rho\colon C \to \mathbb{C}$ by $\rho(a_1a_2) = i$. Then the the character ring of W_j (resp. C) is generated by η and γ_j (resp. ρ).

$$\begin{array}{lll}
\mathrm{Res}_{W_1}(\gamma_1) = \gamma_1 & \mathrm{Res}_{W_1}(\gamma_2) = 1 & \mathrm{Res}_{W_1}(\Delta) = \eta + \eta\gamma_1 \\
\mathrm{Res}_{W_2}(\gamma_1) = 1 & \mathrm{Res}_{W_2}(\gamma_2) = \gamma_2 & \mathrm{Res}_{W_2}(\Delta) = \eta + \eta\gamma_2 \\
\mathrm{Res}_{C}(\gamma_1) = \rho^2 & \mathrm{Res}_{C}(\gamma_2) = \rho^2 & \mathrm{Res}_{C}(\Delta) = \rho + \rho^3
\end{array}$$

Define $u = c_1(\eta)$ and $z = c_1(\rho)$ as elements of the Morava K-theory of the appropriate subgroups. Then

$$\begin{aligned}
K(n)^*(BW_j) &\cong K(n)^*[y_j, u]/(y_j^{2^n}, u^{2^n}) \quad (j = 1, 2),\\
K(n)^*(BC) &\cong K(n)^*[z]/(z^{4^n})\,.
\end{aligned}$$

We shall also use the following Lemma about the formal group law for $K(n)$, which is slightly more precise than Proposition I.3.1. For a proof, see [BP1].

Lemma 1.3. $x +_{K(n)} y = x + y + (xy)^{2^{n-1}} \; mod \; (x^{2^{2(n-1)}}, y^{2^{2(n-1)}})$. □

Thus, if $n > 1$,

$$[2](z) = z^{2^n} \quad \text{and} \quad [3](z) = z + z^{2^n} + z^{2^{n-1}(2^n+1)}$$

in $K(n)^*(BC)$. For example, $[3](z) = z + z^4 + z^{10}$ for $n = 2$.
A basis for $K(n)^4(BD_8)$ consists of the union of the following three sets:

$$A_j = \{y_j^2\} \cup \{y_j^{2^n-2k+1} c_2^k \mid 1 \le k \le 2^{n-1} - 1\} \; (j = 1, 2), \text{ and}$$
$$B = \{c_2^{1+\ell(2^n-1)} \mid 0 \le \ell \le 2^{n-1}\}.$$

Clearly $\mathrm{Res}_{W_1}(A_2) = 0 = \mathrm{Res}_{W_2}(A_1)$. Now, since for W_1 we may compute modulo u^{2^n} and $y_1^{2^n}$,

$$\begin{aligned}\mathrm{Res}_{W_1}(y_1^{2^n-2k+1} c_2^k) &= y_1^{2^n-2k+1}\big(u^2 + uy_1 + u(uy_1)^{2^{n-1}}\big)^k \\ &= y_1^{2^n-2k+1}\big((u^2+uy_1)^k + k(u^2+uy_1)^{k-1} u(uy_1)^{2^{n-1}}\big) \\ &= \sum_{i=0}^{k} \binom{k}{i} u^{2k-i} y_1^{2^n-2k+1+i} + k \sum_{j=0}^{k-1} \binom{k-1}{j} u^{2^{n-1}+(2k-j-1)} y_1^{2^n+2^{n-1}-(2k-j-1)}\end{aligned}$$

The second sum vanishes, since the sum of exponents is $2 \cdot 2^n$. Thus

$$\mathrm{Res}_{W_1}(y_1^{2^n-2k+1} c_2^k) = u^{2k} y_1^{2^n-2k+1} + \text{ monomials with lower exponents for } u.$$

Similarly for $\mathrm{Res}_{W_2}(y_2^{2^n-2k+1} c_2^k)$. Finally,

$$\mathrm{Res}_C(c_2^{1+\ell(2^n-1)}) = z^{2+2\ell(2^n-1)} + \text{ terms of higher order.}$$

Thus if $\xi = \sum_{a \in A_1} \lambda_a a + \sum_{a' \in A_2} \mu_{a'} a' + \sum_{b \in B} \nu_b b$ is an element restricting to zero on W_i and C, then $\lambda_a = 0$ follows from looking at Res_{W_1}, and so on.
This means we can solve the extension problems by restriction. From

$$\begin{aligned}&\mathrm{Res}_{W_1}(y_1y_2) = \mathrm{Res}_{W_2}(y_1y_2) = 0, \quad \mathrm{Res}_C(y_1y_2) = z^{2^{n+1}} \quad \text{resp.} \\ &\mathrm{Res}_{W_1}(c_2^{2^n}) = \mathrm{Res}_{W_2}(c_2^{2^n}) = 0, \qquad \mathrm{Res}_C(c_2^{2^n}) = z^{2^{n+1}}\end{aligned}$$

we obtain $y_1y_2 = c_2^{2^n}$. Secondly, we claim

$$y_i c_2^{2^{n-1}} = c_2^{2^n} + \sum_{j=1}^{n-1} y_i^{2^n-2^j+1} c_2^{2^{j-1}} \quad (i = 1, 2).$$

This can again be checked by restriction. In a later section we shall see how this relation arises, so we omit the proof here.
We arrive at the relations

$$y_i^{2^n} = c_2^{2^{2n-1}+2^{n-1}} = 0, \quad y_1y_2 = c_2^{2^n} \quad y_i c_2^{2^{n-1}} = c_2^{2^n} + \sum_{j=1}^{n-1} y_i^{2^n-2^j+1} c_2^{2^{j-1}}. \qquad (1.6)$$

Remark. (a) To justify our earlier assertion that $K(n)^*(BD_8)$ is not detected in other degrees, it suffices to look at the classes c_2^k for $2^{2n-1} \leq k \leq 2^{2n-1} + 2^{n-1}$: they restrict to zero on any subgroup.
(b) Later we shall describe how to obtain the relations from the representation theory of D_8.

Next, consider dihedral groups $D_{2^{m+2}}$ of order 2^{m+2}. Their mod 2 cohomologies

$$H^*(D_{2^{m+2}};\mathbb{F}_2) \cong \mathbb{F}_2[x_1, x_2, w_2]/(x_1x_2)$$

are isomorphic to $H^*(D_8;\mathbb{F}_2)$ as algebras over the Steenrod algebra; thus the action of the first differential is as described in Lemma 1.2. w_2^2 is a permanent cycle, whence the next differential — which has to exist to arrive at a finite rank object — will do away with ζ, hitting an appropriate power of w_2^2. This will result in an even degree page and collapsing of the spectral sequence at this stage. The power of w_2^2 hit is then easily calculated from the Euler characteristic formula, or alternatively we can detect the differential by restriction to the maximal cyclic subgroup $C_{2^{m+1}}$:

$$d_{2\cdot 2^{n(m+1)}-1}\zeta = v_n^{k_m} w_2^{2^{n(m+1)}+2^n} \qquad \text{with } k_m = (2^{n(m+1)} - 1)/(2^n - 1)\,.$$

We shall be less explicit about the relations, however. Denoting by y_1, y_2, c_2 the Euler classes of the analogous representations $\gamma_1, \gamma_2, \Delta$, we first note that in the degree of the relations one again has detection on subgroups. This enables us in principle to determine them: one needs to compute enough terms of the formal group law to calculate $z \cdot [-1](z) \in K(n)^*[z]/(z^{2^{n(m+1)}})$.
We did not try to produce a closed formula for general m and n, but just looking at the first few terms one gets

$$\begin{aligned} y_i^{2^n} &= c_2^{2^{n(m+1)-1}+2^{n-1}} = 0\,, \quad y_1y_2 = c_2^{2^{nm}} \\ y_ic_2^{2^{n-1}} &= \sum_{j=1}^{n-1} y_i^{2^n-2^j+1} c_2^{2^{j-1}} \quad \text{mod } c_2^{2^n}\,. \end{aligned}$$

As examples, we offer the following relations of the last kind for D_{16} and $n = 2, 3$:

$$\begin{aligned} y_ic_2^2 + y_i^3c_2 &= c_2^{10} + c_2^{22} + c_2^{25} + c_2^{28} + c_2^{31} \quad (n = 2) \\ y_ic_2^4 + y_i^5c_2^2 + y_i^7c_2 &= c_2^{36} + c_2^{148} + c_2^{162} + c_2^{204} + c_2^{232} + c_2^{246} + c_2^{253} \quad (n = 3) \end{aligned}$$

The lengthy computations in this section should convince the reader that for effective calculation, one had better turn to other methods. Nevertheless, we shall find them useful later on.

2 Wreath products

2.1 Wreath products with the cyclic group of order p

Let H be a finite group whose integral Morava K-theory is torsion free. Then a simple application of Kriz's theorem shows that the wreath product of H with a cyclic group of order p enjoys the same property. Such a result was first obtained by Hunton [Hu1], who used 'unitary-like embeddings' to show that if $K(n)^*(BH)$ is concentrated in even degrees, then so is $K(n)^*(BH \wr C_p)$. An independent proof was given in [HKR]; the authors there actually showed something stronger: if $K(n)^*(BH)$ has a basis consisting of transferred Euler classes of complex representations of subgroups of H, then the same holds for $H \wr C_p$.

Theorem 2.1 ([Hu1, HKR]). *Let H be a finite group with torsion free integral Morava K-theory. Then the integral Morava K-theory of the wreath product $H \wr C_p$ is also torsion free.*

Proof. Since $\widetilde{K}(n)^*(BH)$ has no torsion, the Künneth theorem implies that $\widetilde{K}(n)^*(BH^p)$ is a permutation module for C_p. The claim thus follows from Theorem II.4.6. □

2.2 A generalisation

Under suitable hypotheses on the cohomology theory E, the wreath product theorem generalises to other extended power spaces. Let G be a subgroup of the symmetric group Σ_m on m letters, and X a pointed space. Then the space

$$D_G(X) = EG_+ \wedge_G X^{\wedge m}$$

is called the extended power space of X and G. In particular, if $X = BL_+$ is itself a classifying space, then $D_G(X)$ is a model for $B(L \wr G)_+$, the classifying space of the wreath product.

One has a fibration

$$X^{\wedge m} \longrightarrow D_G(X) \longrightarrow BG$$

and an associated Serre spectral sequence (we shall use the homological version below)

$$E^2 = H_*(BG; E_*(X^{\wedge m})) \Longrightarrow E_* D_G(X)\,. \tag{2.1}$$

Assume that E_*X is a free E_*-module. Then the E^2-page decomposes algebraically into a direct sum of terms isomorphic to the E^2-pages of the Atiyah-Hirzebruch spectral sequences associated to the subgroups of G:

$$E^2 = \bigoplus_{H \leq G} H_*(BH; E_*) \otimes M_H$$

where M_H is the image of the function $x \mapsto \sum_{g\in G/H} g_* x$ defined on the coinvariants of $E_*(X^{\wedge m})$. Now if E is an H_∞ ring spectrum, the algebraic decomposition actually has a basis in geometry. This observation is essentially due to McClure (see [B], Chapter IX in particular):

Proposition 2.2. *Suppose E is an H_∞ ring spectrum and $E_*(X)$ is free over E_*. Then the spectral sequence* (2.1) *splits up as the direct sum of Atiyah-Hirzebruch spectral sequences for $E_*(BH)$.*

Proof. That $E_*(X) = \pi_*(E \wedge X)$ is E_* free means that there is a stable map $f\colon W \longrightarrow E \wedge X$ such that W is a wedge of spheres and f induces an equivalence $E \wedge W \longrightarrow E \wedge X$ (using the ring structure of E). The H_∞ structure provides a map $D_G(E) \longrightarrow E$ and thus f induces a map $D_G(W) \longrightarrow E \wedge D_G(X)$ which in turn induces an equivalence $E \wedge D_G(W) \longrightarrow E \wedge D_G(X)$, as in [B, IX §2 or §4]. However, $D_G(W)$ is just a wedge of Thom spectra and this decomposition gives rise to the algebraic decomposition of the spectral sequence. $\square$

As MU and E_n are H_∞, one has such splittings for these theories. By exactness, the same holds for $E(n)$: since

$$E_{n*}(X) = E_{n*} \otimes_{E(n)_*} E(n)_*(X)\,,$$

any space X with $E(n)_*$-free $E(n)$-homology has E_{n*}-free E_{n*}-homology, too. The natural map $E_n \to E(n)$ induces a map of spectral sequences. Since the Atiyah=Hirzebruch spectral sequence for $E_{n*}(BH)$ is the tensored down version of the $E(n)$ spectral sequence, this determines the latter. The same conlusion is valid for the Baker-Würgler completion $\widehat{E(n)}$ of $E(n)$ (since we are assuming X is a space).
Arguing with connective covers (in particular, connective covers of H_∞ ring spectra are H_∞), one sees that the same holds for $E = BP\langle n\rangle$.
Summing up:

Theorem 2.3. *Suppose X is a space and $E = MU$ or one of the Johnson-Wilson theories $E(n)$, its connective cover $BP\langle n\rangle$ or its I_n-adic completion $\widehat{E(n)}$, and that $E_*(X)$ is free as an E_*-module. Then the spectral sequence* (2.1) *is isomorphic to the direct sum of Atiyah-Hirzebruch spectral sequences*

$$E^*_{*,*} = \bigoplus_{1\leq H\leq G} \left\{E^*_{*,*}(H) \otimes_{E_*} F_H\right\}$$

where $E^*_{*,*}(H)$ is the Atiyah-Hirzebruch spectral sequence for $E_*(BH)$. $\square$

Corollary 2.4. *Suppose $K(n)_{odd}(BH) = 0$ for all subgroups $H \leq G$ and that X is a space for which $K(n)_{odd}(X) = 0$. Then the spectral sequence* (2.1) *with $E = K(n)$ is isomorphic to the corresponding direct sum of Atiyah-Hirzebruch spectral sequences for $K(n)_*(BH)$; in particular, $K(n)_{odd}(D_G(X)) = 0$.*

Proof. If $K(n)_{\mathrm{odd}}(X) = 0$ then the Bockstein spectral sequence $K(n)_*(Y) \Longrightarrow \widehat{E(n)}_*(Y)$ of [BW] collapses and $\widehat{E(n)}_*(X)$ is concentrated in even dimensions and is free over $\widehat{E(n)}_*$. The spectral sequence for $\widehat{E(n)}_*(D_G(X))$ is now described by Theorem 2.3. If for each subgroup $H \leq G$ we have $K(n)_{\mathrm{odd}}(BH) = 0$ then the Atiyah-Hirzebruch spectral sequence for $K(n)_*(BH)$ is just that for $\widehat{E(n)}_*(BH)$ tensored down by the coefficients, and hence the same applies for the sequence (2.1) for $K(n)_*(D_G(X))$. □

3 Groups of p-rank 2

For primes greater than 3, these groups were shown to have even Morava K-theory by Tezuka-Yagita [TY2, TY3] and Yagita [Y4]. Furthermore, Yagita also proved that these groups are generated by transfered Euler classes and are thus 'good' in the sense of Hopkins-Kuhn-Ravenel.
These results were obtained case by case using Blackburn's classification of groups of p-rank 2: for $p > 3$, any such group belongs to one of the following classes (see [Hp, III, 12.4]):

(1) $G = \langle a, b \mid a^{p^r} = 1,\ b^{p^s} = a^{p^t},\ b^{-1}ab = a^k \rangle$
with $t \geq 0$, $k^{p^s} \equiv 1\ (p^r)$, and $p^t(k-1) \equiv 0\ (p^r)$;

(2) $G = \langle a, b, c \mid a^p = b^p = c^{p^k} = [a,c] = [b,c] = 1,\ a^{-1}ba = bc^{p^{k-1}} \rangle$;

(3) $G = \langle a, b, c \mid a^p = b^p = c^{p^r} = [b,c] = 1,\ a^{-1}ba = bc^{sp^r},\ a^{-1}ca = bc \rangle$
where $r \geq 2$ and s is either 1 or a quadratic non-residue modulo p.

For $p = 2, 3$ there are other groups; in fact, for $p = 2$ no such classification is known. Nevertheless, Tezuka-Yagita's and Yagita's results are valid for groups with such presentations regardless of the prime.
The groups in (1) are metacyclic; they fit into an extension

$$1 \longrightarrow C_{p^r} \longrightarrow G \longrightarrow C_{p^s} \longrightarrow 1\,. \tag{3.1}$$

Tezuka-Yagita calculate the BP-theory Serre spectral sequence associated to this extension. It turns out that the E_2-page is concentrated in even degrees, and that $BP^*(BG)$ is torsion-free. They further identify the BP^*-algebra generators as Chern classes, thus proving

Theorem 3.1 (Tezuka-Yagita [TY3]). *Let G be a metacyclic p-group. Then $BP^*(BG)$ is multiplicatively generated by Chern classes of representations.* □

Consequently, such groups have even Morava K-theory.

Remark. In [B3], Brunetti attempts to calculate the multiplicative structure of the Morava K-theory of the so-called modular groups, i.e., extensions of C_{p^m} by C_p, drawing on results of Tezuka-Yagita. He does succeed in writing down an algebra structure, but the generators are not geometric, since they are derived from the E_∞-page of the spectral sequence.

The groups (2) can either be described as a central extension

$$1 \longrightarrow C_p \times C_{p^s} \longrightarrow G \longrightarrow C_p \longrightarrow 1 \tag{3.2}$$

or as the central product of a cyclic group with the nonabelian groups of order p^3 and exponent p (resp. D_8 for $p = 2$).
Tezuka-Yagita's argument is again by calculating the Serre spectral sequence for BP-theory. As for metacyclic groups, they can show

Theorem 3.2 (Tezuka-Yagita [TY3])**.** *If G is of type (2), then $BP^*(BG)$ is generated by Chern classes.* □

Alternatively, we can use the other description as a central product, and appeal to Corollary II.5.3. This reduces the problem to calculating the Morava K-theory of the nonabelian group $H := p_+^{1+2}$ of order p^3 and exponent p, which is an instance of applying Theorem 5.1 (compare Example 5.2).
One can also do this differently: consider the central extension

$$1 \longrightarrow C_p \longrightarrow H \longrightarrow C_p \times C_p \longrightarrow 1 \tag{3.3}$$

and the associated Serre spectral sequence

$$E_2 = H^*(C_p \times C_p; K(n)^*(BC_p)) \cong \Lambda(x_1, x_2) \otimes \mathbb{F}_p[y_1, y_2] \otimes K(n)^*[z]/(z^{p^n}) .$$

Then

Lemma 3.3. *The Serre spectral sequence for the extension* (3.3) *has the following differentials:*

(i) $d_3 z = x_2 y_1 - x_1 y_2$;

(ii) $d_{2p-1}((x_2 y_1 - x_1 y_2) z^{p-1}) = y_1^p y_2 - y_1 y_2^p$;

(iii) $d_{2p-1}(x_1 x_2 z^{p-1}) = x_2 y_1^p - x_1 y_2^p$;

(iv) $d_{2p^n-1} x_i = v_n y_i^{p^n}$ $(i = 1, 2)$.

Proof. This is an exercise in exhaustion. (i) follows by comparing to the mod p cohomology spectral sequence via connective Morava K-theory (recall that the extension class is $x_1 x_2$), and (ii) from Kudo's transgression theorem and again comparison. For (iii), observe that z^p is the restriction of the Euler class of a certain representation of G, namely $z^p = i^* c_p(\mathrm{Ind}_A^G \lambda)$ where A is a maximal abelian subgroup of G and λ a character of A restricting to the generator of the character group of the center. By comparison to mod p cohomology again, one sees that z^p should transgress (via d_{2p+1}) to $x_1 y_2^p - x_2 y_1^p$. Thus this element has to be killed by an earlier differential. The only way this can happen is if we have a differential as stated in (iii). Finally, (iv) is the differential inherited from the Atiyah-Hirzebruch spectral sequence for the base. To see that these are the only differentials, one only has to observe that they turn the next page of the spectral sequence into an even degree object. □

Remark. Anyone comparing this calculation to the original in [TY3] will be convinced that their method of using BP is more effective. Working with BP instead of $K(n)$ has its advantages, since BP is both connective and 'integral', making the spectral sequences sparser.

One can generalise the above lemma to show that the Morava K-theory of any minimal non-abelian p-group (a p-group all of whose maximal subgroups are abelian) is concentrated in even degrees. This was first observed by Yagita [Y3], who again worked with BP (the result for Morava K-theory follows).
We include a proof for future reference. Recall that minimal non-abelian p-groups G were classified by Rédei [Re]; we quote from [Hp, p. 309]. G is one of the following groups:

(i) quaternion of order 8;

(ii) $G = \langle a, b \mid a^{p^r} = b^{p^s} = 1,\ b^{-1}ab = a^{p^{r-1}+1} \rangle$, i.e. split metacyclic;

(iii) $G = \langle a, b, c \mid a^{p^r} = b^{p^s} = c^p = [a, c] = [b, c] = 1,\ [a, b] = c \rangle$.

We only consider (iii), the other cases being covered elsewhere. Such G has centre $Z = \langle a^p, b^p, c \rangle \cong C_{p^{r-1}} \times C_{p^{s-1}} \times C_p$, with quotient $C_p \times C_p$. The Serre spectral sequence for the central extension $1 \to Z \to G \to C_p \times C_p \to 1$ now behaves just as described in the lemma: one has

$$E_2 = K(n)^*[x, y, z]/(x^{p^{r-1}n}, y^{p^{s-1}n}, z^{p^n}) \otimes H^*(BC_p \times C_p; \mathbb{F}_p)$$

where x, y, z are the Euler classes of the representations α, β, γ of Z defined in the obvious way: $\alpha(a^p) = \exp(2\pi i p^{1-r})$, $\alpha(b^p) = \alpha(c) = 1$, and so on. Then α and β are restrictions of representations $\tilde{\alpha}$ and $\tilde{\beta}$ of G, implying that x and y are permanent cycles. z is not a cycle, but z^p is: consider the maximal subgroup $A = \langle a, b^p, c \rangle$ of G, and define $\varepsilon \in RA$ by $\varepsilon(a) = \varepsilon(b^p) = 1$, $\varepsilon(c) = \exp(2\pi i/p)$. Set $\sigma = \mathrm{Ind}_A^G(\varepsilon)$, then by the double coset formula,

$$\mathrm{Res}_Z^G(\sigma) = \mathrm{Res}_Z^A\big((1 + b + b^2 + \cdots + b^{p-1})^*(\sigma)\big) = p \cdot \mathrm{Res}_Z^A(\varepsilon) = p \cdot \gamma\,,$$

whence the claim. Together with the result for metacyclic groups, this implies

Theorem 3.4 (Yagita [Y3]). *Let G be a p-group all of whose maximal subgroups are abelian. Then $K(n)^{odd}(BG) = 0$.* □

Finally, in [Y4] Yagita calculates BP-cohomology of the groups of type (3). He uses an extension of the form

$$1 \longrightarrow H \longrightarrow G \longrightarrow C_p \longrightarrow 1 \tag{3.4}$$

where $H = \langle a, b, c^p \rangle$ is a group of type (2). His result is

Theorem 3.5 (Yagita [Y4]). *If G is a group of type (3) and $p > 2$, then $BP^*(BG)$ is generated by transfered Euler classes of complex representations of G.* □

4 Tanabe's work on Chevalley groups

In [T], Tanabe proves a beautiful theorem about the Morava K-theory of Chevalley groups:

Theorem 4.1 (Tanabe [T]). *Let G be a connected reductive $\mathbb{Z}$-group scheme such that $H^*(G(\mathbb{C}); \mathbb{Z})$ has no p-torsion. Let $\widetilde{K}(n)$ denote p-complete Morava K-theory, ℓ a prime distinct from p, and $q = \ell^m$, $m > 0$. Then $\widetilde{K}(n)^*(BG(\mathbb{F}_q))$ is torsion free and concentrated in even dimensions.* □

This theorem applies for example to $GL_k(\mathbb{F}_q)$ or $SL_k(\mathbb{F}_q)$.

The method of proof involves the homotopy pullback diagram

$$\begin{array}{ccc} BG(\mathbb{F}_q)^\wedge & \xrightarrow{\ i_q\ } & BG(\mathbb{F})^\wedge \\ \Big\downarrow{\scriptstyle i_q} & & \Big\downarrow{\scriptstyle \Delta} \\ BG(\mathbb{F})^\wedge & \xrightarrow[1\times\phi^q]{} & (BG(\mathbb{F}) \times BG(\mathbb{F}))^\wedge \end{array}$$

from [FM], where $(\ )^\wedge$ denotes completion in the sense of Bousfield-Kan, Δ is the diagonal map, and ϕ^q is induced by the Frobenius. He then constructs a strongly convergent Eilenberg-Moore spectral sequence

$$\mathrm{Tor}^{K(n)^*(BG(\mathbb{F})^2)}\big(K(n)^*(BG(\mathbb{F})), K(n)^*(BG(\mathbb{F}))\big) \Longrightarrow K(n)^*(BG(\mathbb{F}_q))$$

from which he derives his result.

5 Elementary abelian by cyclic groups

Kriz in [K] and Yagita in [Y5] show that an extension of an elementary abelian p group by a cyclic p-group satisfies the even-dimensionality conjecture.

