The concept of duality: developments in Italian textbooks

Marta Menghini

Sapienza University of Roma

<u>Texts considered</u>:

Pr_1871.

Ministero di agricoltura, industria e commercio. (1871). *Ordinamenti degli istituti tecnici*. Firenze: Claudiana.

Programmes for the physics-mathematics classes of the Technical Institutes (age 14-18)

Cre_1873.

Luigi Cremona (1873). *Elementi di geometria projettiva*. Vol. I, Roma: G.B. Paravia.

Textbook for the physics-mathematics classes of the Technical Institutes, later for Polytechnics and Universities

Luigi Cremona (1875). *Éléments de géométrie projective, traduits, avec la collaboration de l'auteur, par Éd. Dewulf* , 1 ptie. Paris: Gauthier-Villars.

Luigi Cremona (1882). *Elemente der projectivischen Geometrie; unter Mitwirkung des Verfassers übertragen von Fr. R. Trautvetter*. Stuttgart: J. G. Cotta.

Luigi Cremona (1885). *Elements of projective geometry,* translated by Charles Leudesdorf. Oxford: At the Clarendon Press.

Textbooks for Polytechnics and Universities

SO_1871.

Enrico D'Ovidio e Achille Sannia (1871) *Elementi di geometria*, Napoli, tipografia A. Trani

Textbook for gymnasia and Technical Institutes

Pi_1891.

Mario Pieri (1891). *Geometria proiettiva*. Torino, G. Candeletti

Textbook for students of the Military Academy

BL_1891.

Anselmo Bassani e Giulio Lazzeri (1891). *Elementi di geometria*, Livorno, Giusti

Textbook for students of the Naval Academy in Livorno (secondary school, then Polytechnic school)

The programmes of 1871

- explicit introduction of the fundamental principles of projective geometry, seen as a necessary preamble to descriptive geometry.

- the reform recognised the need for a general literary and scientific education in technical education and instituted a Physics-Mathematics Section (*Sezione fisico – matematica*). This section permitted university entrance, and could be seen as the scientific alternative to the Lycée.

Luigi Cremona (1830-1903)

- after the political unification of Italy in 1861, Cremona can be considered one of the period's leading mathematicians.

- geometric studies of a synthetic nature, within the classical school of projective geometry, with particular attention to Poncelet and Chasles, Von Staudt, Plücker, Möbius, Steiner and Clebsch.

- research into birational transformations basis for successive studies carried out in Italy in algebraic geometry, culminated in the famous (Roman) school of geometry at the beginning of the 20^{th} century.

- born in Pavia in 1830, graduated in civil engineering and architecture. In 1860 Chair of Advanced Geometry at the University of Bologna, where he also taught Descriptive Geometry. Milano in 1866, taught Graphical Statics at the Polytechnic and Advanced Geometry at the Scuola Normale (annexed to the Polytechnic itself to train Technical Institute teachers). In 1873 transferred to Rome to head the School for Engineers, in whose reconstruction he played a fundamental role. He became a senator in 1879, was made vice-president of the Senate and in 1898, for just one month, was Minister for Education.

Achille Sannia (1823 - 1892), Enrico D'Ovidio (1843 - 1933)

- in the 1850s Sannia directed a private school of mathematics in Naples which had – for a certain period - greater prestige and efficiency than the university.
- in 1865 he started to teach projective geometry at the University if Naples. Few publications.

- D'Ovidio was his student in the private school and obtained then the degree in 1869. From 1872 he taught Algebra and Analytic geometry at the University of Turin. He also became rector of the University and director of the Polytechnic.
His research concerns Euclidean and non-euclidean metrics; he started the research then developed by Corrado Segre, who was his student.

Mario Pieri (1860 - 1913)
Degree in mathematics at the University of Pisa (SNS). He became professor of projective geometry at the Military Academy and assistant at the Turin University. Then he was transferred to Catania. Influenced by Peano he left the reserch in geometry to cultivate logic.

Anselmo Bassani (1856 - 1911) and Giulio Lazzeri (1861 - 1935)
Both teachers at the Naval Academy of Livorno. Very active in Mathesis, the association of Mathematics teachers.
Their book, based on the fusion of plane and solid geometry, was translated in German by Treutlein.

From Cremona's introduction

This book is not … for those whose … mission is … the progress of science; they would find … nothing new, neither as regards principles, nor as regards methods. The propositions are all old; in fact, not a few of them owe their origin to mathematicians of the most remote antiquity. They may be traced back to EUCLID (*285* B.C.), to APOLLONIUS of Perga (*247* B.C.), to PAPPUS of Alexandria (4th century a. C.); to DESARGUES of Lyons (1593-1662) ; to PASCAL (1623–1662); to DE LA HIRE (1640-1718) ; to NEWTON (1642-1727); to MACLAURIN (1698-1746); to J.H. LAMBERT (1728-1777), &c. The theories and methods which make of these propositions a homogeneous and harmonious whole it is usual to call *modern*, because they have been discovered or perfected by mathematicians of an age nearer to ours, such as CARNOT, BRIANCHON, PONCELET, MöBIUS, STEINER, CHASLES, STAUDT, etc., whose works were published in the earlier half of the present century.