Theorem 5.1 (Kriz [K], Yagita [Y5])**.** *Suppose G fits into an extension $V \to G \to C$ where V is an elementary-abelian p-group and C a cyclic p-group. Then $K(n)^{odd}(BG) = 0$.* □

Kriz's proof is based on an explicit calculation of the structure of $\widetilde{K}(n)^*(BV)$ as a module first over C_p, and then induction on the order of the cyclic quotient. It turns out that the action is a permutation action, and he then appeals to Theorem II.4.6. (He only states the above theorem for semidirect products, but for the action on the Morava K-theory of BV it is irrelevant whether the extension is split or not.) Yagita calculates with Brown-Peterson cohomology BP using a clever filtration on $BP^*(BV)$.

Example 5.2. Since it is needed for Kriz's counterexample, we shall examine the extraspecial group of order p^3 and exponent p in some detail. This calculation is entirely due to Kriz, and we follow his reasoning very closely.
H fits into a split extension

$$1 \longrightarrow \langle a, b\rangle \longrightarrow H \longrightarrow \langle c\rangle \longrightarrow 1\,; \tag{5.1}$$

with $V := \langle a, b\rangle \cong C_p \times C_p$ and $\langle c\rangle \cong C_p$. The element b is central, and a and c commute according to the rule $c^{-1}ac = ab$.
Define representations μ, ν, ρ of H as follows:

$$\begin{aligned} \mu\colon H &\twoheadrightarrow \langle a\rangle \hookrightarrow \mathbb{C}^\times, \qquad \mu(a) = \zeta_p,\\ \nu\colon H &\twoheadrightarrow \langle c\rangle \hookrightarrow \mathbb{C}^\times, \qquad \nu(c) = \zeta_p \end{aligned}$$

and $\rho = \mathrm{Ind}_V^H \beta$ where $\beta(a) = 1$, $\beta(b) = \zeta_p$, and ζ_p denotes a p-th root of unity: Set

$$z = c_1(\mu) \quad , \quad w = c_1(\nu) \quad , \quad u = c_p(\rho)\,.$$

Then a permutation basis for $\widetilde{K}(n)^*(BV)$ consists of free C_p-orbits and the invariant elements $u'^i y^j$, $0 \le i \le p^{n-1} - 1$, $0 \le j \le p-1$, where u' is the restriction of u to V. Furthermore, Theorem 5.1 (or rather its proof, which we have not reproduced here) implies that $\widetilde{K}(n)^*(BH)/\operatorname{Im}(\mathrm{Tr})$ is generated as an algebra by u, w, and z. Here and below be write just Tr for Tr_V^H. It remains to determine the relations between the generators; this can be done by restricting to maximal abelian subgroups and character theory.
There are $(p + 1)$ such subgroups of H, all elementary abelian of rank two. A complete list consists of

$$E_\infty := V\ , \quad E_\lambda := \langle a^\lambda c, b\rangle\ ,\ 0 \le \lambda \le p-1\,.$$

Let $E = \langle a', b \rangle$ be any group from this list. If α and β denote the representations of E with $\alpha(b) = \beta(a') = 1$, $\alpha(a') = \beta(b) = \zeta_p$, and $x_1 = c_1(\alpha)$, $x_2 = c_1(\beta)$, then we get the following restrictions.

	μ	ν	ρ	z	w	u
E_∞	α	1	$\sum_{k=0}^{p-1} \beta \otimes \alpha^k$	x_2	0	$\prod_{i=0}^{p-1} (x_2 +_F [i]x_1)$
E_λ	α^λ	α	$\sum_{k=0}^{p-1} \beta \otimes \alpha^k$	$[\lambda]x_2$	x_2	$\prod_{i=0}^{p-1} (x_2 +_F [i]x_1)$

We claim the following identities:

(i) $wz^p = w^p$ (in $\widetilde{K}(n)^*(BH)$);

(ii) $w^{p^n} = pw$;

(iii) $u^{p^n} = p^p u$.

We check these equality on the maximal abelian subgroups E of H, beginning with (i). For $E = E_\infty$, the equation is trivial. Otherwise the restriction of $wz^p - w^p z$ equals

$$([\lambda]x_1)^p x_1 - x_1^p([\lambda]x_1) \, .$$

By Theorem II.6.3 we can verify the resulting equation using generalised characters.

Remark. When the groups in question have exponent p, calculating with generalised characters becomes particularly easy: In such cases the characters take values in an extension of $W\mathbb{F}_{p^n}$ where the equation $p - x^{p^n-1} = 0$ has a solution. The p-torsion subgroup consists of 0 and the solutions to the equation $p - x^{p^n-1} = 0$. If π is a uniformising element, then all other solutions are of the form $\zeta\pi$ for a $(p^n - 1)$-st root of unity ζ.

So let χ be a character of E. If $\chi(x_1) = 0$ there is nothing to show. Thus we can assume without loss of generality that χ is a character taking the value π on x_1. Then $[\lambda]\pi = \varepsilon_{p-1}\pi$ for a $(p-1)$-st root of unity $\varepsilon_{p-1}$and (i) follows.
(ii) is clear; for (iii) note that the restriction of u is a product of p Euler classes of one dimensional representations of E, see the table above. Each such Euler class satisfies $[p]e = 0$, i.e. $e^{p^n} = pe$; the claim follows. □
The next claim is

$$pw \sum_{i=0}^{n-1}{}^F w \Big(\frac{u}{w^p}\Big)^{p^i} = 0 \tag{5.2}$$

in $\widetilde{K}(n)^*(BH)$. We first should check that the left hand side makes sense: by (ii) and (iii) above,

$$\left(\frac{u}{w^p}\right)^{p^n} = \frac{u}{w^p}\,.$$

The formal group law for $\widetilde{K}(n)^*$ is of the form

$$x +_F y = x + y + \sum_{k\geq 1} C_k(x,y)\,,$$

where the C_k are homogeneous polynomials of degree $k(p^n-1)+1$. Hence $C_k(wx,wy)$ is divisible by $w^{k(p^n-1)+1}$. Furthermore, (ii) implies that the exponents of w in the denominators of the higher order terms of the formal sum do not exceed p^n-1. Since $pw = w^{p^n}$, it follows that the expression in the claim is integral. Moreover, the higher order terms are divisible by increasing powers of p, thus the power series converges in the p-adic topology and thus constitutes an element of $\widetilde{K}(n)^*(BH)$. More precisely,

$$pw\sum_{i=0}^{n-1}{}^{F}\, w\Big(\frac{u}{w^p}\Big)^{p^i} \equiv pw\sum_{i=0}^{n-1} w\Big(\frac{u}{w^p}\Big)^{p^i} \quad \bmod p$$

where the right hand side is not divisible by p, and, since $pw = w^{p^n}$, equals

$$(*) \qquad w(u^{p^{n-1}} + w^{p^n-p^{n-1}}u^{p^{n-2}} + \cdots + w^{p^n-p^{n-i+1}}u^{p^{n-i}} + \cdots + w^{p^n-p}u)\,.$$

Proof of (5.2). Restriction to E_∞ is again 0. Let $E = E_\lambda$ be one of the other subgroups in the list. Then

$$\operatorname{Res}^H_E u = x_2\prod_{i=1}^{p-1}(x_2 +_F [i]x_1)\,, \quad \operatorname{Res}^H_E w = x_2\,.$$

Let χ be a generalized character of E. If $\chi(x_1)=0$ or $\chi(x_2)=0$ there is nothing to prove. Without loss of generality we may thus assume $\chi(x_1)=\pi$. Then $\chi(x_2)=\gamma\pi=[\gamma]\pi$ for some (p^n-1)-st root of unity, and

$$\begin{aligned}\chi(\operatorname{Res}^H_E u) &= \gamma\pi\prod_{i=1}^{p-1}(\gamma\pi +_F [i]\pi) = \gamma\pi\prod_{i=1}^{p-1}(\gamma\pi +_F \varepsilon_i\pi)\\ &= \gamma\pi\prod_{i=1}^{p-1}[\gamma+\varepsilon_i]\pi\,.\end{aligned}$$

where the ε_i range over the $(p-1)$-st roots of unity. Setting $\varepsilon_0 = 0$ we conclude

$$\begin{aligned}\chi\Big(\mathrm{Res}^H_E\, pw \sum_{i=0}^{n-1}{}^F\, w\Big(\frac{u}{w^p}\Big)^{p^i}\Big) &= p\pi \sum_{i=0}^{n-1}{}^F \Big[\prod_{k=0}^{p-1}(\gamma+\varepsilon_k)\Big]\pi \\ &= p\pi\Big[\sum_{i=0}^{n-1}\prod_{k=0}^{p-1}(\gamma+\varepsilon_k)\Big]\,.\end{aligned}$$

Inside the square brackets we may compute modulo p, since $[p]\pi = 0$. Now

$$\prod_{k=0}^{p-1}(\gamma+\varepsilon_k) = \gamma^p - \gamma\,;$$

since

$$\sum_{i=0}^{n-1}(\gamma^p - \gamma) = \gamma^{p^n} - \gamma = 0$$

modulo p, the claim follows. □

Now let x be a class in $\widetilde{K}(n)^*(BH)$. If $x = \mathrm{Tr}(y)$, then $wx = w\,\mathrm{Tr}(y) = \mathrm{Tr}(\mathrm{Res}^H_E(w)\cdot y) = 0$ by Frobenius reciprocity and since $\mathrm{Res}^H_V\, w = 0$. Conversely, suppose that $wx = 0$. Then

$$\mathrm{Tr}\,\mathrm{Res}^H_V(x) = x\cdot\mathrm{Tr}(1) = x(p - w^{p^n-1}) = px$$

implies $px \in \mathrm{Im}(\mathrm{Tr})$. Since $\widetilde{K}(n)^*(BH)/\,\mathrm{Im}(\mathrm{Tr})$ is torsion free (this follows from Theorem II.4.6), x is in the image of Tr. Thus we have established:

$$wx = 0 \Longleftrightarrow x \in \mathrm{Im}(\mathrm{Tr})\,. \tag{5.3}$$

Remark. A similar argument works for any cyclic covering $Y \to X \to BC_{p^r}$.

Corollary 5.3. *$K(n)^*(BH)/\,\mathrm{Im}(\mathrm{Tr}) \otimes_{K(n)^*} \mathbb{F}_p$ is isomorphic to*

$$\mathbb{F}_p[z,w,u]/(z^p - w^{p-1}z, w^{p^n-1}, u^{p^{n-1}} + w^{p^n-p^{n-1}}u^{p^{n-2}} + \cdots + w^{p^n-p}u)\,.$$

Proof. It is easy to check that the relations proved above give a module of the correct rank. □

6 Kriz's counterexample

Kriz's counterexample is the 3-Sylow subgroup of $GL_4(\mathbb{F}_3)$, which we shall denote by P. He shows that the second Morava K-theory of BP at the prime 3 contains elements of odd degree. He considers an extension

$$1 \longrightarrow G \longrightarrow P \longrightarrow C_p \longrightarrow 1$$

and shows that there is an element $0 \neq \zeta \in H^1(C_3; \widehat{K}(2)^*BG)$; that suffices by Theorem II.4.6. To that end one has to determine $\widetilde{K}(n)^*(BG)$, and furthermore the action of the quotient C_3 on $\widetilde{K}(n)^*(BG)$, or at least part of it (this will be made more precise below). To do that one uses the calculations of the previous section and generalised characters again. We shall give a rather a detailed account of his work in this section.

From now on let p denote an odd prime. Let P be the goup of unipotent upper triangular (4×4)-matrices, i.e. the p-Sylow subgroup of $GL_4(\mathbb{F}_p)$:

$$P = \left\{ \begin{pmatrix} 1 & c & b_2 & b_1 \\ 0 & 1 & a_2 & a_1 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; a_i, b_i, c, d \in \mathbb{F}_p \right\}.$$

By abuse of notation, a_1 shall also denote the matrix whose entry in position a_1 is 1 and $a_2 = b_i = c = d = 0$; similarly for a_2 etc.. Then we have an extension

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G & \longrightarrow & P & \longrightarrow & C_p & \longrightarrow & 1 \\ & & \| & & & & \| & & \\ & & \langle a_i, b_i, c \rangle & & & & \langle d \rangle & & \end{array}$$

The group G has an elementary abelian subgroup $A = \langle a_i, b_i \rangle$ of rank four, i.e., there is an extension

$$1 \longrightarrow A \longrightarrow G \longrightarrow \langle c \rangle \longrightarrow 1\,.$$

Thus G is a group to which the results of Section 5 apply.
Let H be the group of Example 5.2, i.e., the nonabelian group of order p^3 and exponent p. There are two homomorphisms $\pi_1, \pi_2 : G \to H$, with

$$\begin{array}{llll} \pi_1(a_1) = a, & \pi_1(b_1) = b, & \pi_1(a_2) = 1, & \pi_1(b_2) = 1, \\ \pi_2(a_1) = 1, & \pi_2(b_1) = 1, & \pi_2(a_2) = a, & \pi_2(b_2) = b, \end{array} \quad \pi_i(c) = c\,.$$

We shall study $\widetilde{K}(n)^*(BG)$ via π_i and $\widetilde{K}(n)^*(BH)$.
A is isomorphic as C_p-module to a sum of two copies of V, which implies

$$\widetilde{K}(n)^*(BV) \cong \widetilde{K}(n)^*(BV) \otimes_{\widetilde{K}(n)^*} \widetilde{K}(n)^*(BV)$$

as $C_p = \langle c \rangle$-module; consequently

$$\hat{H}^*(C_p; \widetilde{K}(n)^*(BA)) \cong \hat{H}^*(C_p; \widetilde{K}(n)^*(BV))^{\otimes 2}$$

where $\hat{H}^*$ stands for Tate cohomology; cf. the remark after Lemma II.4.5.

Denote by u, z (abusing notation) and v, t the images of u, z under π_1^* and π_2^*, respectively. From now on Tr will stand for the transfer from A to G. Let

$$\begin{aligned}\hat{M} &:= \widetilde{K}(n)^*(BG)/\operatorname{Im}(\mathrm{Tr})\,,\\ M &:= K(n)^*(BG)/\operatorname{Im}(\mathrm{Tr}) \otimes_{K(n)^*} \mathbb{F}_p\,.\end{aligned}$$

Then naturality of the transfer and Corollary 5.3 imply

Corollary 6.1. *$M = K(n)^*(BG)/\operatorname{Im}(\mathrm{Tr}) \otimes_{K(n)^*} \mathbb{F}_p \cong \mathbb{F}_p[t, u, v, w, z]/R$ where R is generated by*

$$z^p - w^{p-1}z,\ t^p - w^{p-1}t, w^{p^n-1},\ \textstyle\sum u^{p^{n-i}} w^{p^n-p^{i-1}},\ \sum v^{p^{n-i}} w^{p^n-p^{i-1}}\,. \qquad \square$$

This sums up what we need to know about $\widetilde{K}(n)^*(BG)$.
Next we need to compute the action of the element d of P on the Morava K-theory of BG. More precisely, we shall determine its action on u, ua, z, t, and w, considered as elements of $K(n)^*(BG)/\operatorname{Im}(\mathrm{Tr}) \otimes_{K(n)^*} \mathbb{F}_p$. Conjugation by d changes only entries in the last column of a matrix in P, so v, t, and w are obviously invariant. Also clear is the effect on z:

$$dz = z +_F t\,.$$

It remains to compute du.

Lemma 6.2. *In $\widetilde{K}(n)^*(BG) \otimes \mathbb{Q}$, d acts on u via*

$$du = w^{p-1}\Big(\frac{u}{w^{p-1}} +_F \frac{v}{w^{p-1}}\Big)\,.$$

Remark. Since $(\frac{u}{w^p})^{p^n} = \frac{u}{w^p}$ (see Example 5.2), and similarly $(\frac{v}{w^p})^{p^n} = \frac{v}{w^p}$, the right hand side becomes integral after multiplication by p^{p-1}, see also the remarks following (5.2).

Proof. Once more we restrict to maximal abelian subgroups. Restricting to A results in a trivial equation, since $\operatorname{Res}^G_A w = 0$. The remaining maximal abelian subgroups are of the form

$$A_{rs} := \langle b_1, b_2, ca_1^r a_2^s\rangle\,, \quad 0 \le r, s \le p-1\,.$$

Let β_1, β_2, α be one dimensional representations of A_{rs} with β_i dual to b_i and α dual to $ca_1^r a_2^s$. u and v are the Euler classes of the representations $\rho_1 = \rho \circ \pi_1$ and $\rho_2 = \rho \circ \pi_2$, and du is the Euler class of the conjugate ρ_1^d of ρ_1. The groups A_{rs} are elementary abelian of rank three; their Morava K-theory is isomorphic to

$$\widetilde{K}(n)^*[x_1, x_2, x_3]/([p]x_1, [p]x_2, [p]x_3)\,,$$

with $x_1 = c_1(\beta_1)$, $x_2 = c_2(\beta_2)$ und $x_3 = c_1(\alpha)$. We have the following restrictions of representations and generators.

$$\begin{aligned} \operatorname{Res}^G_{A_{rs}}(\rho_1) &= \textstyle\sum_{i=1}^{p-1} \beta_1 \otimes \alpha^i\,, & \operatorname{Res}^G_{A_{rs}}(u) &= \textstyle\prod_{i=0}^{p-1}(x_1 +_F [i]x_3) \\ \operatorname{Res}^G_{A_{rs}}(\rho_2) &= \textstyle\sum_{i=0}^{p-1} \beta_2 \otimes \alpha^i\,, & \operatorname{Res}^G_{A_{rs}}(v) &= \textstyle\prod_{i=0}^{p-1}(x_2 +_F [i]x_3) \\ \operatorname{Res}^G_{A_{rs}}(\rho_1^d) &= \textstyle\sum_{i=0}^{p-1} \beta_1 \otimes \beta_2 \otimes \alpha^i\,, & \operatorname{Res}^G_{A_{rs}}(du) &= \textstyle\prod_{i=0}^{p-1}(x_1 +_F x_2 +_F [i]x_3) \end{aligned}$$

Let χ be a generalised character, which (without loss of generality) takes the value π on x_3. Then

$$\chi(x_1) = \varepsilon_1 \pi\,, \quad \chi(x_2) = \varepsilon_2 \pi\,,$$

where the ε_i are either (p^n-1)-st roots of unity or 0. The lemma is then equivalent to

$$\frac{\prod_{i=0}^{p-1}[\varepsilon_1+\varepsilon_2+i]\pi}{\pi^{p-1}} = \frac{\prod_{i=0}^{p-1}[\varepsilon_1+i]\pi}{\pi^{p-1}} +_F \frac{\prod_{i=0}^{p-1}[\varepsilon_2+i]\pi}{\pi^{p-1}}\,.$$

Now $[p]\pi = 0$ implies that the left hand side equals $[(\varepsilon_1+\varepsilon_2)^p - (\varepsilon_1+\varepsilon_2)]\pi$, and the right hand side

$$[\varepsilon_1^p - \varepsilon_1]\pi +_F [\varepsilon_2^p - \varepsilon_2]\pi = [\varepsilon_1^p - \varepsilon_1 + \varepsilon_2^p - \varepsilon_2]\pi\,.$$

The expressions in $[\ldots]$ agree modulo p. □

The formulae obtained so far lend themselves to concrete calculations for $p = 3$ and $n = 2$. From Proposition I.3.1 and Lemma 6.2 one obtains

$$\begin{aligned} dz &= z + t + w^6(z^2t + zt^2)\,, \\ du &= u + v - w^2(u^2v + uv^2)\,; \end{aligned}$$

these equations hold in $\hat{M}$. Consequently

$$(1-d)(tu - zv) \equiv tw^2(u^2v + uv^2) \mod w^3$$

in M, hence

$$(1-d)w(tu - zv) \equiv tw^3(u^2v + uv^2) \mod (w^4, 3w)$$

in $\hat{M}$. Filter $\hat{M}$ by powers of w; then $3w = w^9$ implies

$$(1-d)w(tu - zv) \equiv tw^3(u^2v + uv^2) + s\,,$$

where s is in a higher filtration. Thus

$$(1-d)w(tu - zv) - tw^3(u^2v + uv^2) - s$$

is an element of $\operatorname{Im}(\mathrm{Tr})$. On the other hand, it lies in a positive filtration, since $1-d$ preserves filtration. The following variant of (5.3) guarantees that such an element has to be zero.

Lemma 6.3. $\mathrm{Im}(\mathrm{Tr})$ *is in* w*-filtration zero.*

PROOF. Let $y \in \widetilde{K}(n)^*(BA)$ and $x \in \widetilde{K}(n)^*(BG)$ with $\mathrm{Tr}(y) = wx$. (5.3) holds analogously when H is replaced by G and V by A, whence $0 = w\,\mathrm{Tr}(y) = w^2x$ and therefore $0 = w^{p^n}x = pwx = p\,\mathrm{Tr}(y)$. The claim follows from the fact that $\widetilde{K}(n)^*(BG)$ has no p-torsion. □

Now let $Z \in \widetilde{K}(n)^*(BG)$ be an element with $w^2Z = tw^3(u^2v + uv^2) + s$. Such an element exists, since s is divisible by at least w^4, and Z itself will still be divisible by w. We conclude

$$\begin{aligned} 3N(Z) &= 3(1 + d + d^2)Z = w^8(1 + d + d^2)Z \\ &= w^6(1 + d + d^2)(tw^3(u^2v + uv^2) + s) = 0. \end{aligned}$$

The last equality holds since $tw^3(u^2v + uv^2) + s$ lies in the image of $(1 - d)$. Then $N(Z) = 0$ because $\widetilde{K}(n)^*(BG)$ has no 3-torsion, and it remains to show that Z is non-zero.

Lemma 6.4. $0 \neq Z \in H^1(C_p; \widetilde{K}(n)^*(BG))$.

PROOF. It is enough to prove $0 \neq Z \in H^1(C_3; M)$. Consider the w-filtration on M; this is a finite filtration since $w^8 = 0$. From the spectral sequence associated to this filtration one sees that it suffices to check that Z does not vanish in $H^1(C_3; \mathrm{gr}(M))$, where $\mathrm{gr}(M)$ denotes the associated graded of M. Setting $D = 1 - d$, the structure of $\mathrm{gr}(M)$ as D-module is as follows.

$$\begin{aligned} \mathrm{gr}(M) \cong \mathbb{F}_p[w] \otimes \big(&\mathbb{F}_p[D]/(D)\{1, u^2v^2\} \oplus \mathbb{F}_p[D]/(D^2)\{u, u^2v\} \oplus \mathbb{F}_p[D]/(D^3)\{u^2\}\big) \\ \otimes \big(&\mathbb{F}_p[D]/(D)\{1, z^2t^2\} \oplus \mathbb{F}_p[D]/(D^2)\{z, z^2t\} \oplus \mathbb{F}_p[D]/(D^3)\{z^2\}\big) \end{aligned}$$

One sees that u^2v and uv^2 are in different summands: $uv^2 = D(u^2v) \mod w$, and $u^2v \notin \mathrm{Im}(D)$. □

Chapter IV

Examples of Chern approximations

1 The examples D_8 and Q_8

These groups are among the few examples where one can completely calculate the Chern approximation. It turns out to agree with the Morava K-theory of the group in question, giving an explicit desripction of the algebra structure in terms of natural generators. Another way to obtain such a description would be by Bakuradze-Priddy's method, but relations obtained from the transfer alone do not suffice: one has to throw in exterior power relations.

We use the presentation of D_8 given in Section III.1, and

$$Q_8 = \langle g_1, g_2 \mid g_1^4,\ g_1^2 g_2^2,\ g_1 g_2^{-1} g_1 g_2 \rangle\,;$$

this allows us to treat both groups at the same time. We recall the representation theory: There are 4 one-dimensional irreducible complex representations and one two-dimensional. Let γ_j be defined by $\gamma_j(g_k) = (-1)^{\delta_{jk}}$ $(j,k = 1,2)$, and $\Delta = \mathrm{Ind}^G_{\langle g_1 g_2\rangle}(\beta)$ where $\beta(g_1 g_2) = i$. Then one has $\gamma_j^2 = 1$, $\gamma_j \Delta = \Delta$, $\Delta^2 = 1 + \gamma_1 + \gamma_2 + \gamma_1\gamma_2$, and

$$\lambda^2\Delta = \begin{cases} \gamma_1\gamma_2 & \text{for } D_8, \\ 1 & \text{for } Q_8. \end{cases}$$

We shall also use a further refinement of the formal group law, due to Bakuradze and Vershinin [BV]:

Lemma 1.1. $x +_{K(n)} y = x + y + \Big(xy + (x+y)(xy)^{2^{n-1}}\Big)^{2^{n-1}} \mod (xy)^{2^{2n-2}}(x+y)^{2^{2n-2}}$. □

Here we have suppressed the appropriate powers of v_n from the notation, as we shall keep doing below.

Now let $y_i = c_1(\gamma_i)$ $(i = 1,2)$ and $c_j = c_j(\Delta)$ $(j = 1,2)$. Then we know that $K(n)^*(BG)$ is multiplicatively generated by y_1, y_2, and c_2, see [TY2, SY] or Section 1 for D_8.