… I have chosen the name of *Projective Geometry*, as expressing the true nature of the methods, which are based essentially on central projection or perspective. One reason … is that the great PONCELET, the chief creator of the modern methods, gave to his immortal book the title of *Traité des propriétés projectives des figures* (1822).

> ... I have laid more stress on descriptive properties than on metrical ones; and have followed rather the methods of the Geometrie der Lage of STAUDT than those of the Géométrie supérieure of CHASLES.

... I have made use of central projection in order to establish the idea of infinitely distant elements; and, following … STEINER and STAUDT, I have placed the law of duality quite at the beginning of the book, as being a logical fact which arises immediately and naturally from the possibility of constructing space by taking either the point or the plane as element. The enunciations and proofs which correspond to one another by virtue of this law have often been placed in parallel columns; occasionally however this arrangement has been departed from, in order to give to students the opportunity of practising themselves in deducing from a theorem its correlative. Professor REYE remarks, with justice, … that Geometry affords nothing so stirring to a beginner, nothing so likely to stimulate him to original work, as the principle of duality; and for this reason it is very important to make him acquainted with it as soon as possible, and to accustom him to employ it with confidence.

... I have taken from [the masters], …the proofs of many theorems and the solutions of many problems…

Historical development of the concept of duality and place in the texts considered

Spherical triangles: duality point-great circle; and polar (or dual) triangle (Viète, 1650s)

Pr_1871 and other programs for technical institutes:

Spherical triangles: equality, symmetry; perimeter, area.

Polar figures - stereographic projection.

BL_1891 and others:

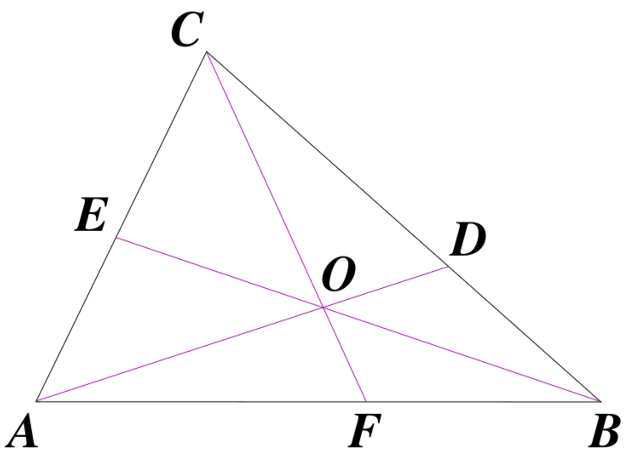
Duality (not explicit) of side and angle in the congruence of spherical triangles. Congruence of the polar triangle.

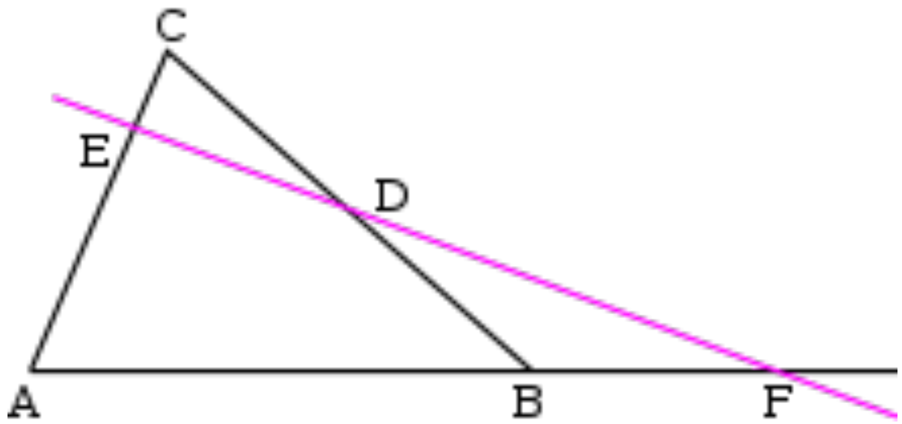
Menelaus (1st century a.C. - translated from Arab by Maurolico in 1500s) – Ceva (1678)

Ceva's theorem: Given a triangle ABC, let the lines AO, BO and CO be drawn from the vertices to a common point O (not on one of the sides of ABC), to meet opposite sides at D, E and F respectively,	Menelaus's theorem: Given a triangle ABC, and a transversal line that crosses BC, AC and AB at points D, E and F respectively, with D, E, and F distinct from A, B and C,