The first relations are easy: from $\gamma_i^2 = 1$ we immediately obtain

$$y_1^{2^n} = 0\,, \quad y_2^{2^n} = 0\,. \tag{1.1}$$

Now to $\gamma_i\Delta = \Delta$: by (1.1), the formula for $c_1(\gamma_i\Delta)$ from Proposition 2.1 simplifies to

$$c_1(\gamma_i\Delta) = c_1 + y_i^{2^{n-1}} c_1^{2^{n-1}} ,$$

whence

$$(y_i c_1)^{2^{n-1}} = 0 \,. \tag{1.2}$$

Similarly, the formula for $c_2(\gamma_i\Delta)$ simplifies to

$$\begin{aligned} c_2(\gamma_i\Delta) &= y_i^2 + y_i c_1 + c_2 + y_i(y_i c_1)^{2^{n-1}} + y_i^{2^{n-1}} \sum_{k=1}^{n-1} c_1^{2^{n-1}-2^k+1} c_2^{2^{k-1}} \\ &= y_i^2 + y_i c_1 + c_2 + y_i^{2^{n-1}} \sum_{k=1}^{n-1} c_1^{2^{n-1}-2^k+1} c_2^{2^{k-1}} \end{aligned}$$

where we used (1.2). Now since this is to equal c_2, we obtain, by repeated application of the ensuing formula,

$$y_i c_1 = y_i^2 + \sum_{k=1}^{n-1} y_i^{2^n-2^k+1} c_2^{2^{k-1}} \,. \tag{1.3}$$

We intend to use $\Delta^2 = 1 + \gamma_1 + \gamma_2 + \gamma_1\gamma_2$ next: one has $c_1(\Delta^2) = c_1(\Delta)^{2^n}$, hence

$$c_1^{2^n} = y_1 + y_2 + (y_1 +_{K(n)} y_2) = (y_1 y_2)^{2^{n-1}} \,. \tag{1.4}$$

By (1.2) and (1.3), this implies

$$\begin{aligned} y_1(y_1y_2)^{2^{n-1}} &= y_1 c_1^{2^n} = \Big(y_1^2 + \sum_{k=1}^{n-1} y_1^{2^n-2^k+1} c_2^{2^{k-1}} \Big) c_1^{2^n-1} \\ &= y_1^2 c_1^{2^n-1} = y_1^4 c_1^{2^n-3} = \cdots = y_1^{2^n} c_1 = 0 \end{aligned}$$

and thus

$$c_1^{2^n+1} = (y_1y_2)^{2^{n-1}} c_1 = y_1^{2^{n-1}} y_2^{2^{n-1}-1} \Big(y_2^2 + \sum_{k=1}^{n-1} y_2^{2^n-2^k+1} c_2^{2^{k-1}} \Big) = 0 \tag{1.5}$$

Consequently, using (1.3) repeatedly,

$$\begin{aligned} y_i c_1^2 &= y_i^2 c_1 + \sum_{k=1}^{n-1} y_i^{2^n-2^k} (y_i c_1) c_2^{2^{k-1}} \\ &= y_i^3 + \sum_{k=1}^{n-1} y_i^{2^n-2^k+2} c_2^{2^{k-1}} + \sum_{k=1}^{n-1} y_i^{2^n-2^k} c_2^{2^{k-1}} \Big(y_i^2 + \sum_{l=1}^{n-1} y_i^{2^n-2^l+1} c_2^{2^{l-1}} \Big) = y_i^3 \end{aligned}$$

which implies $y_1^3 y_2 = y_1 y_2 c_1^2 = y_1 y_2^3$. This gives

$$\begin{aligned} y_1 y_2 c_1 &= y_1^2 y_2 + \sum_{k=1}^{n-1} y_1^{2^n - 2^k + 1} y_2 c_2^{2^{k-1}} = y_1^2 y_2 + \sum_{k=1}^{n-1} y_1 y_2^{2^n - 2^k - 1} c_2^{2^{k-1}} \\ &= y_1^2 y_2 + y_1 y_2 c_1 + y_1 y_2^2 , \end{aligned}$$

hence

$$y_1^2 y_2 = y_1 y_2^2 . \tag{1.6}$$

Furthermore, since we may calculate modulo $c_1^{2^n+1}$ by (1.5), we have

$$c_2(\Delta^2) = c_1^2 + c_1^{2^n} c_2^{2^{2n-1}} .$$

On the other hand

$$c_2(1+\gamma_1+\gamma_2+\gamma_1\gamma_2) = y_1 y_2 + (y_1+y_2)(y_1+y_2+y_1^{2^{n-1}} y_2^{2^{n-1}}) = y_1^2 + y_1 y_2 + y_2^2$$

using (1.6), thus

$$c_1^2 = y_1^2 + y_1 y_2 + y_2^2 + (y_1 y_2)^{2^{n-1}} c_2^{2^{2n-1}} . \tag{1.7}$$

Also, modulo $c_1^{2^n+1}$ one has $c_3(\Delta^2) = c_1^{2^n} c_2^{2^n}$ and

$$c_3(1+\gamma_1+\gamma_2+\gamma_1\gamma_2) = y_1 y_2 (y_1 + y_2 + (y_1 y_2)^{2^{n-1}}) = y_1^2 y_2 + y_1 y_2^2 + c_1^{2^n+1} = 0 ,$$

leading to

$$(y_1 y_2)^{2^{n-1}} c_2^{2^n} = 0 \quad \text{and} \quad c_1^2 = y_1^2 + y_1 y_2 + y_2^2 . \tag{1.8}$$

So far, everything worked for either D_8 or Q_8. Now that we shall use exterior powers, things will start to differ. We have

$$c_1(\lambda^2 \Delta) = c_1 + c_2^{2^{n-1}} + c_1^{2^{n-1}} c_2^{2^{2n-2}} \tag{1.9}$$

since we may calculate modulo $c_1^{2^n} c_2^{2^n}$ by (1.8), and

$$c_1(\gamma_1\gamma_2) = y_1 + y_2 + (y_1 y_2)^{2^{n-1}} .$$

Together with (1.8) this gives

$$y_1^2 + y_1 y_2 + y_2^2 = c_1^2 = \begin{cases} c_2^{2^n} + y_1^2 + y_2^2 & \text{for } D_8, \\ c_2^{2^n} & \text{for } Q_8; \end{cases} \tag{1.10}$$

hence

$$c_2^{2^n} = \begin{cases} y_1 y_2 & \text{for } D_8, \\ y_1^2 + y_1 y_2 + y_2^2 & \text{for } Q_8. \end{cases} \tag{1.11}$$

Equations (1.8) - (1.11) furthermore imply

$$c_1 = \begin{cases} y_1 + y_2 + c_2^{2^{n-1}} & \text{for } D_8, \\ c_2^{2^{n-1}} + c_2^{2^{2n-1}} & \text{for } Q_8. \end{cases} \tag{1.12}$$

Finally, plugging all this into (1.3) results in

$$\sum_{k=1}^{n} y_i^{2^n - 2^k + 1} c_2^{2^{k-1}} = \begin{cases} y_1 y_2 & \text{for } D_8, \\ y_i^2 & \text{for } Q_8. \end{cases} \tag{1.13}$$

To complete the calculation of the Chern approximation, it is easy to check that $c_4(\Delta^2) = 0$ follows from the relations already proved and thus does not give rise to a new one.
Summing up, we get the following relations (which in the case D_8 indeed coincide with those obtained earlier):

(i) $y_i^{2^n} = 0$;

(ii) $c_2^{2^n} = \begin{cases} y_1 y_2 & \text{for } D_8, \\ y_1^2 + y_1 y_2 + y_2^2 & \text{for } Q_8; \end{cases}$

(iii) $\sum_{k=1}^{n} y_i^{2^n - 2^k + 1} c_2^{2^{k-1}} = \begin{cases} y_1 y_2 & \text{for } D_8, \\ y_i^2 & \text{for } Q_8. \end{cases}$

Furthermore, in (1.12) we have also identified c_1, which can not be done by restriction methods. Note that these relations imply all the others proved along the way, as well as $c_2^{2^{2n-1}+2^{n-1}} = 0$.
It remains to check that the Chern approximation has the correct rank, which according to the Euler chracteristic formula from [HKR] should be $\frac{3}{2}4^n - \frac{1}{2}2^n$. From the relations one easily reads off the basis (which works for either group)

$$\left\{ y_1^i y_2^\varepsilon c_2^k \,,\; y_2^j c_2^l \,,\; c_2^m \;\middle|\; \begin{array}{l} 1 \le i < 2^n,\ \epsilon \in \{0,1\},\ 0 \le j < 2^n, \\ 0 \le k, l < 2^{n-1},\ 2^{n-1} \le m < 2^n \end{array} \right\}$$

which has indeed the right length. We thus have:

Theorem 1.2. *Let G be either D_8 or Q_8.*

(a) $K(n)^*(BG) \cong C(G; K(n))$.

(b) $K(n)^*(BG)$ *is multiplicatively generated by the classes* y_1, y_2, c_2 *subject to the relations (i)-(iii) above.* □

Remark. Note that our relations coincide with those obtained in [BV]. The authors use slightly different generators there, their x corresponds to our y_1 and c to $y_1 +_{K(n)} y_2$.

2 Groups with dihedral and quaternion Sylow subgroups

As corollaries of the calculations of the previous section one can determine the Morava K-theory of groups having Q_8 or D_8 as Sylow subgroup. We shall do this for $SL_2(\mathbb{F}_3)$ for all n and Σ_4 for $n = 2$.

Let $G = SL_2(\mathbb{F}_3) \cong Q_8 \rtimes C_3$. This group has three one-dimensional representations $1, \varepsilon, \varepsilon^2$ factoring through the quotient C_3, three two-dimensional representations $\Delta, \varepsilon\Delta, \varepsilon^2\Delta$, where Δ restricts to the representation of the same name of Q_8 (which extends to G), and one three-dimensional representation, which is obtained by inducing a nontrivial linear character of Q_8 up to G. One has the following product relations, Adams operations, and exterior powers, easily calculated from the character table:

$$\beta^3 = 1, \quad \Delta^2 = 1 + \sigma, \quad \sigma^2 = 1 + \beta + \beta^2 + 2\sigma, \quad \beta\sigma = \sigma, \quad \Delta\sigma = (1 + \beta + \beta^2)\Delta$$

$$\psi^k\Delta = \begin{cases} 2 & k = 0, 1\ (12) \\ \Delta & k = 1, 5, 7, 11\ (12) \\ (-1 + \beta + \beta^2)\Delta & k = 2, 10\ (12) \\ \beta + \beta^2 & k = 4, 8\ (12) \\ 1 - \beta - \beta^2 + \sigma & k = 6\ (12) \end{cases}$$

$$\psi^k\sigma = \begin{cases} 3 & k = 0\ (6) \\ \sigma & k = 1, 5\ (6) \\ 1 + \beta + \beta^2 & k = 2, 4\ (6) \\ 2 - \beta - \beta^2 + \sigma & k = 3\ (6) \end{cases}$$

$$\lambda^2\Delta = 1\,, \quad \lambda^2\sigma = \sigma\,, \quad \lambda^3\sigma = 1\,.$$

Since the Morava K-theory of the 2-Sylow subgroup Q_8 is concentrated in even degrees, the same is true for $SL_2(\mathbb{F}_3)$. The rank of $K(n)^*(BG)$ is readily computed using the Euler characteristic formula,

$$\chi_{n,2}(SL_2(\mathbb{F}_3)) = \frac{1}{2}4^n + \frac{1}{2}2^n\,.$$

We proceed to calculate $C(SL_2(\mathbb{F}_3); K(n))$, although in less detail than above. Clearly, $c_k(\beta) = 0$ for $k > 0$. Let $c_i = c_i(\Delta)$ and $d_i = c_i(\sigma)$. From $\psi^4\Delta = \beta + \beta^2$ one obtains $c_i^{4^n} = 0$. (This implies in particular that in calculating Chern classes of operations on Δ, the terms of the formal group law given by the Bakuradze-Vershinin formula (1.1) suffice.) Secondly, $\psi^2\Delta + 1 = \sigma$ gives

$$d_1 = c_1^{2^n}\,, \quad , d_2 = c_2^{2^n} \text{ and } d_3 = 0\,,$$

whence the Chern classes of Δ generate – which is what one would expect, see below. Next, $\lambda^2\Delta = 1$ gives $c_1 = c_2^{2^n} + c_2^{2^{2n-1}}$. Finally, consider $\psi^3\Delta + \Delta = (\beta+\beta^2)\Delta$. Since $c_i(\beta\Delta) = c_i$, one concludes $c_1(\psi^3\Delta) = c_1$ and $c_2(\Psi^3\Delta+\Delta) = 0$, hence

$$c_2 = c_2(\psi^3\Delta) = c_2 + c_2^{2^{2n-1}+2^{n-1}}.$$

Thus $C(SL_2(\mathbb{F}_3); K(n))$ is generated by c_2 and has rank $2^{2n-1} + 2^{n-1}$. Since this coincides with the Euler characteristic (which equals the rank by virtue of there being no elements of odd dimension), we have proved:

Theorem 2.1. *Let $p = 2$ and $c_2 = c_2(\Delta)$. Then*

$$K(n)^*(BSL_2(\mathbb{F}_3)) \cong C(SL_2(\mathbb{F}_3); K(n)) \cong K(n)^*[c_2]/(c_2^{2^{2n-1}+2^{n-1}}) \qquad \square$$

Remark. The calculation of the Morava K-theory of this group (at $p = 2$) is of course much simpler than this: It follows immediately from the Serre spectral sequence that $K(n)^*(BG) = K(n)^*(BQ_8)^{C_3}$. The representation Δ is invariant under the C_3-action whence the same is true for its Chern classes. Now the submodule of $K(n)^*(BQ_8)$ generated by $\mathrm{Res}(c_2(\Delta))$ ($= c_2$ in the notation of Theorem 1.2) has the correct rank, as one immediately reads off the relations for Q_8.

Remark. Of course, this result can also (and arguably more easily) be obtained from Tanabe's theorem.

Next, consider the group Σ_4. The following calculation is a variant of Strickland's calculation in [St4], Section 14. The symmetric group Σ_4 has 5 irreducible representations $1, \varepsilon, \delta, \rho, \varepsilon\rho$ of dimensions $1, 1, 2, 3, 3$, respectivley, subject to relations

$$\varepsilon^2 = 1, \quad \varepsilon\delta = \delta, \quad \delta^2 = 1 + \varepsilon + \delta,$$

$$\rho^2 = 1 + \delta + \rho + \varepsilon\rho, \quad \delta\rho = \rho + \varepsilon\rho.$$

The Adams operations are easily calculated from the character table:

$$\psi^k\delta = \begin{cases} 2 & k \equiv 0 \mod 6 \\ \delta & k \equiv \pm 1 \mod 6 \\ 1 - \varepsilon + \delta & k \equiv \pm 2 \mod 6 \\ 1 + \varepsilon & k \equiv 3 \mod 6 \end{cases}$$

$$\psi^k\rho = \begin{cases} 3 & k \equiv 0 \mod 12 \\ \rho & k \equiv \pm 1, \pm 5 \mod 12 \\ 1 + \delta + \rho - \varepsilon\rho & k \equiv \pm 2 \mod 12 \\ 1 + \varepsilon - \delta + \rho & k \equiv \pm 3 \mod 12 \\ 2 - \varepsilon + \delta & k \equiv \pm 4 \mod 12 \\ 2 + \delta + \rho - \varepsilon\rho & k \equiv 6 \mod 12 \end{cases}$$

and $\psi^k(\varepsilon\rho) = \psi^j(\rho)$. Thus the non-trivial exterior power operations are

$$\lambda^2\delta = \varepsilon, \quad \lambda^2\rho = \lambda^2(\varepsilon\rho) = \varepsilon\rho, \quad \lambda^3\rho = \varepsilon, \quad \lambda^3(\varepsilon\rho) = 1\,.$$

Conceivably one can calculate the Morava K-theory of Σ_4 from these relations and operations, but we shall not do so. Instead, we use the embedding $K(n)^*(B\Sigma_4) \hookrightarrow K(n)^*(BD_8)$. Furthermore, we only consider the case $n = 2$.
Let $d_2 := c_2(\varepsilon\rho)$, $d_3 := c_3(\varepsilon\rho)$, and $y_1 := c_1(\varepsilon)$. We claim that $K(2)^*(B\Sigma_4)$ is generated by these three classes. Denote their images in $K(2)^*(BD_8)$ by the same names; since $\mathrm{Res}^{\Sigma_4}_{D_8}(\varepsilon) = \gamma_1$ and $\mathrm{Res}^{\Sigma_4}_{D_8}(\varepsilon\rho) = \gamma_1\gamma_2 + \Delta$ (using the same notation for representations of D_8 as before), y_1 coincides with the class also called y_1 earlier, whereas

$$\begin{aligned} d_2 &= c_2 + y_1^2 + y_2^2 + (y_1^3 + y_2^3)c_2 \\ d_3 &= (y_1 + y_2)c_2 + c_2^9 \end{aligned}$$

Using the relations in $K(2)^*(BD_8)$, we first see $d_2^i = c_2^i$ for $i \geq 2$, and

$$d_2 d_3 = 0, \quad d_3^2 = 0, \quad y_1 d_2^2 + y_1^3 d_2 = d_2^4, \quad y_1 d_3 = y_1^2 d_2 + y_1 d_2^3\,. \tag{2.1}$$

This gives a basis of 17 elements, namely

$$\{y_1^i d_2^j \mid 0 \leq i,j \leq 3\} \cup \{d_3\}\,.$$

Since 17 is the rank of $K(2)^*(B\Sigma_4)$, we conclude

Theorem 2.2 (Strickland)**.** *$K(2)^*(B\Sigma_4)$ is generated by y_1, d_2, d_3, subject to the relations* (2.1)*. All the relations can be obtained formally from the representation ring. Thus $K(2)^*(B\Sigma_4) \cong C(\Sigma_4; K(2))$.* □

The second sentence in the theorem follows from the result for D_8. This is not to say that $C(G; K(n))$ is isomorphic to $K(n)^*(BG)$ whenever this holds for a Sylow p subgroup P. Indeed, such a statement is plainly false: when $K(n)^*(BP) \cong C(P; K(n))$, one cannot even conclude that $K(n)^*(BG)$ is generated by Chern classes of (irreducible) representations of G. For an example here, consider the alternating group A_4: a simple consideration of invariants shows that $K(n)^*(BA_4)$ is not generated by Chern classes, one needs transfer classes.

3 Dihedral and quaternion groups of larger order

In order to prove a statement like part (a) of Theorem 1.2, it is not always necessary to determine all multiplicative relations.
Suppose one already knew that $K(n)^*(BG)$ was generated by Chern classes of representations. It then suffices to produce, using only formal consequences of the ring structure of RG plus Adams operations, enough relations among the

Chern classes of *all* irreducible representations so that the rank of the result is equal to the Euler characteristic of G. This is the course we shall follow for dihedral and generalised quaternion groups of larger order; for the assumption on generation by Chern classes we refer to the main result of [Sc], reproduced here as Corollary V.1.3.

Let

$$G = \langle a, b \mid a^{2^{m+1}} = 1,\ b^2 = a^e,\ bab^{-1} = a^{-1} \rangle$$

with $G \cong D_{2^{m+2}}$ for $e = 1$ and $G \cong Q_{2^{m+2}}$ for $e = 2^m$.

The next few lemmas, recording the structure of the representation ring RG and the Adams operations, are routine. Note that $RD_{2^{m+2}}$ and $RQ_{2^{m+2}}$ are isomorphic as rings.

Define representations γ_j $(j = 1, 2)$ and σ_k $(0 \le k < 2^{m+1})$ of G by

$$\gamma_1(a) = \gamma_2(a) = -1\,, \quad \gamma_1(b) = 1\,,\ \gamma_2(b) = -1$$

and

$$\sigma_k = \mathrm{Ind}_{\langle a \rangle}^G(\rho^k)$$

where $\rho\colon \langle a \rangle \hookrightarrow \mathbb{C}^\times$ is given by $\rho(a) = \exp(2^{-m}\pi i)$. The choices made for the γ_j are consistent with earlier notation, and will result in more symmetric looking formulas later on.

Lemma 3.1. *The irreducible representations of G are 1, γ_1, γ_2, $\gamma_1\gamma_2$, and σ_k for $1 \le k < 2^m$.* □

Note that $\sigma_0 = 1 + \gamma_1\gamma_2$ and $\sigma_{2^m} = \gamma_1 + \gamma_2$, as well as $\sigma_{2^m+r} = \sigma_{2^m-r}$.

Lemma 3.2. *The ring structure of RG is given by*

(a) $\gamma_j\sigma_k = \sigma_{2^m-k}$ $(0 \le k \le 2^m)$;

(b) $\sigma_j\sigma_k = \sigma_{k+j} + \sigma_{k-j}$ *for* $j \le k$. □

Adams operations and exterior powers differ for the two types; they are determined by

Lemma 3.3. *(a)* $\psi^k\sigma_1 = \sigma_k$ *for k odd;*

(b) $$\psi^2\sigma_k = \begin{cases} 1 - \gamma_1\gamma_2 + \sigma_{2k} & \text{for } G \text{ dihedral,} \\ (-1)^k(1 - \gamma_1\gamma_2) + \sigma_{2k} & \text{for } G \text{ quaternion;} \end{cases}$$

$$\lambda^2\sigma_k = \begin{cases} \gamma_1\gamma_2 & \text{for } G \text{ dihedral, or } G \text{ quaternion and } k \text{ even;} \\ 1 & \text{for } G \text{ quaternion and } k \text{ odd.} \end{cases}$$

□

The central extension $Z \to G \to D' \cong D_{2^{m+1}}$ (for either group) gives an 'inflation' map $RD' \to RG$ with image generated by the γ's and the σ_{2k}. This means we can inductively assume the relations among the Chern classes of these representations.

Theorem 3.4. *Let G be either $D_{2^{m+2}}$ or $Q_{2^{m+2}}$. Then $K(n)^*(BG) \cong C(G; K(n))$.*

Proof. We shall give fewer details than before; the arguments closely resemble those of Section 1. Also, we only give the proof for dihedral groups, the other case being similar.
Let $c_k = c_k(\sigma_1)$ and $y_j = c_1(\gamma_j)$. We shall prove:

(i) all other Chern classes can be expressed in terms of y_1, y_2, and c_2;

(ii) $y_1^{2^n} = y_2^{2^n} = 0$;

(iii) $y_1^2 y_2 = y_1 y_2^2$;

(iv) $y_1 y_2 = c_2^{2^{mn}}$

(v) $c_1 = y_1 + y_2 + c_2^{2^{n-1}} \bmod c_2^{2^n}$;

(vi) $c_2^{2^{(m+1)n-1}+2^{n-1}} = 0$;

(vii) $y_j c_2^{2^{n-1}} = f(y_j, c_2) + g(c_2)$ $(j = 1, 2)$ for certain polynomials f and g, where the c_2-degree of f is less than 2^{n-1}.

This suffices, since $K(n)^*(BG)$ is generated by Chern classes, and modulo the relations (i) - (vii) one has the basis

$$\{y_j^r c_2^s \mid j = 1, 2,\ 1 \le r < 2^n,\ 0 \le s < 2^{n-1}\} \cup \{c_2^t \mid 0 \le t < 2^{(m+1)n-1} + 2^{n-1}\}$$

of cardinality

$$\frac{1}{2} 2^{(m+1)n} + 4^n - \frac{1}{2} 2^n = \chi_{n,2}(G)\,.$$

Assertion (i) follows directly from Lemma 3.3, except for the claim about c_1, to which we shall return later. (ii) is obvious. Most of the others are proved by induction on m, based on the calculations for D_8. Let $d_k = c_k(\sigma_2)$. Then we can inductively assume (i) - (vii) for the d_k (where (i) refers to all σ_{2k}) with m replaced by $m-1$. Then (iii) is immediate, and (iv) follows from $\psi^{2^m}\sigma_1 + \gamma_1\gamma_2 = 1 + \gamma_1 + \gamma_2$, which gives $c_1^{2^{mn}} = (y_1 y_2)^{2^{n-1}}$ and then

$$c_2^{2^{mn}} = c_1^{2^{mn}}(y_1 +_{K(n)} y_2) + y_1 y_2 = (y_1 y_2)^{2^{n-1}}\left(y_1 + y_2 + (y_1 y_2)^{2^{n-1}}\right) + y_1 y_2 = y_1 y_2$$

where we have used (ii) and (iii).
Next, consider $\psi^2\sigma_1 + \gamma_1\gamma_2 = 1 + \sigma_2$. Applying first and second Chern classes to this identity yields

$$c_1^{2^n} = d_2^{2^{n-1}} \quad \bmod d_2^{2^n} \quad \text{and} \quad d_2 = c_2^{2^n} + (y_1 + y_2) c_2^{2^{2n-1}} \quad \bmod c_2^{2^{2n}}\,;$$

where the second equality should then be plugged into the first. Moreover, if we write z for $y_1 +_{K(n)} y_2$, then $\gamma_1\gamma_2\sigma_1 = \sigma_1$ implies

$$zc_1 = z^2 + \sum_{k=1}^{n-1} z^{2^n-2^k+1} c_2^{2^{k-1}} \quad \text{and} \quad zc_1^2 = z^3$$

since we can copy the proof of (1.3). Thus

$$(y_1+y_2)c_1 = (y_1+y_2)^2 + (y_1y_2)^{2^{n-1}} c_1 + \sum_{k=1}^{n-1} (y_1+y_2)^{2^n-2^k+1} c_2^{2^{k-1}} \tag{3.1}$$

(note that by virtue of (iii), $z^r = (y_1+y_2)^r$ for $r > 1$).
From $\lambda^2\sigma_1 = \gamma_1\gamma_2$ one obtains

$$c_1 = (y_1+y_2) + c_2^{2^{n-1}} + c_1^{2^{n-1}} c_2^{2^n} \mod c_1^{2^{2n-2}} c_2^{2^{2n-2}} \tag{3.2}$$

which by repeated application proves (i) for c_1 and (v). Together with (3.1) this implies

$$(y_1+y_2)c_2^{2^{n-1}} = \sum_{k=1}^{n-1} (y_1+y_2)^{2^n-2^k+1} c_2^{2^{k-1}} \tag{3.3}$$

which in turn gives (using (iii) and (iv))

$$\begin{aligned} c_2^{2^{(m+1)n-1}+2^{n-1}} &= (y_1y_2)^{2^{n-1}} c_2^{2^{n-1}} = y_1^{2^n-1} y_2 c_2^{2^{n-1}} \\ &= y_1^{2^n-1}\Big(y_1 c_2^{2^{n-1}} + \sum_{k=1}^{n-1} (y_1+y_2)^{2^n-2^k-1} c_2^{2^{k-1}}\Big) = 0\,, \end{aligned}$$

i.e., (vi). Finally, (vii) follows from

$$\sigma_1 + \gamma_1\sigma_1 = \sigma_{2^{m-1}-1}\sigma_{2^{m-1}} = \psi^{2^{m-1}-1}\sigma_1 \cdot \psi^{2^{m-1}}\sigma_1\,,$$

a general formula for $c_1(\gamma\sigma)$ as in Section 1, the formula for c_1, and the fact that by all of the equalities proved already, $c_2(\psi^{2^{m-1}-1}\sigma_1 \cdot \psi^{2^{m-1}}\sigma_1)$ can be expressed as a polynomial in y_j and c_2. □

Remark. (a) With more effort, the actual relations can be derived using this method, but we were more interested in the fact that the Morava K-theory of these groups is completely determined by $K(1)$. Also, there are more efficient ways to find the relations; one was hinted at in III.1, another consists of a combination of the above with transfer methods, see [BV].

(b) An analogous theorem can be proved for semidihedral groups.

Chapter V

Calculations at the prime 2

This chapter is concerned with the prime 2, as the title says. There are some ways in which this prime is different. For example, there is no complex orientation x such that $[-1](x) = -x$, which makes some arguments harder.
The first section is an outgrowth (and slight generalisation) of [Sc]; it deals with groups having a maximal cyclic subgroup and other related groups.
Section 2 covers groups of order 16. These calculations are implicit in Yagita's papers, but we chose to include them for illustration and reference.
Section 3 gives calulations of the Morava K-theory of the groups of order 32. Many of those are new, and make use of the tools developed in the first two sections of this chapter.
In this chapter we assume always $p = 2$.

1 Central extensions with dihedral quotients

In [Sc], we calculated $K(n)^*(BG)$ for groups G possessing a maximal cyclic subgroup; these are the dihedral, semidihedral, generalised quaternion, and quasidihedral groups of 2-power order. The method of calculation was in all instances the same: in each case there is a central extension

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{\pi} D \longrightarrow 1 \tag{1.1}$$

with D a dihedral group (it makes sense to identify $C_2 \times C_2$ with D_4). Then N is cyclic of order 2 for dihedral, semidihedral or generalised quaternion groups, and cyclic of index 4 in the quasidihedral case. It turned out that the Serre spectral sequence

$$E_2 = H^*(D; \mathbb{F}_2) \otimes K(n)^*(BN) \Longrightarrow K(n)^*(BG) \tag{1.2}$$

associated to (1.1) had only three differentials, namely $d_3\colon E_2^{0,3} \to E_2^{3,0}$, then $d_{2^{n+1}-1}$ given by Q_n, and one more, which is determined by the maximal cyclic subgroup.
After a slight generalisation of the method, we shall give a shortened account of this calculation.
The reason it is so easy is that one has perfect control over the Q_n-homology of quotients of $H^*(D)$ by non-zero divisors; thus this part of the argument can be

made a little bit more general. An 'integral' variant for the special case $D = D_4$ was originally considered in [SY].
Suppose that in the extension (1.1), D 'acts trivially' on N, by which we just mean that every element of D has a preimage in G that centralises N. This certainly happens when the extension is central, or when G is the central product of N with D. In general, when N is not abelian, D does not act on N, but only up to inner automorphisms of N, i.e., one has a homomorphism $\psi\colon D \to \mathrm{Out}(N)$; this homomorphism should be trivial. A set theoretic splitting s of π gives rise to a set theoretic lift $\phi\colon D \to \mathrm{Aut}(N)$ of ψ, and we further require the existence of a trivial lift.
Under these circumstances, we have

Theorem 1.1. *Let G be as above. Suppose that $K(n)^*(BN)$ is concentrated in even degrees, and that in the Serre spectral sequence* (1.2), *all elements in $E_4^{0,*}$ are permanent cycles. Then $K(n)^*(BG)$ is concentrated in even degrees.*

Proof. We first prove the statement when D has order 4 (this is the case treated in [SY]). Consider the inverse images H of any $C_2 \subset D_4$. Such H is either abelian or a central product; in any case, the associated Serre spectral sequence has only one differential $d_{2^{n+1}-1} = v_n Q_n$. This implies that the first potentially nontrivial differential has to be of the form $d_3 z = x_1^2 x_2 + x_1 x_2^2 \mod v_n$, where $H^*(D_4; \mathbb{F}_2) = \mathbb{F}_2[x_1, x_2]$. Thus we obtain an isomorphism

$$E_4 \cong K \otimes \mathbb{F}_2[x_1, x_2]/(x_1 x_2(x_1 + x_2)) \oplus H \otimes \mathbb{F}_2[x_1, x_2]\{x_1 x_2(x_1 + x_2)\}$$

where $K = \mathrm{Ker}(d_3|_{K(n)^*(BN)})$ and $H = H(K(n)^*(BN), d_3 \otimes (x_1 x_2(x_1 + x_2))^{-1})$. By assumption on E_4, the next differential is $d_{2^{n+1}-1} = v_n Q_n$. It is now easy to verify that the Q_n-homology of $M' = \mathbb{F}_2[x_1, x_2]x_1 x_2(x_1 + x_2)$ and $M'' = \mathbb{F}_2[x_1, x_2]/(x_1 x_2(x_1 + x_2))$ is finite and concentrated in even degrees: look at the short exact sequence $0 \to M' \to M \to M'' \to 0$ with $M = H^*(BV)$ and the induced long exact sequence(s) in Q_n-homology (one for each degree modulo $|v_n|$). The Q_n-homology of M'' is even and concentrated in degrees at most 2^{n+1}, and the map $H(M; Q_n) \to H(M''; Q_n)$ is onto, rendering the connecting homomorphisms trivial, whence the claim. This finishes the proof for this case.
For the case $D = D_{2^{m+1}}$ with $m > 1$, first recall the cohomology of D (e.g. from III.1):

$$H^*(D; \mathbb{F}_2) \cong \mathbb{F}_2[x_1, x_2, w_2]/(x_1 x_2)$$

As before, we sometimes write w_1 for $x_1 + x_2$. There are two conjugacy classes of maximal elementary abelian subgroups, represented (say) by K and T, both of rank two. Restricting to these and applying the special case just proved, one sees that d_3 is either trivial, or has image $(w_1 w_2) \mod v_n$. If d_3 is trivial, we are done by the assumption on E_4 and (the calculation of) the Atiyah-Hirzebruch spectral sequence for the dihedral quotient. Otherwise, we have

$$E_4 \cong K \otimes M'' \oplus H \otimes M'$$

where K and H are defined as before as the kernel and homology of d_3 and $d_3 \otimes (w_1w_2)^{-1}$ on $K(n)^*(BN)$, respectively. Here $M' = M\{w_1w_2\}$ and $M'' = M/(w_1w_2)$, where $M = H^*(D;\mathbb{F}_2)$ (note that w_1w_2 is not a zero divisor in M). The next differential being $d_{2^{n+1}-1} = v_n \otimes Q_n$, we need to calculate the Q_n-homology of M' and M''. For M'' this is easy: modulo w_1w_2, one has

$$Q_n(x_i) = x_i^{2^{n+1}}\,, \quad Q_n(w_2) = \sum_{r=0}^{n} w_1^{2^{n+1}-2^{r+1}+1} w_2^{2^r} = 0\,, \quad Q_n(x_1w_2) = 0\,,$$

giving

$$H(M'';Q_n) = \mathbb{F}_2[w_2]\{1, x_1w_2\} \oplus \mathbb{F}_2[x_1^2, x_2^2]/(x_1^2x_2^2, x_2^{2^{n+1}}, x_2^{2^{n+1}})\,.$$

For M', one could either do this directly, or, since we already calculated the Q_n-homology of M in III.1, by means of the short exact sequence of Q_n-modules $0 \to M' \to M \to M'' \to 0$ and the associated long exact sequence(s)

$$\cdots \longrightarrow H^s(M';Q_n) \xrightarrow{\iota} H^s(M;Q_n) \xrightarrow{\kappa} H^s(M'';Q_n) \xrightarrow{\delta} H^{s+|Q_n|}(M';Q_n) \longrightarrow \cdots\,.$$

Recall from Lemma III.1.2

$$H(M;Q_n) = \mathbb{F}_2[x_1^2, x_2^2]/(x_1^2x_2^2, x_2^{2^{n+1}}, x_2^{2^{n+1}}) \otimes \mathbb{F}_2[w_2^2]/(w_2^{2^n}) \oplus \mathbb{F}_2[w_2^2]\{w_2^{2^n}, \zeta\}$$

with $\zeta = \sum_{r=0}^{n} x_1^{2^{n+1}-2^{r+2}+1} w_2^{2^r}$. We need to determine δ and κ. The formulas for the action of Q_n give

$$\begin{aligned} \delta(x_i^{2k}) &= \delta(w_2^{2l}) = 0\,,\\ \delta(w_2^{2k+1}) &= w_2^{2k} Q_n w_2 \quad \text{(note that } Q_nw_2 \text{ is not a boundary in } M'),\\ \delta(x_1w_2^{2k}) &= x_1^{2^{n+1}} w_2^{2k}\,, \quad \delta(x_1w_2^{2k+1}) = Q_n(x_1w_2)w_2^{2k} = \sum_{r=1}^{n} x_1^{2^{n+1}-2^{r+1}+2} w_2^{2^r+2k}\,. \end{aligned}$$

Reduction modulo w_1w_2 gives

$$\mathrm{Ker}(\kappa) = \mathbb{F}_2\{x_1^{2i}w_2^{2k},\ x_2^{2j}w_2^{2l} \mid 1 \le i,j < 2^n\,,\ 0 \le k,l < 2^{n-1}\}\,.$$

Splitting the long exact homology sequences into short exact sequences $0 \to \mathrm{Im}(\delta) \to H(M';Q_n) \to \mathrm{Ker}(\kappa) \to 0$ thus yields an additive isomorphism

$$\begin{aligned} H(M';Q_n) \cong\ & \mathbb{F}_2[w_2^2]\{x_1^{2^{n+1}},\ Q_n(x_1w_2),\ Q_nw_2\}\\ & \oplus \mathbb{F}_2\{x_1^{2i}w_2^{2k},\ x_2^{2j}w_2^{2l} \mid 1 \le i,j < 2^n\,,\ 0 \le k,l < 2^{n-1}\}\,. \end{aligned}$$

Thus the $E_{2^{n+1}}$-page

$$E_{2^{n+1}} \cong K \otimes H(M''; Q_n) \oplus H \otimes H(M'; Q_n)$$

is still infinite, and there must be further differentials. The only way to arrive at a finite E_∞-page is when ζ and $Q_n w_2$ support differentials, and we obtain an E_∞-page concentrated in even degrees. □

Corollary 1.2. *Suppose in addition that all elements in $E_4^{0,*}$ are restrictions of good elements of $K(n)^*(BG)$. Then $K(n)^*(BG)$ is good.*

Proof. This is a consequence of the following facts: (i) the x_i^2 are clearly represented by Euler classes of one-dimensional representations of G; (ii) there is an extension problem identifying w_2 as a polynomial in elements of $E_4^{0,*}$: this can be seen either by restriction to subgroups, or by appealing to the extension class in (ordinary) cohomology. □

As a first application, we immediately recover the aforementioned results of [Sc]:

Corollary 1.3. *Let G be either dihedral, semidihedral, generalised quaternion, or quasidihedral. Then*

(a) $K(n)^{odd}(BG) = 0$;

(b) $K(n)^*(BG)$ is generated by Euler classes of complex representations.

Proof. For each of the four types, one has a central extension

$$1 \longrightarrow C \longrightarrow G \longrightarrow D \longrightarrow 1$$

as in the lemma, with $C = C_2$ for the first three types, and C an index four cyclic subgroup for quadidihedral G.
It remains to check the condition on the E_4-page of the Serre spectral sequence. It is easily verified that each of these groups has a two-dimensional complex representation restricting to a sum of two copies of a generator μ of the representation ring of the (cyclic) centre; thus if $z = c_1(\mu)$ denotes the generator of $K(n)^*(BC)$, then z^2 is the restriction of an Euler class of G. □

Remark. The multiplicative relations given in [Sc] contain errors, since we missed another extension problem there. For D_8 and Q_8, the correct relations were derived from the Chern approximation in IV.1. The same method should work for the bigger groups; compare the remark at the end of IV.3.

Remark. Since these groups are metacyclic, one could equally well derive the corollary from the results of Tezuka-Yagita [TY3].

There are other instances when the lemma is useful, as shall be seen in the next sections.

2 Groups of order 16

There are 11 nonabelian groups of order 16,

(a) $D_8 \times C_2$, $Q_8 \times C_2$,

(b) D_{16}, Q_{16}, the semidihedral group SD_{16}, the quasidihedral group QD_{16},

(c) the central product $C_4 \circ D_8$, also known as almost extraspecial group,

(d) $G_1 = \langle a, b, c \mid a^4 = b^2 = c^2 = 1, cac = ab, [a, b] = [b, c] = 1\rangle \cong (C_4 \times C_2) \rtimes C_2$,

(e) $G_2 = \langle a, b \mid a^4 = b^4 = 1, b^{-1}ab = a^{-1}\rangle \cong C_4 \rtimes C_4$.

The groups in (a) and (b) have been dealt with in previous sections. The central product is described additively by Corollary II.5.3, so just (d) and (e) remain: both are minimal non-abelian 2-groups, i.e., all of their maximal subgroups are abelian. Thus they also have even Morava K-theory by Theorem III.3.4.

3 Groups of order 32

In this section we calculate $K(n)^*(BG)$ additively for G of order 32. The original motivation for doing so was the faint hope of finding a 2-primary counterexample to the Hopkins-Kuhn-Ravenel conjecture, i.e., a 2-group G with odd Morava K-theory. In this we failed — the groups considered are probably way too small. The problem thus remains open.

The theory as described in Chapter III covers 33 groups, leaving 18 to be calculated. For some groups we only consider the second Morava K-theory $K(2)$; the reason for this restriction is that in these cases we rely on computer calculations (with MAPLE), being unable to cope without. This concerns the groups #38-41 and #44-48, which shall be dealt with at the very end of the section.
We shall use the Hall-Senior list for the 51 groups of order 32, and denote the group number i by G_i. The groups are ordered by their central quotients.

3.1 Groups 1-15

The first 7 groups are abelian, and G_8 - G_{15} have an abelian factor, so they are all good by the results of Section 2.

3.2 Groups 16-22

These all have central quotient $C_2 \times C_2$, so Theorem 1.1 applies, once one has checked the hypothesis on the E_4-page of the Serre spectral sequence. In fact, we only need to do this for G_{16} and G_{17}, since the others are metacyclic and/or minimal non-abelian:

$$G_{18} = \langle a, b, c \mid a^4 = b^4 = c^2 = [a, c] = [b, c] = 1,\ a^{-1}ba = ac\rangle \cong (C_4 \times C_2) \rtimes C_4$$

and

$$G_{20} = \langle a, b, c \mid a^8 = b^2 = c^2 = [a, c] = [b, c] = 1,\ bab = ac\rangle \cong (C_8 \times C_2) \rtimes C_2$$

are minimal non-abelian and hence good by Theorem III.3.4;

$$\begin{aligned} G_{19} &= \langle a, b \mid a^8 = b^4 = 1,\ b^{-1}ab = a^5\rangle \cong C_8 \rtimes C_4 \\ G_{21} &= \langle a, b \mid a^4 = b^8 = 1,\ b^{-1}ab = a^3\rangle \cong C_4 \rtimes C_8 \\ G_{22} &= QD_{32} \end{aligned}$$

are split metacyclic.
G_{16} is a semidirect product $(C_4 \times C_4) \rtimes C_2$; a presentation is

$$G = G_{16} = \langle a, b, c \mid a^4 = b^4 = c^2 = [a, b] = [b, c] = 1,\ cac = ab^2\rangle\,.$$

a^2 and b generate the centre $Z \cong C_4 \times C_2$, and we consider the Serre spectral sequence for the central extension $1 \to Z \to G \to C_2 \times C_2 \to 1$ with

$$E_2 \cong K(n)^*[y, z]/(y^{2^n}, z^{4^n}) \otimes H^*(C_2 \times C_2; \mathbb{F}_2)$$

where y and z are the Euler classes of the representations η and λ of Z defined by $\eta(a^2) = -1$, $\eta(b) = 1$, and $\lambda(a^2) = 1$, $\lambda(b) = i$, respectively. Then η extends to a representation $\tilde\eta$ of G_{16} by setting $\tilde\eta(a) = i$ and $\tilde\eta(c) = 1$; thus y is a permanent cycle. z however is not, but we need to check that z^2 is. Define a representation σ of G by

$$\sigma = \mathrm{Ind}^G_{\langle a,b\rangle}(\mu) \quad \text{where } \mu(a) = 1,\ \mu(b) = i.$$

Then $\mathrm{Res}^G_Z(\sigma) = 2\lambda$, and we are done.

G_{17} has a presentation

$$G = G_{17} = \langle a, b, c \mid a^8 = c^2 = [a, c] = [a, b] = 1,\ b^2 = a^2,\ cbc = a^4 b\rangle\,;$$

the centre $Z = \langle a\rangle$ is cyclic of order 8 with quotient $C_2 \times C_2$. The E_2-term of the Serre spectral sequence of the corresponding central extension is then

$$E_2 = K(n)^*[z]/(z^{8^n}) \otimes H^*(C_2 \times C_2; \mathbb{F}_2)$$

with $z = e(\rho)$, $\rho(a) = \exp(\pi i/4)$. Now ρ extends to a representation $\tilde\rho$ of $\langle a, c\rangle$ by setting $\tilde\rho(c) = 1$. Since a is central, one has $\tilde\rho^b(a) = \tilde\rho(a)$, whence

$$\mathrm{Res}^G_Z \mathrm{Ind}^G_{\langle a,c\rangle}(\tilde\rho) = 2\rho\,,$$

implying that z^2 is a permanent cycle. Summing up:

Theorem 3.1. *The groups #16-22 are good.* □

3.3 Groups 23-33

These groups all have central quotient D_8, so this will be another instance of Theorem 1.1 and henceforth an easy exercise in representation theory.
They also share the same Morava K-theory Euler characteristic

$$\chi_{n,2}(G) = \frac{1}{2}16^n + 8^n - \frac{1}{2}4^n .$$

The first three (groups 23-25 or Γa_i, $i = 1, 2, 3$) have a direct factor C_2: they are isomorphic to $C_2 \times D_{16}$, $C_2 \times SD_{16}$, and $C_2 \times Q_{16}$, respectively.

$$\begin{aligned} G_{29} &= \langle a, b \mid a^8 = b^4 = 1,\ b^{-1}ab = a^{-1}\rangle \quad \text{and} \\ G_{30} &= \langle a, b \mid a^8 = b^4 = 1,\ b^{-1}ab = a^3\rangle \end{aligned}$$

are split metacyclic, whereas $G_{31} \cong C_4 \wr C_2$ and $G_{33} = (C_2 \times C_2) \wr C_2$. Thus we are left with numbers 26-28 and 32.

Groups 26-28 and 32. Presentations are e.g. as follows:

$$\begin{aligned} G_{26} &= \langle a, b, c \mid a^8 = b^2 = c^2 = [a, b] = 1,\ cac = a^{-1},\ cbc = a^4 b\rangle \\ G_{27} &= \langle a, b, c \mid a^8 = b^2 = c^2 = [a, b] = [b, c] = 1,\ cac = a^{-1} b\rangle \\ G_{28} &= \langle a, b, c \mid a^8 = b^2 = [a, b] = [b, c] = 1,\ c^2 = a^4,\ c^{-1}ac = a^3 b\rangle \\ G_{32} &= \langle a, b, c \mid a^8 = 1,\ b = c^2,\ c^4 = a^4,\ [a, b] = [b, c] = 1,\ c^{-1}ac = a^3\rangle \end{aligned}$$

We chose these presentations in order to facilitate unified treatment: for example, a and b generate a maximal abelian subgroup $A \cong C_8 \times C_2$ in all four of these groups.

We start with G_{26}: this group is a central product of a cyclic group of order four (generated by $a^2 b$) with a dihedral group of order 16, and we are done by Corollary II.5.3.

Next, G_{32} is (nonsplit) metacyclic, thus already covered.

Finally, let G be either G_{27} or G_{28}. Both groups have centre $Z = \langle a^4, b\rangle \cong C_2 \times C_2$ (with quotient D_8, as remarked earlier). Mandated by Theorem 1.1, we consider the Serre spectral sequence of the central extension

$$1 \longrightarrow \langle a^4, b\rangle \longrightarrow G \longrightarrow \langle \bar{a}, c\rangle \longrightarrow 1$$

with

$$E_2 = H^*(D_8; K(n)^*(BZ)) \cong \mathbb{F}_2[x_1, w_1, w_2]/(x_1^2 + x_1 w_1) \otimes K(n)^*[z_1, z_2]/(z_1^{2^n}, z_2^{2^n}) .$$

Here z_1 and z_2 are the Euler classes of λ_1, λ_2 corresponding to a^4 and b, respectively, while we keep the notation for $H^*(D_8; \mathbb{F}_2)$ from previous sections.

Since $[G,G] = \langle a^2 b\rangle \cong C_4$, we have a one-dimensional representation β of G with $\beta(b) = -1$ (and $\beta(a) = \beta(c) = 1$); this restricts to λ_2 on the centre. Thus z_2 is a permanent cycle. Now let $A = \langle a, b\rangle \cong C_8 \times C_2$ as above, and define $\rho \in RA$ by $\rho(a) = \exp(\pi i/4)$ and $\rho(b) = 1$. Then ρ^c is either ρ^{-1} (for G_{27}) or ρ^3 (for G_{28}); in any case,

$$\mathrm{Res}^G_Z \mathrm{Ind}^G_A(\rho) = \mathrm{Res}^A_Z(\rho + \rho^c) = 2\lambda_1\,,$$

so z_1^2 is a permanent cycle, too.

Theorem 3.2. *The groups #23-33 are good.* □

3.4 Groups 34-37

Presentations of G_{34}-G_{37} are as follows:

$$G_{34} = \langle a, b, c \mid a^4 = b^4 = c^2 = [a, b] = 1,\ cac = a^{-1},\ cbc = b^{-1}\rangle$$
$$G_{35} = \langle a, b, c \mid a^4 = b^4 = [a, b] = 1,\ c^2 = a^2, cac = a^{-1},\ cbc = b^{-1}\rangle$$
$$G_{36} = \langle a, b, c \mid a^4 = b^4 = c^2 = [b, c] = 1,\ a^{-1}ba = b^{-1},\ cac = a^{-1}\rangle$$
$$G_{37} = \langle a, b, c \mid a^4 = c^2 = d^2 = [b, c] = 1,\ d = [a, c],\ b^2 = a^2,\ bab^{-1} = a^{-1}\rangle$$

All four groups have centre $Z \cong C_2\times C_2$ with quotient C_2^3, and Euler characteristic

$$\chi_{n,2} = \frac{1}{2}16^n + 8^n - \frac{1}{2}4^n\,.$$

G_{34} and G_{35} have the maximal abelian subgroup $A = \langle a, b\rangle \cong C_4 \times C_4$, on which the quotient acts (diagonally) by inverting a and b. From the result for D_8 we know that $M := \widetilde{K}(n)^*(BC_4)$ is a permutation module for the automorphism inverting the generator of the group, thus $\widetilde{K}(n)^*(BA) \cong M \otimes M$ is again a permutation module. It follows that G_{34} and G_{35} are both good.
G_{36} contains the maximal abelian aubgroup $A = \langle b, a^2, c\rangle \cong C_4 \times C_2 \times C_2$. From the relations one reads off that $\widetilde{K}(n)^*(BA) \cong M \otimes N$, where $N = \widetilde{K}(n)^*(BC_2 \times C_2)$ with the switch action, so this is again a permutation module; the situation is similar for G_{37} and the maximal abelian subgroup $A = \langle b, c, d\rangle$.

Theorem 3.3. *The groups #34-37 are good.* □

3.5 Extraspecial groups

There are two of those, namely the central products $D_8 \circ D_8$ and $D_8 \circ Q_8$, and they carry the numbers 42 and 43. Both groups were treated originally in [SY] using integral Morava K-theory. Here we present a mod 2 calculation; this has its advantages, as we shall see.
In both cases, G is generated by elements $a_1, \ldots, a_4$ of order 2, and we have an extension

$$1 \longrightarrow G' \longrightarrow G \longrightarrow V \longrightarrow 1$$

with $G' \cong D_8$ and trivial V-action on G'. Thus the strategy will be to use Theorem 1.1.
Set $G_{ij} = \langle a_i, a_j \rangle \subset G$, numbering the generators a_i such that $G' = G_{12}$, and $A_i = \langle a_i \rangle$. Then $G_{34} \cong D_8$ or Q_8, and $G_{34}/C = V$ for $C =$ centre of G. This allows us to keep the notation for $K(n)^*(BD_8)$ from the earlier section. Furthermore, let $H^*(BV; \mathbb{F}_2) = \mathbb{F}_2[x_1, x_2]$, and set for convenience $\alpha = x_1^2 x_2 + x_1 x_2^2$. We consider the spectral sequence

$$E_2^{*,*} = H^*(BV; K(n)^*(BD_8)) \Longrightarrow K(n)^*(BG)\,. \qquad (3.1)$$

Lemma 3.4. *In the above spectral sequence, we have*

$$d_3 c_2 = c_1 \otimes \alpha \quad \mathrm{mod}\ (y_1, y_2)^2\,.$$

Before beginning with the proof, recall that $c_1 = y_1 + y_2 + v_n c_2^{2^{n-1}}$; we use c_1 here for notational simplicity. Also, all $K(n)^*$ generators of cyclic groups will indiscriminately be called u.

Proof. For dimensional reasons, $d_3 c_2 = (\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 c_1) \otimes \alpha \mod (y_1, y_2)^2$ with $\lambda_i \in \mathbb{F}_2$. Consider the map of spectral sequences induced by

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A_1 \times C & \longrightarrow & A_1 \times G_{34} & \longrightarrow & V = G_{34}/C & \longrightarrow & 0 \\ & & \downarrow i & & \downarrow i & & \| & & \\ 1 & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & V & \longrightarrow & 0 \end{array}$$

Since $\mathrm{Res}_{A_1 \times C}(c_2) = u^2 + u y_1 \mod (y_1^2)$ and $d_3 u = 1 \otimes \alpha$, we get

$$i^*(d_3 c_2) = d_3(u^2 + u y_1) = y_1 \otimes \alpha \quad \mathrm{mod}\ (y_1^2)$$

and hence $\lambda_1 + \lambda_3 = 1$. Similarly, replacing A_1 with A_2, we get $\lambda_2 + \lambda_3 = 1$. Finally, consider the inclusion of $A = \langle a_1 a_2 \rangle \cong C_4$ into G_{12}:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & A \circ G_{34} & \longrightarrow & V & \longrightarrow & 0 \\ & & \downarrow j & & \downarrow j & & \| & & \\ 1 & \longrightarrow & G_{12} & \longrightarrow & G & \longrightarrow & V & \longrightarrow & 0 \end{array}$$

Now modulo $u^{2^{n+1}}$, we have $\mathrm{Res}_A(c_2) = u^2 + v_n u^{2^n+1}$ and thus $j^*(d_3 c_2) = v_n u^{2^n} \otimes \alpha$. Since $\mathrm{Res}_A(c_1) = v_n(\mathrm{Res}_A(c_2))^{2^{n-1}} = v_n u^{2^n}$, we get $\lambda_1 + \lambda_2 + \lambda_3 = 1$, too. □

Therefore

$$d_3(y_i c_2) = y_i c_1 \otimes \alpha = y_i^2 \otimes \alpha \quad \mathrm{mod}\ (y_1, y_2)^3\,.$$

Using this formula, it is easy to see that $K := \mathrm{Ker}(d_3|_{K(n)^*(BD_8)})$ is generated as $K(n)^*$-algebra by

$$y_1\,,\ y_2\,,\ c_2^2\,,\ b_1 = y_1^{2^n-1} c_2\,,\ b_2 = y_2^{2^n-1} c_2\,,\ y_2 b_1 = y_1 b_2 = y_1 y_2^{2^n-1} c_2 = c_2^{2^{2n-1}+1}\,.$$

The last three terms are in K since $v_n y_i^{2^n} = 0$ in $K(n)^*(BD_8)$. More precisely, we have

Lemma 3.5. *In the spectral sequence* (3.1)*, the kernel K and the homology H with respect to $d_3 \otimes \alpha^{-1}$ are given additively by*

$$\begin{aligned} K &\cong K(n)^*[c_2^2]/(c_2^{2^{n-1}})\{b_1, b_2, y_2b_1, y_1^i, y_2^j(1 \le i,j < 2^n)\} \\ &\quad \oplus \; K(n)^*[c_2^2]/(c_2^{2^{2n-1}+2^{n-1}}), \\ H &\cong K(n)^*\{1, y_1, y_2, b_1, b_2, y_1b_2\}[c_2^2]/(c_2^{2^{n-1}}) \end{aligned}$$

Proof. This is a simple calculation using the relations in $K(n)^*(BD_8)$. □

We want to show that all elements in K are permanent cycles. This is clear for y_1 and y_2, since they are Euler classes of linear characters of G itself. Furthermore, c_2^2 is the restriction of the Euler class of the spin representation of G: it restricts to 2Δ on D_8. Finally, using the relation

$$\sum_{k=1}^{n} y_i^{2^n-2^k+1} c_2^{2^{k-1}} = y_1 y_2$$

we see that b_1, b_2, and y_2b_1 are polynomials in y_i and c_2^2 (so we might as well have said that K is generated by y_i and c_2^2, but the b_i show up as generators for homology). Note how much easier this argument is compared to the laboured character considerations in [SY]. We even get a slightly better result: all generators are restrictions of Euler classes of representations of G proper – no need for transfers.

Thus the assumptions of Theorem 1.1 hold, yielding (for $D_8 \circ Q_8$ the reasoning is completely analogous)

Theorem 3.6. *Let G be an extraspecial group of order* 32*. Then $K(n)^*(BG)$ is concentrated in even degrees, and generated by Euler classes.* □

Remark. One might try other extensions as well, e.g., the one with a single C_2 on top. Morally, the associated spectral sequence should have no differentials at all, if one works integrally, or only the one inherited from the Atiyah-Hirzebruch spectral sequence for BC_2 in the mod 2 case. One would however need a calculation of invariants, which (at least in the case of Morava K-theory) seems fairly complicated. At the other extreme, one could attempt to use Quillen's original approach for mod 2 Morava K-theory. One gets a similar picture with powers of the multiplicative generator of $K(n)^*(BC_2)$ transgressing and killing a regular sequence in the cohomology of BV, but is left with the problem of computing the Q_n-homology of the quotient. Computer calculations for $n = 2, 3$ suggest that this is indeed finite, concentrated in even degrees, and gives rise to the correct rank, but we have been unable to prove it in general. Conceptually it would be the most satisfying approach.

Remark. Yet another possible approach is using the Rothenberg-Steenrod spectral sequence of II.5 for the description of the group as a central product $D_8 \circ D_8$. To that end, one first has to calculate the structure of $K(n)^*(BD_8)$ as a comodule over the Morava K-theory of the centre Z: let $K(n)^*(BZ) = K(n)^*[u]/(u^{2^n})$ with u the Euler class of η, say. Keeping the notation established previously for D_8, the composite of the multiplication map $\mu\colon Z \times D_8 \to D_8$ with the representations Δ and γ_i of D_8 gives

$$\gamma_i \circ \mu = 1 \otimes \gamma_i\,, \quad \Delta \circ \mu = \eta \otimes \Delta\,.$$

From now on let $n = 2$; then one obtains

$$\begin{aligned} \mu^*(y_i) &= 1 \otimes y_i \quad (i = 1, 2) \\ \mu^*(c_1) &= 1 \otimes c_1 + u^2 \otimes c_1^2 \\ \mu^*(c_2) &= 1 \otimes c_2 + u \otimes c_1 + u^2 \otimes (1 + c_1 c_2) + u^3 \otimes c_1^2 \end{aligned}$$

Using this coaction, we used a MAPLE program to calculate first the cotensor product of two copies of $K(2)^*(BD_8)$ over $K(2)^*(BZ)$. This turns out to have rank 148 ('equidistributed' over the degrees 0, 2, 4, i.e., one extra in degree 0). To calculate

$$\mathrm{Cotor}_{K(2)^*(BZ)}\big(K(2)^*(BD_8), K(2)^*(BD_8)\big)\,,$$

we would have to construct a resolution e.g. by extended comodules. We found it easier to dualise everything and build a free resolution of $K(2)_*(BD_8)$ as a $K(2)_*(BZ)$-module. The $K(2)$-homology of BZ is again a truncated polynomial algebra, generated by ξ dual to u^2 (in the basis $\{1, , u, u^2, u^3\}$), cf. Theorem II.5.1. Feeding this into MAPLE, one sees that the 22 dimensional module $K(2)_*(BD_8)$ splits as a sum of four free and six trivial modules (this can be done by computing the Jordan normal form for the action of ξ.) Thus Tor_k (Cotor^k) has rank 148 for $k = 0$ and rank 6 in every positive degree. This means that the spectral sequence cannot collapse on E_2, there must be differentials. Having another argument for this group, we have not pursued this line further.

3.6 Groups 49-51

These are the dihedral, semidihedral, and generalised quaternion groups of order 32 and thus covered by Corollary 1.3.

3.7 Groups 38-41

The results in this and the next subsection are for $n = 2$ only and were obtained with the aid of computer calculations. The method is simple: choose an index 2 subgroup H whose Morava K-theory you know everything about, calculate invariants, and try to represent them as restrictions to H of transfers of Chern classes of subgroups of G.

The easy bit is to calculate invariants. The image of $\mathrm{Res}_H^G \mathrm{Tr}_H^G$ clearly gives the invariants corresponding to the free summands (recall that $K(n)^*(BH)$ is a direct sum of free and trivial C_2-modules), so we need to find representatives for the trivial summands. Often these are restrictions of Chern classes of G itself, but not always, and one has to caclulate the transfer from other subgroups of G.
We did this with MAPLE, which limited us to $n = 2$: calculating the image of $C(G; K(n))$ tends to get large; without human intervention one would have to handle square matrices of roughly 100MB in size even for $n = 2$ — small fry for a serious computational effort, which we however shied.
For each group, we supply the following information: a presentation, the $K(n)$ Euler characteristic, the index 2 subgroup H used in the calculation, the irreducible representations of G and their restrictions to H, the action of G/H on the representation ring RH of H and the resulting action on $K(n)^*(BH)$, and finally the result of our MAPLE manipulations.

Presentations.

$$G_{38} = \langle a, b, c \mid a^4 = b^2 = c^4 = [a, b] = 1,\ cac^{-1} = ac^2,\ cbc^{-1} = a^2 b\rangle$$
$$G_{39} = \langle a, b, c \mid a^4 = b^4 = c^2 = [a, b] = 1,\ cac = a^3,\ cbc = a^2 b^3\rangle$$
$$G_{40} = \langle a, b, c \mid a^4 = b^4 = 1,\ c^2 = b^2,\ [a, b] = 1,\ c^{-1}ac = a^3,\ c^{-1}bc = a^2 b^3\rangle$$
$$G_{41} = \langle a, b, c \mid a^4 = b^4 = c^2 = [a, b] = 1,\ cac = a^3 b^2,\ cbc = a^2 b\rangle$$

Each of these groups has an index 2 abelian subgroup A: for G_{38}, this is the subgroup $\langle a, b, c^2\rangle \cong C_4 \times C_2 \times C_2$, whereas all the others contain a copy of $C_4 \times C_4$ generated by a and b. Note that $G_{39} \cong A \rtimes C_2$ is a semidirect product, and G_{40} is a non-split version of G_{39}. Thus if we can establish G_{39} to be good, the same holds for G_{40}: the action of the quotient C_2 on the (integral) Morava K-theory is the same for both groups.

Euler characteristics. All these groups have a unique index 2 abelian subgroup, centre of order 4, and 14 conjugacy classes of elements. This suffices to conclude that they all have the same Euler characteristic

$$\chi_{n,2} = \frac{1}{2}16^n + 8^n - \frac{1}{2}4^n\,.$$

The subgroup H. One can always take the index 2 abelian subgroup A. In the case of G_{38}, one might alternatively use the subgroup $H := \langle ac, b, c^2\rangle \cong D_8 \times C_2$; given the smaller size of its Morava K-theory, the MAPLE programme is faster by a factor of about 10. To make presenting the calculation easier, we stick with A, though.

Irreducible representations. For all of these groups, there are 8 irreducible representations of dimension 1 and 6 of dimension 2. In detail:

- G_{38}: the commutator subgroup equals the centre $Z = \langle a^c, c^2\rangle \cong C_2 \times C_2$ with quotient $\langle \bar{a}, \bar{b}, \bar{c}\rangle$. Thus the 1-dimensional representations are $\alpha^i\beta^j\gamma^k$, $i, j, k \in \{0, 1\}$, with $\alpha(a) = -1$, $\alpha(b) = \alpha(c) = 1$; $\beta(b) = -1$, $\beta(a) = \beta(c) = 1$; $\gamma(a) = \gamma(b) = 1$, $\gamma(c) = -1$.
  The 2-dimensional irreducibles are induced representations from certain subgroups of G. Let $A = \langle a, b, c^2\rangle$ as above with RA generated by ζ, η, ξ corresponding to the generators (in this order). Secondly, let $K_1 = \langle a, c\rangle = \langle c\rangle \rtimes \langle a\rangle$ (with $a^{-1}ca = c^{-1}$), and define $\lambda \in RK_1$ by $\lambda(c) = i$, $\lambda(a) = 1$. Thirdly, let $K_2 = \langle ab, abc, c^2\rangle$ (this subgroup is isomorphic to group (d) from the previous section), and set $\mu(ab) = i$, $\mu(abc) = 1$, $\mu(c^2) = -1$. Then
  $$\begin{aligned} \sigma_1 &= \mathrm{Ind}_A^G(\xi)\,, & \sigma_2 &= \mathrm{Ind}_{K_1}^G(\lambda)\,, & \sigma_3 &= \mathrm{Ind}_{K_2}^G(\mu)\,,\\ \sigma_1' &= \beta\sigma_1\,, & \sigma_2' &= \alpha\sigma_2\,, & \sigma_3' &= \alpha\sigma_3 \end{aligned}$$
  are distinct irreducible representations of dimension 2.

- G_{39} and G_{41} can be treated simultaneously. The 1-dimensional irreducibles are given by $\alpha^i\beta^j\gamma^k$ defined as above. The 2-dimensional representations can be obtained by induction from the subgroup $A = \langle a, b\rangle \cong C_4 \times C_4$: let $\lambda_j \in RA$ be defined by $\lambda_1(a) = i$, $\lambda_1(b) = 1$, and $\lambda_2(a) = 1$, $\lambda_2(b) = -1$. Set
  $$\sigma_1 = \mathrm{Ind}_A^G(\lambda_1)\,, \quad \sigma_2 = \mathrm{Ind}_A^G(\lambda_1)\,, \quad \sigma_3 = \mathrm{Ind}_A^G(\lambda_1\lambda_1)\,.$$
  For G_{39}, the 2-dimensional irreducibles are $\sigma_1, \alpha\sigma_1, \sigma_2, \alpha\sigma_2, \sigma_3, \beta\sigma_3$, for G_{41} they are $\sigma_1, \alpha\sigma_1, \sigma_2, \beta\sigma_2, \sigma_3, \alpha\sigma_3$.

The action of G/A on RA and restriction to A. In all cases, $G/A = \langle\bar{c}\rangle$. For G_{38}, the action on ζ, η, ξ is given by
$$\zeta^c = \zeta\eta\,, \quad \eta^c = \eta\,, \quad \xi^c = \zeta^2\xi\,.$$
Define $z, y, x \in K(n)^*(BA)$ by $z = c_1(\zeta)$, $y = c_1(\eta)$, and $x = c_1(\xi)$. Then c acts on the Morava K-theory of A by
$$c^*(z) = y +_F z\,, \quad c^*(y) = y\,, \quad c^*(x) = x +_F [2](z)\,;$$
this was fed into a MAPLE routine to caclulate invariants. The generators of the representation ring of G_{38} restrict to A as follows:
$$\begin{aligned} &\mathrm{Res}(\alpha) = \zeta^2, & &\mathrm{Res}(\beta) = \eta, & &\mathrm{Res}(\gamma) = 1,\\ &\mathrm{Res}(\sigma_1) = (1+\zeta^2)\xi, & &\mathrm{Res}(\sigma_2) = (1+\eta)\zeta, & &\mathrm{Res}(\sigma_3) = (1+\zeta^2\eta)\zeta\xi, \end{aligned}$$

from which one readily reads off the restrictions of the Euler classes of these representations. Note that the simplest form of the formal group law suffices, since only $[2](z)$ appears in all expressions.

For G_{39} and G_{41}, let $x, y \in K(n)^*(BA)$ be defined as the Euler classes of λ_1 and λ_2, respectively. The following table gives the action of c on RA and $K(n)^*(BA)$:

	λ_1^c	λ_2^c	$c^*(x)$	$c^*(y)$
G_{39}	$\lambda_1^3\lambda_2^2$	λ_2^3	$[3](x) +_F [2](y)$	$[3](y)$
G_{41}	$\lambda_1^3\lambda_2^2$	$\lambda_1^2\lambda_2$	$[3](x) +_F [2](y)$	$[2](x) +_F (y)$

Finally, the restrictions of the generators of RG are as follows:

	α	β	γ	σ_1	σ_2	σ_3
G_{39}	λ_1^2	λ_2^2	1	$\lambda_1(1+\lambda_1^2\lambda_2^2)$	$\lambda_2+\lambda_2^3$	$\lambda_1\lambda_2(1+\lambda_1^2)$
G_{41}	λ_1^2	λ_2^2	1	$\lambda_1(1+\lambda_1^2\lambda_2^2)$	$\lambda_2+\lambda_2^3$	$\lambda_1\lambda_2(1+\lambda_2^2)$

MAPLE results. A short description of the MAPLE programmes might be in order. We first created a basis of monomials for $K(2)^*(BH)$, and calculated the matrix of $(c-1)$ with respect to this basis. The dimension of the nullspace of the matrix gives the rank of the invariants

$$r_I := \operatorname{rank}_{K(2)^*} K(2)^*(BH)^{C_2}\,.$$

We also produced an explicit basis for the image of the matrix, which represents the image of $\operatorname{Res}_H^G \operatorname{Tr}_H^G$ (i.e., the free summands); this meant no extra cost and turned out to be useful.

Next, we calculated the restriction of the Chern approximation of G to H. To keep the size manageable, we did this one representation at a time, checking for each representation individually how many powers of its Chern classes would be needed. This resulted in a second submodule of rank

$$r_C := \operatorname{rank}_{K(2)^*}\bigl(\operatorname{Im}(\operatorname{Res}_H^G C(G;K(2))\bigr)\,.$$

Finally, we added the image of $\operatorname{Res}_H^G \operatorname{Tr}_H^G$, to obtain a third submodule of rank

$$r_{HC} = \operatorname{rank}_{K(2)^*}\bigl(\operatorname{Im}(\operatorname{Res}_H^G C(G;K(2)) + \operatorname{Im}(\operatorname{Res}_H^G \operatorname{Tr}_H^G K(2)^*(BH))\bigr)\,.$$

For the groups in this subsection, this turned out to be sufficient, but in the next one we shall see that in some cases, more effort is needed to see that G is good. The results of these steps are given in the next table.

	r_I	r_C	r_{HC}
G_{38}	136	136	136
G_{39}	136	132	136
G_{41}	136	132	136

This effectively computes the Serre spectral sequence for the extension $1 \to A \to G \to G/A \to 1$: there is only one differential in

$$E_2 = H^*(G/A; K(2)^*(BA) \cong F^{C_2} \oplus T \otimes H^*(C_2; \mathbb{F}_2)\,,$$

where $K(2)^*(BA) = F \oplus T$ is the decomposition into free and trivial modules, namely $d_{2^{n+1}-1}(t) = v_n t^{2^{n+1}}$ for t the generator of $H^*(C_2;\mathbb{F}_2)$. Since t^2 is represented by $e(\gamma)$, we arrive at

Theorem 3.7. *The groups G_{38}-G_{41} are $K(2)$-good in the sense of Hopkins-Kuhn-Ravenel. More precisely,*

(a) $K(2)^*(BG_{39})$ is multiplicatively generated by Euler classes of irreducible representations;

(b) if G is one of $G_{39} - G_{41}$, then $K(2)^*(BG)$ is generated by transfers of Euler classes.

The statement (a) holds since we only used Euler classes in the calculation.

3.8 Groups 44–48

These groups are the most complicated of all groups of order 32, insofar as they have a single central involution, and 'small' Euler characteristic, indicating more differential action in the Serre spectral sequence for the central extension. On the other hand, they all have a subgroup H isomorphic to either $D_8 \times C_2$ or $Q_8 \times C_2$. Armed with explicit bases and multiplicative relations for the Morava K-theories of those groups, we can let MAPLE calculate the invariants under the quotient C_2 and the image of $C(G; K(2))$ in $K(n)^*(BH)$.

Presentations.

$$G_{44} = \langle a,b,c \mid a^8 = b^2 = c^2 = [b,c] = 1,\ bab = a^{-1},\ cac = a^5 \rangle$$
$$G_{45} = \langle a,b,c \mid a^8 = c^2 = 1,\ b^2 = a^4, [b,c] = 1,\ b^{-1}ab = a^{-1},\ cac = a^5 \rangle$$
$$G_{46} = \langle a,b,c \mid a^4 = b^2 = c^2 = [a,c]^2 = 1,\ [a,[a,c]] = [b,c] = 1,\ bab = ac \rangle$$
$$G_{47} = \langle a,b,c \mid a^8 = b^2 = c^2 = [b,c] = 1,\ bab = ac,\ cac = a^5 \rangle$$
$$G_{48} = \langle a,b,c \mid a^8 = c^2 = 1,\ b^2 = a^4,\ [b,c] = 1,\ b^{-1}ab = ac,\ cac = a^5 \rangle$$

Euler characteristics. All of the above groups have Euler characteristic

$$\chi_{n,2} = \frac{7}{4}8^n - \frac{3}{4}4^n \,.$$

The subgroup H. We take $H = \langle a^2, b\rangle \times \langle c\rangle$; for G_{44}, G_{46}, G_{47} this is isomorphic to $D_8 \times C_2$, and in the remaining cases to $Q_8 \times C_2$.

Irreducible Representations. In all cases, there are eight irreducible representations of dimension 1, two of dimension two, and one of dimension four. The groups fall into two classes:

- G_{44} and G_{45}: The commutator subgroup is $\langle a^2\rangle \cong C_4$ with quotient C_2^3; thus the 1-dimensional representations are $\alpha^i\beta^j\gamma^k$, $i, j, k \in \{0, 1\}$, with $\alpha(a) = -1$, $\alpha(b) = \alpha(c) = 1$, etc. (the notation is meant to be suggestive). For the 2- and 4-dimensional representations, recall the representations γ_i and Δ of D_8 and Q_8 from previous sections (or see below). Let

  $$\sigma = \mathrm{Ind}_H^G(\gamma_1) \quad \text{and} \quad \tau = \mathrm{Ind}_H^G(\Delta)\,.$$

  Then $\sigma_1 = \sigma$ and $\sigma_2 = \gamma\sigma$ are two distinct two-dimensional irreducibles, and τ is an irredicible of dimension 4.

- G_{46} - G_{48}: Here $[G, G]$ is isomorphic to $C_2 \times C_2$ with quotient $C_4 \times C_2$, generated by $\bar{a}$ and $\bar{b}$; let α and β be the obvious characters corresponding to $\bar{a}, \bar{b}$. Then $\alpha^i\beta^j$, $0 \le i << 3$, $0 \le i \le 1$ are the one-dimensional representations of G. Furthermore, let ε be the representation of H given by $\varepsilon(a^2) = \varepsilon(b) = 1$, $\varepsilon(c) = -1$. This time set

  $$\sigma = \mathrm{Ind}_H^G(\varepsilon) \quad \text{and (as before)} \quad \tau = \mathrm{Ind}_H^G(\Delta)\,.$$

  Then $\sigma_1 = \sigma$ and $\sigma_2 = \alpha\sigma$ are the irreducibles of dimension two, and τ is the four-dimensional irreducible.

The abusive notation, giving different representations the same name, is of course deliberate; as before, this allows us to treat several groups at the same time.

The action of G/H on RH and restriction to H. First recall the representation theory of D_8 and Q_8: there are two one-dimensional generators γ_1, γ_2, and one two-dimensional, Δ. We chose γ_i so that they both have the value -1 on the elements of order four of D_8. Furthermore, let ε be the representation that is trivial on $\langle a^2, b\rangle$ and -1 on c, as before. We list the actions and restrictions in three tables, starting with the action.

	γ_1^a	γ_2^a	Δ^a	ε^a
G_{44}	γ_2	γ_1	$\varepsilon\Delta$	ε
G_{45}	$\gamma_1\gamma_2$	γ_2	$\varepsilon\Delta$	ε
G_{46}	γ_1	γ_2	$\varepsilon\Delta$	$\varepsilon\gamma_2$
G_{47}	γ_1	γ_2	$\varepsilon\Delta$	$\varepsilon\gamma_1\gamma_2$
G_{48}	γ_1	γ_2	$\varepsilon\Delta$	$\varepsilon\gamma_2$

Restrictions for G_{44} and G_{45}:

	α	β	γ	σ_1	σ_2	τ
G_{44}	1	$\gamma_1\gamma_2$	ε	$\gamma_1+\gamma_2$	$\varepsilon(\gamma_1+\gamma_2)$	$\Delta+\varepsilon\Delta$
G_{45}	1	γ_2	ε	$\gamma_1(1+\gamma_2)$	$\varepsilon\gamma_1(1+\gamma_2)$	$\Delta+\varepsilon\Delta$

Restrictions for G_{46} - G_{48}:

	α	β	σ_1	σ_2	τ
G_{46}	γ_1	γ_2	$\varepsilon(1+\gamma_2)$	$\varepsilon\gamma_1(1+\gamma_2)$	$\Delta+\varepsilon\Delta$
G_{47}	γ_1	$\gamma_1\gamma_2$	$\varepsilon(1+\gamma_1\gamma_2)$	$\varepsilon(\gamma_1+\gamma_2)$	$\Delta+\varepsilon\Delta$
G_{48}	γ_1	γ_2	$\varepsilon(1+\gamma_2)$	$\varepsilon\gamma_1(1+\gamma_2)$	$\Delta+\varepsilon\Delta$

From the action table one immediately reads off the action on Morava K-theory: recall $y_i = c_1(\gamma_i)$, $c_i = c_i(\Delta)$; set $z := c_1(\varepsilon)$. Then we have

$$K(2)^*(BH) \cong K(2)^*[y_1, y_2, c_2, z]/R$$

with

$$R = y_i^4,\ z^4,\ \text{and} \begin{cases} y_i c_2^2 + y_i^3 c_2 + y_1 y_2,\ c_2^4 + y_1 y_2 & \text{for } D_8 \times C_2; \\ y_i c_2^2 + y_i^3 c_2 + y_i^2,\ c_2^4 + y_1^2 + y_1 y_2 + y_2^2 & \text{for } Q_8 \times C_2. \end{cases}$$

and, e.g. for G_{48}:

$$a^*(z) = y_2 +_F z = y_2 + z + y_2^2 z^2$$

(we again suppress all mention of v_2).

MAPLE results. Again we use a table to record our results, which are for $n = 2$ only.

	r_I	r_C	r_{HC}
G_{44}	52	44	50
G_{45}	52	44	50
G_{46-48}	52	46	52

For G_{44} and G_{45}, one can obtain the invariants not yet seen to be in the image of restriction from G as restrictions of transfers from other subgroups. We exemplify this for G_{44}, where we are missing the span of the invariants $c_2^6 + c_2^9$ and $(c_2^6 + c_2^9)z$. $G = G_{44}$ has a maximal subgroup $K = \langle a^2, ab, c\rangle \cong D_8 \circ C_4$, with a^2c central of order four. K has 10 irreducible complex representations, 8 of dimension 1 and 2 of dimension 2. The one-dimensional representations are $\tilde{\alpha}^k \otimes \tilde{\beta}^l \otimes \tilde{\gamma}^m$ with $0 \le k, l, m \le 1$, where $\tilde{\alpha}(a^2) = -1$, $\tilde{\beta}(ab) = -1$, $\tilde{\gamma}(a^2c) = -1$, and trivial on the remaining generators. (They are the restrictions of the one-dimensional representations of G.) The two-dimensionals can be described as induced representations from the subgroup $\langle ab, a^2c\rangle \cong C_2 \times C_4$: let

$$\delta = \mathrm{Ind}^K_{\langle ab, a^2c\rangle}(\rho) \quad \text{with } \rho(ab) = 1,\ \rho(a^2c) = i.$$

Then δ and $\gamma\delta$ are two distinct irreducibles. Now set

$$\zeta := e(\tau)^3 + e(\tau)^2 e(\gamma) + \mathrm{Tr}^G_K\big(e(\delta)e(\tilde{\alpha})\big) + \mathrm{Tr}^G_K\big(e(\delta)e(\tilde{\gamma})\big)$$

where $e(\ )$ stands for Euler class.

Lemma 3.8. $\mathrm{Res}^G_H(\zeta) = c_2^6 + c_2^9$.

Proof. Classes in $K(2)^0(BH)$ are almost detected on maximal abelian subgroups; the kernel of the detection map is spanned by $c_2^9, c_2^9 z^3$, and $c_2^8 z^2$; this was proved in III.1. Thus we may verify the claim via the double coset formula, while keeping in mind that all restrictions to H have to be invariants — this is how we may conclude that $\mathrm{Tr}^G_K(e(\delta)e(\tilde{\alpha}))$ must contain a summand c_2^9. We leave the details to the reader. □

It follows that $(c_2^6 + c_2^9)z$ is also in the image of restriction from G, and we may conclude that G_{44} is good. A similar analysis can be performed for G_{45}.

Theorem 3.9. *The groups G_{44} - G_{48} are $K(2)$-good, i.e., $K(2)^*(BG)$ is (additively) generated by transfers of Euler classes.*

Chapter VI

Permutation modules

Kriz's theorem II.4.6 leads one to consider the structure of the Morava K-theory of a group G as a module over a subgroup of $\mathrm{Aut}(G)$ of order p. It appears natural to ask the question in more generality: given a group extension of Q by H, what can be said about the structure of $K(n)^*(BH)$ as a $K(n)^*[Q]$-module? In particular, when is it a permutation module? A positive answer has certain computational implications as it makes the E_2-term of the Serre spectral sequence associated to this extension easily computable.
In this chapter we study the problem for linear actions on $\mathbb{F}_p$-vector spaces. The results were originally obtained by I. Leary and the author in [LS]; some preliminary observations may be found in [B1].

1 Preliminaries

For a ring R and a finite group G we call an R-free $R[G]$-module M a permutation module if there is an R-basis for M which is permuted by the action of G; such a basis is called a permutation basis for M. For a G-set S, we write $R[S]$ for the permutation module with permutation basis S. If M is a graded module over the graded ring $R[G]$, call M a graded permutation module if it has a permutation basis consisting of homogeneous elements.

Lemma 1.1. *Let M be a graded $K(n)^*[G]$-module. Then each of the following conditions implies the next:*

(i) M is a graded permutation module;

(ii) M is a permutation module;

(iii) M is a direct summand of a permutation module.

Furthermore, if G is a p-group then (iii) implies (i).

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are clear, and hold for any graded $R[G]$-module. The proof of the last sentence is deferred until Section 4. □

Our question is really a problem in modular representation theory. As we shall need them later, we very briefly recall modular (or Brauer) characters. As a general reference on modular characters, see [CR], in particular § 17.

Fix a prime p. Let G be a finite group and W an $\mathbb{F}_p[G]$-module. Choose an embedding of the multiplicative group of the algebraic closure of $\mathbb{F}_p$ in the group of roots of 1 in $\mathbb{C}$. Let g be a p-regular element of G, i.e., an element whose order is coprime to p, and let $\lambda_1, \lambda_2, \ldots, \lambda_m$ denote the images in $\mathbb{C}$ of the eigenvalues of its action on W. Then the Brauer character of g is

$$\chi_W(g) = \lambda_1 + \lambda_2 + \cdots + \lambda_m \, .$$

Two $\mathbb{F}_p[G]$-modules have the same Brauer character if and only if they have the same composition factors ([CR, Corollary 17.10]). In our context, Brauer characters are useful in proving that a given module is *not* a permutation module, see Section 3. They do not help in establishing positive answers, and say nothing about p-groups.

Now let V be an elementary abelian p-group, or equivalently an $\mathbb{F}_p$-vector space, and $GL(V)$ the group of automorphisms of V.
It would be too much to ask that the $K(n)^*[GL(V]$-module $K(n)^*(BV)$ satisfied condition (i) of the lemma above: if V has dimension at least 3, there are infinitely many indecomposable graded $K(n)^*[GL(V)]$-modules, only finitely many of which occur as summands of modules satisfying condition (iv). Thus it is unlikely that a 'random module' will satisfy any of the conditions. On the other hand, from [HKR] we know that for complex oriented theories E with torsion free coefficients containing an inverse for p, $E^*(BV)$ *is* a permutation module for $E^*[GL(V)]$. N. Kuhn [Ku1] has shown that $K(n)^*(BV)$ has the same Brauer character as the permutation module $K(n)^*[\mathrm{Hom}(V, \mathbb{F}_p^n)]$ for $GL(V)$, but as we shall see below, as *graded* permutation modules the Brauer characters differ. However, Brauer characters give no information for the Sylow p-subgroup of $GL(V)$. In Section 4.1 we present an algorithm to determine whether an $\mathbb{F}_p[G]$-module is a permutation module; this algorithm was put on the computer and we record some of the results.

For a graded $K(n)^*$-module M, let $\overline{M}$ be the $\mathbb{F}_p$-vector space $M/(1-v_n)M$. Then $\overline{M}$ is an $\mathbb{F}_p[G]$-module, naturally graded by the cyclic group $\mathbb{Z}/2(p^n-1)$, and determines M up to isomorphism.
When only interested in composition factors, we may further simplify the problem by neglecting all higher order terms in the formal group law.
Let $d = \dim V$ and write K_V or $K_{n,d}$ for $K(n)^*(BV) \otimes_{K(n)^*} \mathbb{F}_p$, i.e. the $\mathbb{Z}/2(p^n-1)$-graded $\mathbb{F}_p[GL_d(\mathbb{F}_p)]$-module given as an $\mathbb{F}_p$-algebra as

$$K_{n,d} = K_V = \mathbb{F}_p[x_1, \ldots, x_d]/(x_1^{p^n}, \ldots, x_d^{p^n})$$

with each x_i of degree 1. A matrix $(a_{ij}) \in GL_d(\mathbb{F}_p)$ acts via

$$x_j \mapsto \sum_{i=1}^{d}{}^F [a_{ij}]_{K(n)}(x_i) = e'\big(\sum_i l'(x_i)a_{ij}\big)\,,$$

where e' and l' are as in the proof of Proposition I.3.1. (Recall that for any V we take $GL(V)$ to act on the right of V, and hence obtain a left $K(n)^*[GL(V)]$-module structure on $K(n)^*(BV)$.)

Secondly, let L_V denote the algebra of polynomial functions on V (with V^* is degree 2) modulo the ideal of p^nth powers of elements of positive degree. Grade L_V by $\mathbb{Z}/2(p^n-1)$, and let $GL_d(\mathbb{F}_p)$ act on L_V by its natural action on polynomial functions. Thus as a graded algebra, L_V is isomorphic to K_V, but the action is the standard action.

Lemma 1.2. *K_V has a filtration by graded submodules such that the associated graded module is isomorphic to L_V. In particular, K_V and L_V have the same composition factors.*

Proof. For each degree $2k$, take the basis consisting of monomials of degree congruent to $2k$ modulo $2(p^n-1)$, and arrange them in blocks with respect to length. For any g, the matrix of its action on L_V^{2k} with respect to this basis consists of square blocks along the diagonal, whereas the corresponding matrix for the action on K_V^{2k} has some extra entries below the blocks. Thus both modules have the same Brauer character. □

We conclude this section with a few introductory remarks concerning the permutation module $K(n)^*[\mathrm{Hom}(V,\mathbb{F}_p^n)]$. If ϕ is a homomorphism from V to $\mathbb{F}_p^n$, then $g \in GL(V)$ acts by composition, i.e.,

$$g\phi(v) = \phi(vg)\,.$$

Since we view $GL(V)$ as acting on the right of V, this makes $\mathrm{Hom}(V,\mathbb{F}_p^n)$ into a left $GL(V)$-set. The $GL(V)$-orbits in $\mathrm{Hom}(V,\mathbb{F}_p^n)$ may be described as follows. For W a subspace of V, let $H(W) \le GL(V)$ be

$$H(W) = \{g \in GL(V) \mid vg - v \in W \text{ for all } v \in V\}.$$

For example, $H(\{0\}) = \{1\}$, and $H(V) = GL(V)$. For $0 \le i \le \dim(V)$, let H_i be $H(W_i)$ for some W_i of dimension i. Thus H_i is defined only up to conjugacy, but this suffices to determine the isomorphism type of the $GL(V)$-set $GL(V)/H_i$. Now let ϕ be an element of $\mathrm{Hom}(V,\mathbb{F}_p^n)$. The stabilizer of ϕ in $GL(V)$ is the subgroup $H(\ker(\phi))$, and the orbit of ϕ consists of all ϕ' such that $\mathrm{Im}(\phi') = \mathrm{Im}(\phi)$. It follows that as $GL(V)$-sets,

$$\mathrm{Hom}(V,\mathbb{F}_p^n) \cong \coprod_{0\le i\le \dim(V)} m(n,i)\cdot G/H_i,$$

where $m(n,i)$ is the number of subspaces of $\mathbb{F}_p^n$ of dimension i. Thus to decompose the module $\mathbb{F}_p[\mathrm{Hom}(V,\mathbb{F}_p^n)]$, it suffices to decompose each $\mathbb{F}_p[GL(V)/H_i]$.

2 On $K(1)$

The case $n = 1$ is a simple application of Kuhn's description of the mod-p K-theory of finite groups [Ku2].

Theorem 2.1. *The* $K(1)^*[GL(V)]$*-modules* $K(1)^*(BV)$ *and* $K(1)^*[\mathrm{Hom}(V, \mathbb{F}_p)]$ *are (ungraded) isomorphic.*

Remark. For $p = 2$, v_1 has degree -2, so that the 'cyclically graded' modules $K(1)^*(BV)$ are in fact concentrated in a single degree.

Proof. Recall [Wi] that the spectrum representing mod p K-theory splits as a wedge of one copy of each of the 0th, 2nd, $\ldots$, $(2p-4)$th suspensions of the spectrum representing $K(1)^*$. Since $K(1)^*(BV)$ is concentrated in even degrees it follows that K_V is naturally isomorphic to $K^0(BV; \mathbb{F}_p)$. In [Ku2] it is shown that for any p-group G, $K^0(BG; \mathbb{F}_p)$ is naturally isomorphic to $\mathbb{F}_p \otimes R(G)$, where $R(G)$ is the (complex) representation ring of G. The case $G = V$ gives the theorem, because as a $GL(V)$-module, $\mathbb{F}_p \otimes R(V)$ is isomorphic to $\mathbb{F}_p[\mathrm{Hom}(V, \mathbb{F}_p)]$. □

For $p = 2$, there is an 'elementary' proof working directly with the description of K_V. In this case, K_V is isomorphic to an exterior algebra $\Lambda(x_1, \ldots, x_d)$. The monomial 1 generates a trivial $GL(V)$-summand. Let H be the subgroup of $GL(V)$ fixing x_1. Then H is the subgroup of $GL(V)$ stabilising some hyperplane W and inducing the identity map on the quotient V/W. There is a $GL(V)$-set isomorphism

$$\mathrm{Hom}(V, \mathbb{F}_2) \cong GL(V)/GL(V) \amalg GL(V)/H,$$

so it suffices to show that the submodule M generated by x_1 contains each monomial in $\Lambda(x_1, \ldots, x_d)$ of strictly positive length. The permutation matrices permute the monomials of any given length transitively. Assume that M contains all monomials of length i (this holds for $i = 1$), and let $g \in GL(V)$ be such that

$$gx_1 = x_1, \ldots, gx_{i-1} = x_{i-1}, \quad gx_i = x_i +_F x_{i+1} = x_i + x_{i+1} + x_i x_{i+1}\,.$$

Then

$$g(x_1 \ldots x_i) + x_1 \ldots x_i + x_1 \ldots x_{i-1} x_{i+1} = x_1 \ldots x_i x_{i+1} \in M,$$

so M contains all monomials of length $i + 1$.

3 Negative results

In this section we shall prove those of our negative results that do not rely on computer calculations, and analyse the cases $\dim V = 2$, $p = 2, 3$.
We start with an easy observation.

Theorem 3.1. *If p is odd then $K(n)^*(BV)$ is not a graded permutation module for $GL(V)$.*

Proof. Let D be the subgroup of diagonal matrices in $GL_d(\mathbb{F}_p)$, so that D is isomorphic to a direct product of d cyclic groups of order $p-1$. In K_V, each monomial in $x_1, \ldots, x_d$ is an eigenvector for D, and the monomials fixed by D are those in which the exponent of each x_i is divisible by $p-1$. Hence if $p-1$ does not divide k, then K_V^k cannot be a permutation module for D because it contains no D-fixed point. □

For $p = 2$ this argument clearly does not work, we shall use Brauer characters instead. Recall from Lemma 1.2 that the characters of K_V and L_V coincide, so we begin by describing how to compute the latter.
Fix an embedding of the multiplicative group of the algebraic closure of $\mathbb{F}_p$ in the group of roots of unity in $\mathbb{C}$. Let g be a p-regular element of $GL(V)$, and let $\lambda_1, \lambda_2, \ldots, \lambda_d$ denote the images in $\mathbb{C}$ of the eigenvalues of its action on V^*, where $d = \dim V$. Then the Brauer character of g afforded by V^* is

$$\chi_{V^*}(g) = \lambda_1 + \lambda_2 + \cdots + \lambda_d \,.$$

To compute the character of L_V^k we proceed as follows. The proof of Molien's theorem (see e.g. [CR], p. 329) can be adapted to show that the character of a p^n-truncated polynomial algebra has generating function

$$f_g(t) = \prod_{i=1}^{d} \Big(\frac{1 - (\lambda_i t)^{p^n}}{1 - \lambda_i t} \Big) \,.$$

Then the character of L_V evaluated at g is simply $f_g(1)$, whereas for each degree k (recall that we are grading cyclically) one has

$$\chi_{L_V^k}(g) = \frac{1}{p^n - 1} \sum_{\tau} \tau^{-k} f_g(\tau) \,, \tag{3.1}$$

where the sum ranges over all $(p^n - 1)$-st roots of unity—to see this, recall that the sum of λ^k over all mth roots of unity λ is equal to zero if m does not divide k, and equal to m if m does divide k.

Theorem 3.2. *Let $p = 2$ and $d = \dim V$.*

(a) Let $n > 1$. Assume $d \geq 4$ and d is greater or equal to the smallest prime divisor of n. Then $K(n)^*(BV)$ is not a graded permutation module for $GL(V)$.

(b) Assume $d = 3$ and 3 divides n. Then $K(n)^*(BV)$ is not a graded permutation module for $GL(V)$.

(c) *For $d = 2$, $K(n)^*(BV)$ is a graded permutation module for $GL(V)$ if and only if n is odd.*

Proof. Suppose first that d equals a prime divisor q of the fixed number n. By considering the action of the multiplicative group of $\mathbb{F}_{2^q}$ on the additive group of $\mathbb{F}_{2^q}$, we can always construct an element $g_q \in GL_q(\mathbb{F}_2)$ that permutes the $2^q - 1$ nontrivial elements of $\mathbb{F}_2^q$ cyclically. The set of eigenvalues of g_q contains a primitive (2^q-1)st root of unity, and is closed under the action of the Galois group $\mathrm{Gal}(\mathbf{F}_{2^q}/\mathbb{F}_2)$. Hence the Brauer lifts of the eigenvalues of g_q are λ, $\lambda^2, \dots, \lambda^{2^{q-1}}$ for some primitive $(2^q - 1)$-st root of unity $\lambda \in \mathbb{C}$. Consequently, the generating function for the character afforded by L_V is given by

$$f_{g_q}(t) = \prod_{i=0}^{q-1}\Big(\frac{1-(\lambda^{2^i}t)^{2^n}}{1-\lambda^{2^i}t}\Big).$$

If τ is a $(2^n - 1)$-st root of unity, one obtains

$$f_{g_q}(\tau) = \begin{cases} 2^n & \text{if } \tau \in \{\lambda^{-2^i}, \quad i = 0, 1, \dots q-1\} \\ 1 & \text{otherwise.} \end{cases}$$

Thus evaluating the formula (3.1) for the character afforded by L_V^k yields

$$\chi_{L_V^k}(g_q) = \frac{1}{2^n-1}\sum_{\tau \neq \lambda^{-2^i}} \tau^k + \frac{2^n}{2^n-1}\sum_{i=0}^{q-1}\lambda^{2^i k} = \begin{cases} q+1 & \text{for } k = 0 \\ \sum_{i=0}^{q-1} \lambda^{2^i k} & \text{for } k \neq 0. \end{cases}$$

Specializing to the case $k = 1$, this sum is never zero, since the powers λ^i for i coprime to $2^q - 1$ form a $\mathbb{Q}$-basis for $\mathbb{Q}[\lambda]$. For $q > 2$ the sum is not a rational, because it is not fixed by the whole Galois group $\mathrm{Gal}(\mathbb{Q}[\lambda]/\mathbb{Q})$. In the case $q = 2$, one obtains -1 (the sum of the two primitive third roots of unity). Since permutation modules have non-negative integer character values, this shows that $K(n)^*(BV)$ is not a graded $GL_q(\mathbb{F}_2)$-permutation module. To proceed with vector spaces of dimension bigger than q we consider the cases $q > 2$ and $q = 2$ separately. In the first case we use the following lemma to conclude that the character still takes non-integer values on certain elements of $GL(V)$. Let $V = U \oplus W$ with U and W of dimensions d and r, respectively; then clearly $L_V \cong L_U \otimes L_W$. Let g be an arbitrary 2-regular element of $GL(U)$ and denote by $g \times 1$ the element of $GL(V)$ which acts like g on L_U and trivially on L_W.

Lemma 3.3. $\chi_{L_V^k}(g \times 1) = \chi_{L_U^k}(g) + \frac{2^{nr}-1}{2^n-1} \cdot \chi_{L_U}(g)$.

Proof. The generating function for $g \times 1$ is obtained from the one for g as the product with r factors $(1 + t + t^2 + \dots + t^{2^n-1})$, thus

$$\begin{aligned}\chi_{L_V^k}(g\times 1) &= \frac{1}{2^n-1}\sum_{\tau}\tau^{-k}f_{g\times 1}(\tau)\\ &= \frac{1}{2^n-1}\sum_{\tau\neq 1}\tau^{-k}f_g(\tau)\Big(\frac{1-\tau^{2^n}}{1-\tau}\Big)^r + \frac{2^{nr}}{2^n-1}f_g(1)\\ &= \chi_{L_U^k}(g) + \big(\frac{2^{nr}-1}{2^n-1}\big)f_g(1)\,.\end{aligned}$$

□

Thus $g_q \times 1$ will provide the contradiction when V has rank $q+r$. This fails for $q=2$, whence we choose the element g' which consists of $d/2$ copies of $g_2 = \left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$ arranged along the diagonal if d is even, and add an extra diagonal entry 1 if d is odd. Then a similar computation shows that for $k \not\equiv 0 \bmod 3$,

$$\chi_{L_V^k}(g') = \begin{cases} -\dfrac{2^{nd/2}-1}{2^n-1} & \text{if } d \text{ is even,}\\[2ex] -\dfrac{2^{n(d-1)/2}-1}{2^n-1}+1 & \text{if } d \text{ is odd,}\end{cases}$$

For $d > 3$ these numbers are negative.

The other parts of the theorem are proved in a similar fashion. In the case when V has rank 3, evaluation of the Brauer character of an element of $GL(V)$ of order 7 shows that L_V^1 is not a $GL(V)$-permutation module if 3 divides n. Similarly, when V has rank 2, the Brauer character of an element of $GL(V)$ of order 3 on L_V^1 is negative if n is even. When V has rank 2 and n is odd, it may be shown that for each k, any $GL(V)$-module having the same Brauer character as L_V^k is a permutation module. This shows that for n odd, K_V^k is a permutation module, but does not specify which one. In (3.1) we shall describe its isomorphism type, giving an alternative argument. □

3.1 $\dim V = 2$ for $p = 2$

We shall determine the structure of L_V as a $GL_2(\mathbb{F}_2)$-module and deduce that K_V and L_V are isomorphic.

There are three isomorphism types of indecomposable $\mathbb{F}_2[GL(V)]$-modules: the trivial module T, its projective cover N (which is expressible as a non-split extension of T by T), and the tautological module V, which is both simple and projective. All these modules are self-dual. The transitive permutation modules for $GL(V)$ decompose as T, N, $T \oplus V$ and $N \oplus 2V$. Let $S^*[V^*]$ denote the algebra of polynomial functions on V as a graded $GL(V)$-module.

Proposition 3.4. *The generating functions P_T, P_N and P_V for the multiplicities of each of the indecomposable $GL(V)$-summands of $S^*[V^*]$ are*

$$P_T(t) = \frac{1}{1-t^2}, \qquad P_N(t) = \frac{t^3}{(1-t^2)(1-t^3)}, \qquad P_V(t) = \frac{t}{(1-t)(1-t^3)}.$$

Proof. The ring of invariants $S^*[V^*]^{GL(V)}$ is a free polynomial ring on two generators of degrees two and three (see [Wk]). The Poincaré series for $S^*[V^*]$ and the invariants, together with the Brauer character of an element of order three give the equalites

$$\begin{aligned} P_T + 2P_N + 2P_V &= \frac{1}{(1-t)^2} \\ P_T + P_N &= \frac{1}{(1-t^2)(1-t^3)} \\ P_T + 2P_N - P_V &= \frac{1-t}{1-t^3} \end{aligned}$$

Solving for P_T etc. gives the result. □

Proposition 3.5. *Let $k \in \mathbb{Z}/(2^n-1)$. Then*

$$L_V^k \cong \begin{cases} 2T \oplus \frac{2^n-2}{6} N \oplus \frac{2^n+1}{3} V & \text{for } n \text{ odd, } k=0, \\ T \oplus \frac{2^n-2}{6} N \oplus \frac{2^n+1}{3} V & \text{for } n \text{ odd, } k \neq 0, \\ 2T \oplus \frac{2^n+2}{6} N \oplus \frac{2^n-1}{3} V & \text{for } n \text{ even, } k=0, \\ T \oplus \frac{2^n+2}{6} N \oplus \frac{2^n-1}{3} V & \text{for } n \text{ even, } k \neq 0,\ k \equiv 0 \text{ mod } 3, \\ T \oplus \frac{2^n-4}{6} N \oplus \frac{2^n+2}{3} V & \text{for } n \text{ even, } k \not\equiv 0 \text{ mod } 3. \end{cases}$$

Proof. Let $\widetilde{L}$ be the truncated symmetric algebra L_V, but graded over the integers instead of cyclically. Then $\widetilde{L}^k = 0$ for $k > 2(2^n-1)$, and $\widetilde{L}^{2(2^n-1)} \cong T$, generated by $x_1^{2^n-1}x_2^{2^n-1}$. The product structure on $\widetilde{L}$ gives a duality pairing $\widetilde{L}^k \times \widetilde{L}^{2(2^n-1)-k} \to T$, and since all modules are self-dual, it follows that $\widetilde{L}^k \cong S^{2(2^n-1)-k}[V^*]$ for $2^n \leq k \leq 2(2^n-1)$. Since $L_V^k \cong \widetilde{L}^k \oplus \widetilde{L}^{2^n-1+k}$ for $k \neq 0$ (viewing k as either an integer or an integer modulo 2^n-1 as appropriate), and $L_V^0 \cong \widetilde{L}^0 \oplus \widetilde{L}^{2^n-1} \oplus \widetilde{L}^{2(2^n-1)} \cong 2T \oplus \widetilde{L}^{2^n-1}$, the claim follows from Proposition 3.4. □

Corollary 3.6. *Let $\dim V = 2$ and $p = 2$. Then K_V and L_V are isomorphic as graded $\mathbb{F}_2[GL(V)]$-modules.*

Proof. K_V^k has odd dimension for $k \neq 0$ and thus contains at least one trivial summand. K_V^0 contains the trivial summand generated by 1 and then at least another trivial summand, by the same dimension argument. Since N and V are projective, Lemma 1.2 implies that K_V^k has at least as many summands N and V as L_V^k. This accounts for all the summands of K_V^k. □

The proof of Theorem 3.2 (c) follows easily from this description of K_V. Furthermore, the remarks at the end of Section 1 now easily imply:

Corollary 3.7. *Let $p = 2$ and $\dim V = 2$. Then as $K(n)^*[GL(V)]$-modules, $K(n)^*(BV)$ and $K(n)^*[\mathrm{Hom}(V, \mathbb{F}_2^n)]$ are isomorphic.* □

3.2 $\dim V = 2$ for $p = 3$

The argument used in the proof of Corollary 3.6 shows that for $\dim V = 2$ and any p, n, and k such that L_V^k contains at most one non-projective summand, $K_V^k \cong L_V^k$. For odd primes this does not always occur however. If $0 < k < p^n - 1$ then L_V^k splits as a direct sum of submodules of dimensions $k+1$ and $p^n - k - 2$ coming from the standard $\mathbb{Z}$-grading on the truncated polynomial algebra. If k is not congruent to either -1 or -2 modulo p, the dimensions of these summands are not divisible by p, and hence L_V^k contains at least two non-projective indecomposable summands. The calculations described next show that for $p = 3$, $n = 2, 3$ and $0 < k < 3^n - 1$, the module K_V^k has exactly one non-projective indecomposable summand. It follows that for $p = 3$, K_V^k and L_V^k are not isomorphic in general. The results in this subsection were obtained by computer.

For the remainder of this section let $\dim V = 2$ and $p = 3$. There are fourteen indecomposable $GL(V)$-modules in four blocks, two of which contain a single simple projective module. The six indecomposables in the block containig V are not so easy to distinguish, whence we consider only $SL(V)$. Standard representation theory techniques (see e.g. [Al, CR]) give the following facts. There are three simple $\mathbb{F}_3[SL(V)]$-modules: the trivial module T, the natural module V, and a simple projective module $P = S^2(V)$ of dimension three. There are three blocks. The blocks containing T and V each contain three indecomposable modules, each of which is uniserial. This data may be summarised as follows:

block of $T = I_1$:	$T \rightarrowtail I_2 \twoheadrightarrow T$,	$T \rightarrowtail I_3 \twoheadrightarrow I_2$,
block of $V = I_4$:	$V \rightarrowtail I_5 \twoheadrightarrow V$,	$V \rightarrowtail I_6 \twoheadrightarrow I_5$,
block of $P = I_7$:	contains no other indecomposables.	

Letting τ stand for the element of order two in $SL(V)$ and σ for the sum of the six elements of $SL(V)$ of order four, the block idempotents are

$$b_T = 2 + 2\tau + 2\sigma, \qquad b_V = 2 + \tau, \qquad b_P = \sigma.$$

The modules in any single block are distinguishable by their restrictions to a cyclic subgroup of $SL(V)$ of order three. Thus if α is an element of $SL(V)$ of order three, and M is an $SL(V)$-module, the direct summands of M are determined by the ranks of the elements of $\mathrm{End}(M)$ representing the actions of the following seven elements of $\mathbb{F}_3[SL(V)]$:

$$b_T, \quad (1-\alpha)b_T, \quad (1-\alpha)^2 b_T, \quad b_V, \quad (1-\alpha)b_V, \quad (1-\alpha)^2 b_V, \quad b_P.$$

More precisely, if the seven ranks are $r_1, \ldots, r_7$, and n_i stands for the number of factors of M isomorphic to I_i, then

$$n_1 = r_1 - 2r_2 + r_3, \quad n_2 = r_2 - 2r_3, \quad n_3 = r_3, \quad n_4 = r_5 - 2r_6,$$

$$n_5 = r_4 - 2r_5 + r_6, \quad n_6 = (2r_5 - r_4)/2, \quad n_7 = r_7/3.$$

The numbers r_i (and thus n_i) can be calculated on a computer. (We used MAPLE to generate matrices representing the action of a certain pair of generators and fed these into GAP.)
Recall from Section 1 that there is an isomorphism of (right) $GL(V)$-sets

$$\mathrm{Hom}(V, \mathbb{F}_3^n) = GL(V)/GL(V) \amalg \tfrac{3^n-1}{2} \cdot GL(V)/H_1 \amalg \tfrac{(3^n-1)(3^n-3)}{48} \cdot GL(V)/\{1\}$$

where H_1 is the subgroup stabilising a line L in V and acting trivially on V/L. As $SL(V)$-modules, it may be checked that

$$\begin{aligned} \mathbb{F}_3[GL(V)/GL(V)] &\cong I_1, \\ \mathbb{F}_3[GL(V)/H_1] &\cong I_1 \oplus I_5 \oplus I_7, \\ \mathbb{F}_3[GL(V)/\{1\}] &\cong 2I_3 \oplus 4I_6 \oplus 6I_7. \end{aligned}$$

This information together with the results given in Table 1 in Appendix B easily imply:

Proposition 3.8. *For* $\dim V = 2$, $p = 3$, *and* $n = 1, 2, 3$, *the* $K(n)^*[SL(V)]$-*modules* $K(n)^*(BV)$ *and* $K(n)^*[\mathrm{Hom}(V, \mathbb{F}_p^n)]$ *are isomorphic.* □

4 Permutation modules for p-groups

In the previous section our methods made use of the fact that there were only finitely many indecomposable modules. If G is a group whose Sylow p-subgroup is not cyclic, then $\mathbb{F}_p[G]$ has infinitely many indecomposable modules, so the same sort of arguments cannot work.
In this section we shall describe an algorithm for determining, for any p-group G, whether an $\mathbb{F}_p[G]$-module is a permuation module, and if so to decompose

it. The algorithm relies on the following fact [CR, 19.25]: For G a p-group, any transitive permutation module for $\mathbb{F}_p[G]$ has a unique minimal submodule, which is the trivial module generated by the sum of the elements in a permutation basis. This implies that any transitive permutation module is indecomposable. The Krull-Schmidt theorem and the indecomposabilty of transitive permutation modules together imply that if a graded $\mathbb{F}_p[G]$-module is a permutation module, then it is also a graded permutation module. This argument finishes the proof of Lemma 1.1.

Let $G_1, \ldots, G_n$ be subgroups of a p-group G, where the order of G_{i+1} is at least the order of G_i, and let M be a (finitely generated) $\mathbb{F}_p[G]$-module. Construct a sequence of submodules M_i of M as follows. Let M_0 be the zero submodule of M. If M_{i-1} has been defined, let

$$M_i = M_{i-1} + \operatorname{Im}\Big(\big(\sum_{g \in G/G_i} g \big) \colon M^{G_i} \longrightarrow M \Big),$$

where the sum ranges over a transversal to G_i in G, M^{G_i} denotes the G_i-fixed points of M, and the sum is an element of $\mathbb{F}_p[G]$ viewed as an element of $\operatorname{End}(M)$. Now define

$$m_i = \dim M_i - \dim M_{i-1}.$$

Proposition 4.1. *(a) M contains a submodule $M' \cong m_1\mathbb{F}_p[G/G_1] \oplus \cdots \oplus m_n\mathbb{F}_p[G/G_n]$, and M' has maximal dimension among all submodules of M isomorphic to a direct sum of copies of the $\mathbb{F}_p[G/G_i]$.*

(b) If $G_1, \ldots, G_n$ contains a representative of each conjugacy class of subgroups of G, then $\dim M' = \dim M$ *if and only if M is a permutation module.*

Proof. First, recall that the socle, $\operatorname{Soc}(N)$, of a module N is the smallest submodule of N containing every minimal submodule. The following statement is easy to prove, and will be useful below. If L is a submodule of M, and $f \colon N \to M$ is a module homomorphism, then f is injective if and only if its restriction to $\operatorname{Soc}(N)$ is injective. If f is injective, then $\operatorname{Soc}(f(N)) = f(\operatorname{Soc}(N))$, and the sum $L + f(N)$ in M is direct if and only if the sum $\operatorname{Soc}(L) + f(\operatorname{Soc}(N))$ is direct.

Module homomorphisms from $\mathbb{F}_p[G/G_i]$ to M are naturally bijective with elements of M^{G_i}, where the element x corresponds to the homomorphism θ_x sending $1 \cdot G_i$ to x. The socle of $\mathbb{F}_p[G/G_i]$ is a trivial submodule generated by $\sum_{g \in G/G_i} g \cdot G_i$, so its image under θ_x is generated by $\sum_{g \in G/G_i} g \cdot x$. It follows that any submodule of M isomorphic to a direct sum of copies of the modules $\mathbb{F}_p[G/G_1], \ldots, \mathbb{F}_p[G/G_i]$ has socle contained in M_i, and in particular consists of at most $\dim M_i$ summands. This shows that any submodule of M isomorphic to a direct sum of $\mathbb{F}_p[G/G_i]$'s has dimension less than or equal to $\sum_i m_i|G : G_i|$, but it remains to exhibit a submodule M' having this dimension.

Define M'_0 to be the zero submodule of M, and assume that for some j with $1 \leq j \leq n$ we have constructed a submodule M'_{j-1} of M with

$$M'_{j-1} \cong m_1\mathbb{F}_p[G/G_1] \oplus \cdots \oplus m_{j-1}\mathbb{F}_p[G/G_{j-1}] .$$

Let $x_1, \ldots, x_{m_j} \in M^{G_j}$ be such that the images $\sum_{g \in G/G_j} g \cdot x_i$ form a basis for a complement to M_{j-1} in M_j. Taking $L = M_{j-1}$, $N = m_j\mathbb{F}_p[G/G_j]$, and $f \colon N \to M$ the map sending the elements $(0, \ldots, 1 \cdot G_j, \ldots, 0)$ to the x_i's, the statements in the first paragraph of the proof show that f is injective, and that M'_j defined as the submodule of M spanned by M_{j-1} and the x_i's is isomorphic to $M'_{j-1} \oplus m_j\mathbb{F}_p[G/G_j]$. Now M' may be taken to be M'_n. □

In Appendix B we record the results of computer calculations obtained by implementing the algorithm described above. The information contained in Tables 2–6 suffices to show:

Proposition 4.2. *Let* $\dim V = 3$. *Then* $K(n)^*(BV)$ *is not a permutation module for* $K(n)^*[U(V)]$ *in the following cases:*

(a) $p = 3$, $n = 2$,

(b) $p = 5$, $n = 2$.

In the following cases, as well as those implied by Theorems 3.2, 2.1, Corollary 3.7, and Proposition 3.8, $K(n)^*(BV)$ *is a graded permutation module for* $K(n)^*[U(V)]$*:*

(c) $p = 2$, $n = 2$, 3 *or* 4. □

For each n and p considered, K^0_V is a $U(V)$-permutation module, although it is easy to show that usually K^0_V cannot be a $GL(V)$-permutation module by comparing the information in the tables with the information given by Brauer characters. This technique may also be used to prove:

Proposition 4.3. *For* $p = 2$ *and* $\dim V = 3$, $K(n)^*(BV)$ *is not a graded permutation module if* n *is a multiple of three, or if* n *is* 2, 4, *or* 5. □

We have seen in Chapter III that for any V, $K(n)^*(BV)$ is a (graded) permutation module for any subgroup of $GL(V)$ of order p. In the cases covered by Proposition 4.4, the group $U(V)$ has order p^3, and for $p > 2$ it contains no element of order p^2. The gap between Proposition 4.2 and a special case of Kriz's result is filled by:

Theorem 4.4. *Let* $d = 3$, *let* $p = 3$ *or* 5, *and let* H *be any subgroup of* $GL(V)$ *of order* p^2. *Then* $K(2)^*(BV)$ *is not a permutation module for* H.

PROOF. Tables 5 and 6 we give just enough information to prove this. For each subgroup H of $U(V)$ of order p^2, we give the dimension of a maximal H-permutation submodule M'' of K_V^1. The dimension of K_V^1 is 91 for $p = 3$ and 651 for $p = 5$. Only one of the subgroups $\langle AB, C\rangle, \ldots, \langle AB^{p-1}, C\rangle$ is listed in these tables, because these subgroups are all conjugate in $GL(V)$ and so give rise to M'''s of the same dimension. □

Chapter VII

Some remarks on Morava K-theory of discrete groups

Finiteness conditions for discrete groups of different kinds are well documented in the literature, mostly in terms of ordinary group cohomology, or equivalently of projective and free resolutions over group rings. Other formulations use finiteness properties of the classifying space of a group; in fact, many homological concepts can be thought of as algebraists' attempts to capture finite complexes. It is however not clear to what extent these two approaches are equivalent. For example, a group Γ which admits a finite model for its classifying space is always of type FL, but the converse is an open problem.

Here we propose yet another type of finiteness condition, based on Morava K-theory. Since every finitely generated $K(n)^*$-module is free, the following condition, which we intend to study below, makes sense:

Definition 0.1. *A discrete group* Γ *is said to be* $K(n)$*-finite if* $K(n)^*(B\Gamma)$ *has finite rank as a* $K(n)^*$*-module.*

Remark. Whether this is a useful concept or not remains to be seen (and is not yet clear to the author). This chapter should be considered a preliminary report on work in progress.

It is immediately clear that groups having finite mod p cohomology are $K(n)$-finite (such as finitely presented groups of type FL). According to Ravenel's theorem (Corollary II.4.2) all finite groups fall into this class, too. This is the marked difference to cohomological finiteness conditions studied in the past.
Section 2 contains simple-minded applications of results due to Hopkins-Kuhn-Ravenel to a class of discrete groups containing (S-)arithmetic groups. In Section 1, we shall begin the investigation of the class of $K(n)$-finite groups by making a few (more or less obvious) observations about what kind of groups might belong to it. Section 3 is devoted to Euler characteristics. From the very definition, it is clear that a $K(n)$-finite group Γ has an Euler characteristic, defined as the difference in ranks between the even and the odd degree part of its Morava K-theory (this becomes a "classical" Euler characteristic if one uses a

$\mathbb{Z}/2$-graded version of Morava K-theory). We shall see that in certain cases this coincides with a so-called equivariant Euler characteristic; as in the classical case, this follows easily from the Leray spectral sequence.

1 The class of $K(n)$-finite groups: Preliminary observations

We begin with the (obvious) observation that there are finitely presented groups whose Morava K-theory has infinite rank, by the Kan-Thurston theorem.
On the other hand, we have Corollary II.4.3, which says that whenever P is a finite p-group and $F \to E \to BP$ a fibration, then finite generation of $K(n)^*(F)$ implies the same for E.

This argument may be applied to the following situation. Let Γ be a discrete group, N a normal finite index subgroup, and $G = \Gamma/N$. Let P be a Sylow p subgroup of G. Then Γ has a subgroup Γ^p normalizing N with $\Gamma/\Gamma^p \cong P$ which plays the role of "Sylow p subgroup" for Γ, see [Ad]. In particular, the usual transfer argument shows that $B\Gamma$ is a stable summand of $B\Gamma^p$ in the p-local category. Assuming further N to be $K(n)$-finite, the above proof shows that Γ has finite $K(n)^*$-rank, too. In summary:

Corollary 1.1. *Let Γ be a discrete group and Γ' a normal finite index subgroup. If Γ' has finite $K(n)^*$-rank, then so does Γ.* □

The reverse implication is trivially false:

Example 1.2. Consider Morava K-theory at the prime p. Let C be a cyclic group of p' order, Γ a (countably) infinite free product of copies of C, and $\pi\colon \Gamma \to C$ the homomorphism which is the identity on each factor. Then both C and Γ have rank 1 over $K(n)^*$, but the kernel of π is an infinitely generated free group.

Another class of $K(n)$-finite groups may be obtained using the so-called classifying spaces for families of subgroups:

Definition 1.3. *A set $\mathscr{F}$ of subgroups of a group Γ is called a family of subgroups if for $H \in \mathscr{F}$ all subgroups of H and all conjugates of H are in $\mathscr{F}$.*

Definition 1.4. *A classifying space for a family of subgroups $\mathscr{F}$ of Γ is a Γ CW complex $E_{\mathscr{F}}\Gamma$ such that $(E_{\mathscr{F}}\Gamma)^H$ is contractible for $H \in \mathscr{F}$ and empty otherwise. If $\mathscr{F}$ is the family of finite subgroups, also write $\underline{E}\Gamma$ for $E_{\mathscr{F}}\Gamma$.*

Existence and uniqueness of $E_{\mathscr{F}}\Gamma$ are proved in [tD1] or [tD2, I.6]; a combinatorial construction can be found in [DL]. Uniqueness is implied by the fact that any Γ-complex with isotropy in $\mathscr{F}$ admits a map to $E_{\mathscr{F}}\Gamma$ which is unique up to Γ-homotopy.

Now let $\mathscr{F}$ be a family of subgroups of Γ all of whose members are $K(n)$-finite. If there is a finite model for $E_{\mathscr{F}}\Gamma$, then Γ is also $K(n)$-finite. This follows from the Leray spectral sequence of the bundle

$$B\Gamma \simeq E\Gamma \times_\Gamma E_{\mathscr{F}}\Gamma \to E_{\mathscr{F}}\Gamma/\Gamma$$

with E_1-page

$$E_1 = \bigoplus_\sigma K(n)^*(B\Gamma_\sigma)\,;$$

here summation is over the equivariant cells σ of $E_{\mathscr{F}}\Gamma$. In particular, if there is a finite model for $\underline{E}\Gamma$, the Γ is $K(n)$-finite. As above one proves

Theorem 1.5. *Let* $1 \to \Gamma' \to \Gamma \to \Gamma'' \to 1$ *be a group extension satisfying the following conditions:*

(i) $K(n)^*(B\Gamma')$ *has finite rank over* $K(n)^*$*;*

(ii) there is a finite model for $\underline{E}\Gamma''$*.*

Then $K(n)^*(B\Gamma)$ *has finite rank over* $K(n)^*$*.*

Proof. The isotropy groups Γ_σ are extensions of finite groups by Γ'. Hence the claim follows from Corollary 1.1 □

Again the converse is false, which should not be too surprising. The property of having a finite model for $\underline{E}G$ does not pass from a finite index subgroup to the supergroup: I. Leary and B. Nucinkis have constructed counterexamples [LN], such as the the one below.

Example 1.6. ([LN]) There is a group $G = H \rtimes A$ with the following properties:

- there is a finite model for BH;
- A is $C_{30} \times C_{30}$ or A_5 (or any group not of the form p-group $\times$ cyclic group $\times$ q-group ist, for p, q not necessarily distinct primes);
- G has infinitely many conjugacy classes of subgroups isomorphic to A.

The third property implies that such G cannot have a model for $\underline{E}G$ of finite type. On the other hand, G is $K(n)$-finite by Corollary 1.1.

2 Character theory for discrete groups

Inspired by Quillen's method of extending his result to groups of finite vcd [Q4, II, §15], we shall now describe how some of the results of [HKR] can be used to describe Morava K-Theory of a certain class of discrete groups.
Suppose given a contractible complex X with an action of the discrete group Γ such that a finite index normal subgroup Γ' acts freely. Assume further that a subgroup of Γ has non-empty fixed point set if and only if it is finite, and that in that case the fixed point set is contractible. Then X is a classifying space for Γ'. Let us finally assume that $Y = X/\Gamma'$ is a finite complex.

Remark. There are several important examples where these conditions are met:

(a) (S-)arithmetic groups by a result of Borel and Serre [BS],

(b) mapping class groups,

(c) word hyperbolic groups.

The finite group $G = \Gamma/\Gamma'$ acts on Y, and Γ acts on EG via the projection $\pi\colon \Gamma \to G$. The map $r\colon X \to Y$ is a finite covering with G as its group of deck transformations. Since the diagonal action of Γ on $EG \times X$ is free, we see that

$$EG \times_\Gamma X \simeq EG \times_G Y$$

is a model for $B\Gamma$. Thus Theorem II.6.3 applies to the G-complex Y. In order to interpret the answer, a few lemmas due to Quillen may help. Let X, Y, Γ, Γ' be as above.

Lemma 2.1 ([Q4], Lemma 15.1)**.** *Let $x \in X$ and $y = r(x)$. Then r induces an isomorphism of isotropy groups $\Gamma_x \cong G_y$.* □

Lemma 2.2 ([Q4], Lemma 15.3)**.** *Let K be a subgroup of G. Then $r^{-1}(Y^K)$ is the disjoint union of the fixed point sets X^H, where H runs over the subgroups of Γ mapped isomorphically onto K by π.* □

In other words,

$$r^{-1}(Y^K) = \coprod_{H \in \tilde{\mathscr{C}}} X^H$$

where $\tilde{\mathscr{C}}$ is the set of finite subgroups of Γ whose image in G is K.

Corollary 2.3. *Let $\mathscr{C}$ denote a set of representatives for the Γ'-conjugacy classes of finite subgroups of Γ whose image in G is K. Then*

$$Y^K = \coprod_{H \in \mathscr{C}} X^H/\Gamma' \cap NH\,.$$

3 Euler characteristics

We start with the naive generalization of Morava K-theory Euler characteristics:

Definition 3.1. *For a $K(n)$-finite group Γ let*

$$\chi_{n,p}(\Gamma) := \mathrm{rank}_{K(n)^*} K(n)^{\mathrm{ev}}(B\Gamma) - \mathrm{rank}_{K(n)^*} K(n)^{\mathrm{odd}}(B\Gamma)$$

denote the naive $K(n)$ Euler characteristic of Γ.

Example 3.2. Let $p = 2$ and Γ the split extension of C_2 by the integers $\mathbb{Z}$, where C_2 acts by sign change. Then we have a fibration $S^1 \to B\Gamma \to BC_2$. The associated Serre spectral sequence is easily seen to yield an additive isomorphism

$$K(n)^*(B\Gamma) \cong K(n)^*(BC_2) \otimes \Lambda(\alpha) \cong K(n)^*[x]/(x^{2^n}) \otimes \Lambda(\alpha)$$

with x in degree 2 and α in degree 1. Thus $\chi_{n,2} = 0$ here – not very thrilling.

As in the classical case one can try to compute Euler characteristics via actions on CW complexes. To that end we consider the so-called equivariant Euler characteristic defined as follows.

Definition 3.3. *Let X be a Γ-complex satisfying*

(i) X has are only finitely many equivaraint cells, and

(ii) every isotropy group Γ_σ is $K(n)$-finite.

Under these hypotheses set

$$\chi^{\Gamma}_{n,p}(X) := \sum_{\sigma \in \mathscr{E}} (-1)^{\dim \sigma} \chi_{n,p}(\Gamma_\sigma)$$

where $\mathscr{E}$ runs over a set of representatives of the cells mod Γ.

An important example is the situation described in the last section, i.e., where one has an extension

$$1 \to \Gamma' \to \Gamma \to G \to 1 \tag{3.1}$$

with G finite; and where one is given a Γ-complex X with finitely many cells mod Γ', all finite subroups as isotropy groups, and contractible fixed point sets. Then

$$B\Gamma \simeq EG \times_\Gamma X \simeq EG \times_G X/\Gamma' , \tag{3.2}$$

and the (naive) Euler characteristic coincides with the equivariant Euler characteristic of X:

Proposition 3.4. *For such Γ, Γ', and X one has $\chi_{n,p}(\Gamma) = \chi^{\Gamma}_{n,p}(X)$.*

Proof. Clearly Γ is $K(n)$-finite, see section 1. The claimed equality follows from the Cartan-Leray spectral sequence of the bundle $\pi\colon EG \times_\Gamma X \to X/\Gamma$, with

$$E_1^{*,*} \cong \bigoplus_{\sigma\in\mathscr{E}} K(n)^*(\pi^{-1}(\sigma)) \Longrightarrow K(n)^*(X\times_\Gamma E\Gamma) \cong K(n)^*(B\Gamma)\,,$$

where $\mathscr{E}$ denotes as above the set of equivariant cells of X. Then

$$\pi^{-1}(\Gamma_\sigma\sigma) = E\Gamma\times_\Gamma \Gamma/\Gamma_\sigma = E\Gamma\times_{\Gamma_\sigma} * = B\Gamma_\sigma\,,$$

whence the claim. □

Example 3.5. Using Soulé's computation of the 2-local cohomology [So], Tezuka and Yagita calculated BP-cohomology and (Morava) K-theory of $SL_3(\mathbb{Z})$ at the primes 2 and 3 [TY4].
We recall the 2-primary calculation: $\Gamma = SL_3(\mathbb{Z})$ acts on a contractible 3-dimensional complex X (symmetric space respectively upper half plane) with finite isotropy. Consider as above the Leray spectral sequence for the bundle $\pi : X\times_\Gamma E\Gamma \to X/\Gamma$,

$$E_1^{*,*} \cong \bigoplus_{\sigma\in\mathscr{E}} k^*(\pi^{-1}(\sigma) \Longrightarrow k^*(X\times_\Gamma E\Gamma) \cong k^*(B\Gamma)$$

where k is either 2-local cohomology, or BP mod 2, or any other 2-local theory. According to Soulé, locally at 2 one can replace X/Γ by the interval $Y = [0,2]$ considered as a 1-complex with 0-cells $0,1,2$ and two 1-cells σ,τ. The cell stabilisers are Σ_4 for the 0-cells, whereas $\Gamma_\sigma \cong C_2$ and $\Gamma_\tau \cong D_8$. Since Y is 1-dimensional, the spectral sequence degenerates to a Mayer-Vietoris sequence, from which one can read off the result. In particular, one gets

$$\chi_{n,2}^\Gamma = 3\cdot\chi_{n,2}(\Sigma_4) - \chi_{n,2}(D_8) - \chi_{n,2}(C_2) = 2\cdot(4^n - 2^n) + 1\,.$$

Appendix

A Euler characteristics of some groups

In this appendix we give the Euler characteristic of some groups which were omitted for ease of exposition, including those whose Morava K-theory was computed earlier.
We shall compute the Morava K-theory Euler characterstic $\chi_{n,p}(G)$, i.e., the difference between the ranks of the even and odd degree parts of $K(n)^*(BG)$, using Theorem II.6.4 from [HKR]:

$$\chi_{n,p}(G) = \sum_{A<G} \frac{|A|}{|G|} \mu_{\mathcal{A}(G)}(A) \chi_{n,p}(A) \tag{A.1}$$

where the sum is over all abelian subgroups $A < G$ and $\mu_{\mathcal{A}(G)}$ is a Möbius function defined recursively by

$$\sum_{A<A'} \mu_{\mathcal{A}(G)}(A') = 1 \tag{A.2}$$

where the sum is over all abelian subgroups of G containing A (including A). In particular, $\mu_{\mathcal{A}(G)}(A) = 1$ when A is maximal. It is easy to see that one only has to consider subgroups arising as intersections of maximal ones. Furthermore, one clearly has $\chi_{n,p}(A) = |A_{(p)}|^n$ where $A_{(p)}$ denotes the p-part of the abelian group A.

1. We begin with extraspecial p-groups of exponent p. The formula below was obtained by Brunetti [B2]; we give a simplified version of his proof.
The abelian subgroups of $D(m) = p_+^{1+2m}$ are in one-to-one correspondence with the subspaces W of the central quotient $V \cong \mathbb{F}_p^{2m}$ which are isotropic with respect to the bilinear form

$$b(x, y) = x_1 y_2 + x_2 y_1 + \cdots + x_{2m-1} y_{2m} + x_{2m} y_{2m-1}\,.$$

Let $\alpha_{m,i}$ denote the number of such subspaces of dimension i. Note that the maximal dimension of a b-isotropic subspace is m.
The following lemma is an easy exercise in counting:

Lemma A.1. $\alpha_{m,i} = \prod_{j=1}^{i} \frac{p^{2(m-j+1)} - 1}{p^j - 1}$.

Proof. We first count the b-isotropic flags $W_1 \subset W_2 \subset \cdots \subset W_l$ of length l, where W_k has rank k, in V of dimension $2m$: For $l = 1$ this number equals the number of one-dimensional subspaces, hence we get $(p^{2m} - 1)/(p - 1)$. Now fix W_1 and count the complete flags staring with W_1: this is the same as the number of complete flags of length $l - 1$ in $V' = W_1^{\perp}/W_1$ of dimension $2m - 2$. By induction, the number of flags thus becomes

$$\prod_{j=1}^{l} \frac{p^{2(m-j+1)} - 1}{p - 1} .$$

Secondly, the number of complete flags inside a given subspace W_d of dimension d is

$$\prod_{i=1}^{d} \frac{p^i - 1}{p - 1} .$$

This is trivial for $d = 1$; for every $W_1 \subset W_d$, each flag in W_d/W_1 gives rise to one in W_d, and the formula follows again by induction. Finally, the number of b-isotropic subspaces is clearly the quotient of the above numbers. □

The Möbius function on abelian subgroups can be computed via a Möbius function on b-isotropic subspaces defined as in (A.2). Let $\gamma_{m,k}$ denote its value on a subspace of dimension k: by symmetry, it is constant on subspaces of the same rank. Furthermore, it only depends on the *codimension* of a b-isotropic subspace in a maximal one, independent of m; this follows as above by considering $W_k^{\perp}/W_k$. In particular, $\gamma_{m,k} = \gamma_{m-k,0}$.

Lemma A.2. $\gamma_{m,k} = (-p)^{(m-k)^2}$. □

Proof. By the remarks above, it is enough to show

$$\gamma_{m,0} = (-1)^m p^{m^2} ,$$

where we may assume by induction

$$\gamma_{m,d} = (-1)^{m-d} p^{(m-d)^2} .$$

By definition of γ, this means we have to verify the formula

$$s_m := \sum_{j=0}^{m} (-1)^j p^{j^2} \alpha_{m,m-j} = 1 . \tag{A.3}$$

This will be done in several steps. First, we note the obvious identities

$$\alpha_{m-1,m-l} = \frac{p^{2l}-1}{p^{m-l}-1} \cdot \alpha_{m-1,m-l-1} \; ;$$
$$\alpha_{m,m-k} = \frac{p^{2m}-1}{p^{m-k}-1} \cdot \alpha_{m-1,m-k-1} \; .$$

Next, we claim the following auxiliary formula

$$t_l := \sum_{k=0}^{l} (-1)^k p^{k^2} \frac{p^{2m}-p^{m-k}}{p^{m-k}-1} \alpha_{m-1,m-k-1} = (-1)^l p^{m+l(l+1)} \alpha_{m-1,m-l-1} \; . \quad \text{(A.4)}$$

This is an easy induction on l. Clearly $t_0 = \frac{p^{2m}-p^m}{p^m-1}\alpha_{m-1,m-1} = p^m \alpha_{m-1,m-1}$. Assuming (A.4) for $l-1$, we compute

$$\begin{aligned} t_l &= t_{l-1} + (-1)^l p^{l^2} \frac{p^{2m}-p^{m-l}}{p^{m-l}-1} \alpha_{m-1,m-l-1} \\ &= (-1)^{l-1} p^{m+l(l-1)} \alpha_{m-1,m-l} + (-1)^l p^{l^2} \frac{p^{2m}-p^{m-l}}{p^{m-l}-1} \alpha_{m-1,m-l-1} \\ &= \left[(-1)^{l-1} p^{m+l(l-1)} \frac{p^{2l}-1}{p^{m-l}-1} + (-1)^l p^{l^2+m-l} \frac{p^{m+l}-1}{p^{m-l}-1} \right] \alpha_{m-1,m-l-1} \\ &= (-1)^l p^{m+l(l-1)} \frac{(p^{m+l}-1)-(p^{2l}-1)}{p^{m-l}-1} \alpha_{m-1,m-l-1} \\ &= (-1)^l p^{m+l(l+1)} \alpha_{m-1,m-l-1} \; . \end{aligned}$$

In particular, $t_{m-1} = (-1)^{m-1} p^{m^2}$. To finish the proof, subtract s_{m-1} from s_m:

$$\begin{aligned} s_m - s_{m-1} &= \sum_{k=0}^{m} (-1)^j p^{k^2} \alpha_{m,m-k} - \sum_{k=0}^{m-1} (-1)^k p^{k^2} \alpha_{m-1,m-k-1} \\ &= (-1)^m p^{m^2} + \sum_{k=0}^{m-1} (-1)^k p^{k^2} \left(\frac{p^{2m}-1}{p^{m-k}-1} - 1 \right) \alpha_{m-1,m-k-1} \\ &= (-1)^m p^{m^2} + t_{m-1} = 0 \; . \end{aligned}$$

Since clearly $s_0 = 1$, the claim follows. □

Since a b-isotropic subspace W of dimension i gives rise to an abelian subgroup of index $2m - i$ (the corresponding abelian is either $W \times C$ or $W \times_C \mathbb{Z}/4$, where C is the centre), we arrive at

Proposition A.3. *The Morava K-theory Euler characteristic of $G = p_+^{1+2m}$ is given by*

$$\chi_{n,p}(G) = \sum_{i=0}^{m} \frac{\alpha_{m,i}\gamma_{m,i}}{p^{2m-i}} p^{(i+1)n} = \sum_{i=0}^{m} (-1)^{m-i} p^{(m-i-1)^2+(n-1)(i+1)} \alpha_{m,i}$$

with α and γ as above.

For example, for D_8 and $D(2) = 2_+^{1+4}$ we obtain

$$\begin{aligned}\chi_{n,2}(D_8) &= \frac{3}{2}4^n - \frac{1}{2}2^n\,, \quad \text{and}\\ \chi_{n,2}(D(2)) &= \frac{15}{4}(8^n - 4^n) + 2^n\,.\end{aligned}$$

This agrees with the Euler characteristics we can compute from Theorems IV.1.2 and V.3.6, as we shall now see. For D_8 and Q_8 this was done in the text, so consider $D(2)$. From Lemma V.3.5 we have that K (the kernel of d_3) has rank $k := 2^{2n-1}+2^{2n-2}+2^{n-1}$, and H (the homology with respect to d_3) has rank $h := 6\cdot 2^{n-2}$. The E_4-page of the spectral sequence is isomorphic to $K\otimes M_1 \oplus H\otimes M_2$ where $M_1 = \mathbb{F}_2[x_1,x_2]/(x_1^2x_2 + x_1x_2^2)$ and $M_2 = \mathbb{F}_2[x_1,x_2]\{x_1^2x_2 + x_1x_2^2\}$. The Q_n-hmologies of M_1 and M_2 have ranks $m_1 = 3\cdot(2^n-1)$ and $m_2 = 2^{2n} - m1$ (compare V.2), thus the rank of the E_∞-page is

$$k\cdot m_1 + h\cdot m_2 = (15\cdot 2^{2n} - 15\cdot 2^n + 4)\cdot 2^{n-2} = \chi_{n,2}(D(2))\,.$$

2. Let G be dihedral, or semidihedral, or generalized quaternion of order 2^{N+2}. In the dihedral case, the maximal abelian subgroups are the cyclic subgroup $\langle s\rangle$ of order 2^{N+1} and 2^N subgroups $C_2\times C_2$, coming in two conjugacy classes; their common intersection is the centre $\langle s^{2^N}\rangle \cong C_2$ with Möbius function value -2^N. The same pattern holds for semidihedral groups, so we obtain

$$\chi_{n,2}(D_{2^{N+2}}) = \chi_{n,2}(SD_{2^{N+2}}) = \frac{1}{2}2^{(N+1)n} + 4^n - \frac{1}{2}2^n\,.$$

For generalized quaternion groups, there are 2^N subgroups C_4 (two conjugacy classes) and one copy of $C_{2^{N+1}}$, intersecting in the centre C_2, so again we have

$$\chi_{n,2}(Q_{2^{N+2}}) = \frac{1}{2}2^{(N+1)n} + 4^n - \frac{1}{2}2^n\,.$$

Finally, for quasidihedral groups the maximal subgroups are $Z\times\langle t\rangle \cong C_{2^N}\times C_2$, $\langle st\rangle \cong C_{2^{N+1}}$ and $\langle s\rangle \cong C_{2^{N+1}}$, the latter two non-conjugate, all intersecting in the centre $Z = \langle s^2\rangle \cong C_{2^N}$. Thus

$$\chi_{n,2}(QD_{2^{N+2}}) = \frac{3}{2}2^{(N+1)n} - \frac{1}{2}2^{Nn}\,.$$

3. Let $G = PSL_2(\mathbb{F}_q)$ with $q = p^f$ for an odd prime p. According to [Hp, II, 8.5], every non-identity element g of G lies in precisely one conjugate of one of the subgroups P, U, S where P is an elementary abelian p-group of rank f

(a Sylow p subgroup), U is cyclic of order $m = \frac{1}{2}(q-1)$ (U is the subgroup of Möbius transformations leaving 0 and ∞ fixed), and S is cyclic of order $k = \frac{1}{2}(q+1)$. Furthermore P has $q+1$ conjugates, and U and S have index two in their normalisers (which are dihedral groups). Thus

$$\chi_{n,2}(PSL_2(\mathbb{F}_q) = \frac{1}{2}\left(2^{n\cdot\nu_2((q+1)/2)} + 2^{n\cdot\nu_2((q-1)/2)}\right).$$

4. Let P be the 3-group giving rise to Kriz's counterexample, *i.e.*, the 3-Sylow subgroup of $GL_4(\mathbb{F}_3)$. The maximal abelian subgroups of P are

- one elementary abelian subgroup of rank 4, corresponding to matrices of the form $\left(\begin{smallmatrix}1&0&*&*\\&1&*&*\\&&1&0\\&&&1\end{smallmatrix}\right)$;
- 12 subgroups isomorphic to $C_9 \times C_3$, coming in four conjugacy classes;
- 51 elementary abelian subgroups of rank 3.

Furthermore, P has 57 conjugacy classes of elements; since this gives $\chi_{1,3}(P)$, and $\chi_{0,p}(G) = 1$ for any G, the above data suffice to compute

$$\chi_{n,3}(P) = \frac{1}{9}81^n + \frac{7}{3}27^n - \frac{16}{9}9^n + \frac{1}{3}3^n.$$

Just for fun, here is the corresponding result for the prime 2, *i.e.* the group $U_4(\mathbb{F}_2)$ of 4×4 upper triangular matrices over $\mathbb{F}_2$: The number of conjugacy classes is 16, and the maximal abelian subgroups are: one elementary abelian of rank 4 (as above), 15 of type $C_4 \times C_2$, and 5 of type $C_2 \times C_2 \times C_2$. Thus

$$\chi_{n,2}(U_4(\mathbb{F}_2)) = \frac{1}{4}16^n + \frac{5}{2}8^n - \frac{9}{4}4^n + \frac{1}{2}2^n.$$

B Tables

For each p, n, and V, let $\widetilde{K}_V$ be the direct summand of K_V corresponding to the reduced n-th Morava K-theory of BV. Thus $\widetilde{K}_V^k = K_V^k$ for $k \neq 0$, and $K_V^0 = \widetilde{K}_V^0 \oplus T$, where T is the trivial $\mathbb{F}_p[GL(V)]$-submodule of dimension one spanned by the monomial 1. The $\mathbb{F}_p$-dimension of $\tilde{K}_V^k$ is $(p^{nd}-1)/(p^n-1)$, where $d = \dim V$ as before.
Table 1 describes the $SL_2(\mathbb{F}_3)$-module structure of $\widetilde{K}_V^k$ (for $p=3$) in terms of the indecomposable modules $I_1, \ldots, I_7$ as described in VI.3.2.

Tables 2–4 give the maximal rank permutation submodules M' of $\widetilde{K}_V^k$ for $\dim V = 3$ and $p = 2, 3, 5$.
Specifically, let l be a line in V, and let π be a plane in V containing l. The group $GL(V)$ acts on the set of all such pairs, and the stabiliser of the pair (l, π) contains a unique Sylow p-subgroup $U(V)$ of $GL(V)$ (and is in fact equal to the normaliser of $U(V)$). Let C be a generator for the centre of $U(V)$, which is cyclic of order p. Let A be a non-central element of $U(V)$ stabilising every line in π, and let B be a non-central element of $U(V)$ stabilising every plane containing l. Then A and B generate $U(V)$, and after replacing C by a power if necessary, the commutator of A and B is equal to C. If we identify V with $\mathbb{F}_p^3$, and take $U(V)$ to be the upper triangular matrices, then we may take

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

For $p = 2$ the group $U(V)$ has 8 conjugacy classes of subgroups, which we list in the following order:

$$\{1\}, \langle A \rangle, \langle B \rangle, \langle C \rangle, \langle AB \rangle, \langle A, C \rangle, \langle B, C \rangle, U(V).$$

Let $P_1, \ldots, P_8$ be the corresponding transitive permutation modules, so that P_1 is the free module and P_8 is the trivial module. Similarly, for $p > 2$, $U(V)$ has $2p+5$ conjugacy classes of subgroups, which we list as:

$$\{1\}, \langle A \rangle, \langle AB \rangle, \ldots, \langle AB^{p-1} \rangle, \langle B \rangle, \langle C \rangle,$$
$$\langle A, C \rangle, \langle AB, C \rangle, \ldots, \langle AB^{p-1}, C \rangle, \langle B, C \rangle, U(V).$$

Again we let $P_1, \ldots, P_{2p+5}$ be the corresponding transitive permutation modules. Note that for $p = 2$, the computations show this submodule always to coincide with the entire module: we have $\dim P_1 = 8$, $\dim P_2 = \dim P_3 = \dim P_4 = 4$, $\dim P_5 = \dim P_6 = \dim P_7 = 2$, and $\dim P_8 = 1$, and the dimension of each row adds up to $2^{3n} - 1/(2^n - 1)$. (This is the reason $\dim M'$ is omitted from this table.)

Tables 5 and 6 finally give, for $\dim V = 2$, the dimension of a maximal H-permutation submodule M'' of K_V^1, for each conjugacy class (in $GL(V)$) of subgroups H of $U(V)$. The dimension of K_V^1 is 91 for $p = 3$ and 651 for $p = 5$.

Table 1: The $SL_2(\mathbb{F}_3)$-summands of $\widetilde{K}_V^k$.

n	k	I_1	I_2	I_3	I_4	I_5	I_6	I_7
1	0	1	0	0	0	0	0	1
1	1	0	0	0	0	1	0	0
2	0,4	1	0	1	0	0	0	2
2	1,3,5,7	0	0	0	0	1	1	0
2	2,6	1	0	0	0	0	0	3
3	even	1	0	2	0	0	0	7
3	odd	0	0	0	0	1	4	0

Table 2: The D_8-summands of $\widetilde{K}_V^k$.

n	k	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
1	0	0	1	0	0	0	0	1	1
2	0	0	2	2	1	0	0	0	1
2	1,2	1	1	1	0	0	1	1	1
3	0	4	4	4	2	0	0	0	1
3	1,6	6	1	3	0	0	3	1	1
3	2,5	5	3	3	1	0	1	1	1
3	3,4	5	2	4	1	0	2	0	1
4	0	24	8	8	4	0	0	0	1
4	1,14	28	1	7	0	0	7	1	1
4	2,13	25	7	7	3	0	1	1	1
4	3,12	27	2	8	1	0	6	0	1
4	4,11	25	6	8	3	0	2	0	1
4	5,10	27	3	7	1	0	5	1	1
4	6,9	26	5	7	2	0	3	1	1
4	7,8	26	4	8	2	0	4	0	1
5	0	112	16	16	8	0	0	0	1

Table 3: A maximal $\mathrm{Syl}_3(GL_3(\mathbb{F}_3))$-permutation submodule of $\widetilde{K}_V^k$.

n	k	dim. M'	P_1	P_2	P_5	P_6	P_7	P_{10}	P_{11}
1	0,1	13	0	1	0	0	0	1	1
2	0	91	1	3	3	1	0	0	1
2	1	65	1	1	2	0	3	0	2
2	2	71	1	2	2	0	1	1	2
2	3	73	1	2	2	0	1	2	1
2	4	57	1	1	1	0	2	1	3
2	5	65	1	2	1	0	1	2	2
2	6	67	1	2	1	0	2	2	1
2	7	73	1	2	2	0	2	1	1

Table 4: A maximal $\mathrm{Syl}_5(GL_3(\mathbb{F}_5))$-permutation submodule of $\widetilde{K}_V^k$.

n	k	dim. M'	P_1	P_2	P_7	P_8	P_9	P_{14}	P_{15}
1	0–3	31	0	1	0	0	0	1	1
2	0	651	3	5	5	1	0	0	1
2	1	527	3	1	4	0	5	0	2
2	2	447	2	3	4	0	3	1	2
2	3	467	2	4	4	0	2	1	2
2	4	587	3	4	4	0	1	1	2
2	5	591	3	4	4	0	1	2	1

Table 5: A maximal H-permutation submodule of K_V^1 ($p = 3$, $\dim V = 2$).

Subgroup H	dim. M''
$\langle A, C\rangle$	69
$\langle AB, C\rangle$	84
$\langle B, C\rangle$	87

Table 6: A maximal H-permutation submodule of K_V^1 ($p = 5$, $\dim V = 2$).

Subgroup H	dim. M''
$\langle A, C \rangle$	535
$\langle AB, C \rangle$	628
$\langle B, C \rangle$	643

Epilogue

There is much work left to be done. As already mentioned, no example of a 2-group with odd Morava K-theory exists. We did some calculations for the mod 2 analogon of Kriz's example, and our evidence points to this group having even Morava K-theory: this is suggested by the decomposition of the Morava K-theory of a rank 3 elementary abelian group V as D_8-module carried out in Chapter VI, where it is shown that $K(n)^*(BV)$ has a permutation basis. Of course, this is very far from a proof.
Even more pressing would be a complete calculation of $K(n)^*(BG)$ for the group giving rise to the counterexample at odd primes.

Other experimental calculations suggest that any automorphism of order p of a finite abelian p-group A turns $\widetilde{K}(n)^*(BA)$ into a permutation module for C_p. This would imply that any p-group having an index p abelian subgroup would have even Morava K-theory, and render some of the calculations in Chapter V obsolete. We have however not succeeded in proving such a statement, and are not even prepared to make a formal conjecture. Furthermore, there seem to exist only very few possible module structures for such actions. We hope to be able to classify them in the future using suitable filtrations.

We are more confident about a phenomenon sometimes called 'equidistribution'. Any p-group G whose Morava K-theory has been determined (this only concerns G with $K(n)^{\mathrm{odd}}(BG) = 0$) enjoys the property that the rank of $K(n)^{2i}(BG)$ does not depend on i, except for $i = 0$, where the unit gives one extra dimension:

Conjecture. *Let G be a p-group with $K(n)^{odd}(BG) = 0$. Then for $0 < i < p^n - 1$,*

$$\mathrm{rank}_{K(n)^*}\big(K(n)^{2i}(BG)\big) = \mathrm{rank}_{K(n)^*}\big(K(n)^0(BG)\big) - 1\,.$$

A positive resolution to the conjecture would imply that the additive structure of the Morava K-theory of a p-group is determined by a single invariant, its rank.

The treatment of discrete groups given here also leaves much to be desired. Many natural questions were not addressed, leave alone answered. For example, we could show that the property of having finite $K(n)$-rank passes from a finite index subgroup to the supergroup, and had an example showing that the converse can not hold. But this example was highly artificial and depended on mixing different primes. When the index of the subgroup is a power of the defining prime, we do not know what happens.

Finally, the problem in our opinion overriding all others is the quest for a theory explaining generalised characters in a functorial way, i.e., a theory that may be dubbed 'higher representation theory'. We have no idea what this could be.

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