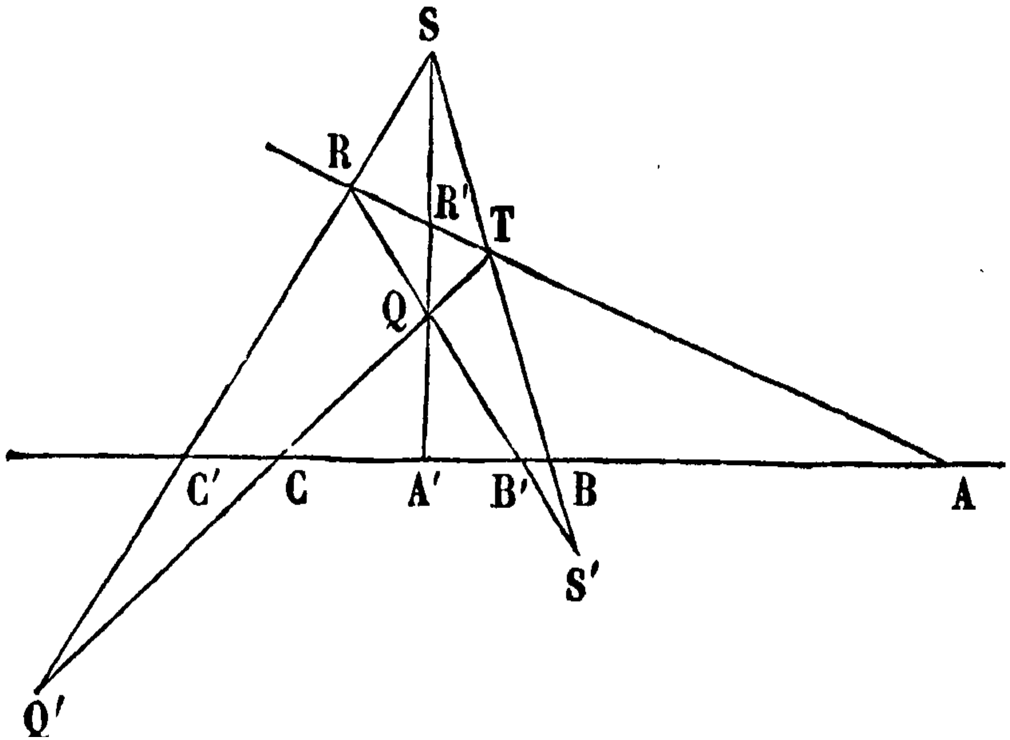
Then, using signed lengths of segments,

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = -1$$





Cre_1873. p. 110. **Theorems of Ceva and Menelaus derived from the theorem of the complete quadrilateral (the duality is not explicit).**



(ACA'B') = (ABA'C') = (A'C'AB)

➔

(projecting from T)

(BCAA')(CABB')(ABCC') = –1

➔

(SQR'A') (QRS'B')(RSQ'C') = –1

Suppose now the transversal to lie at infinity; then the anharmonic ratios *(SQR'A')*, *(QRS'B')*, and *(RSQ'C')* become respectively equal to *SR'* : *QR'*, *QS'*: *RS'*, and *RQ'*: *SQ'*; so that the preceding proposition reduces to the following: (THEOREM of CEVA)

Considering again the triangle QRS, and taking the transversal to be entirely arbitrary, let ST, QT be taken so as to be parallel to QR, RS respectively. Then the figure $QRST$ becomes a parallelogram; the points S' and Q' pass to infinity, and R' (being the point of intersection of the diagonals QS , RT) becomes the middle point of SQ. Consequently the anharmonic ratios $(SQR'A')$, $(QRS'B')$, $(RSQ'C')$ become equal respectively to $-(QA':SA')$, $(RB':QB')$, and $(SC':RC')$. Thus … (THEOREM of MENELAUS)

Pi_1891. **Proves using similarity both theorems (before speaking of duality) in a section concerning oriented segments**.

Desargues theorem (published in 1648)

Cre_1873. **Presented in the first pages**

Fig. 1.

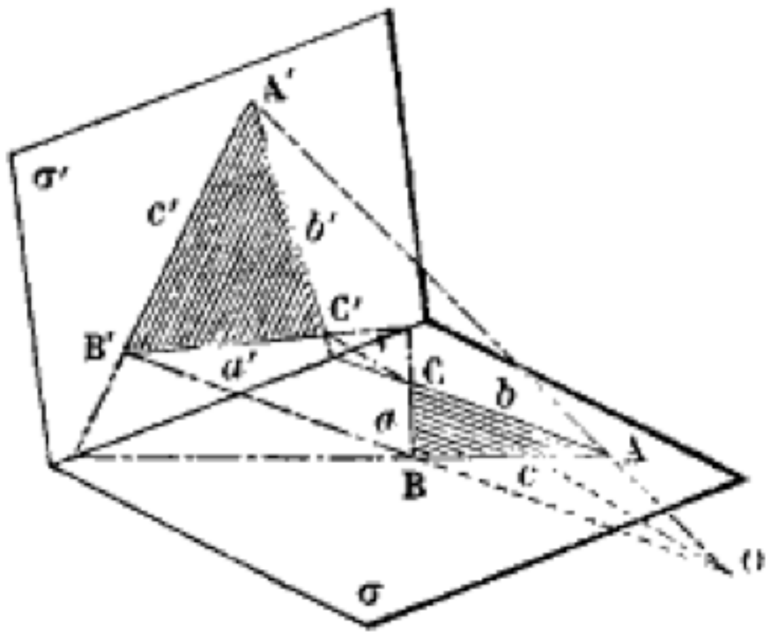
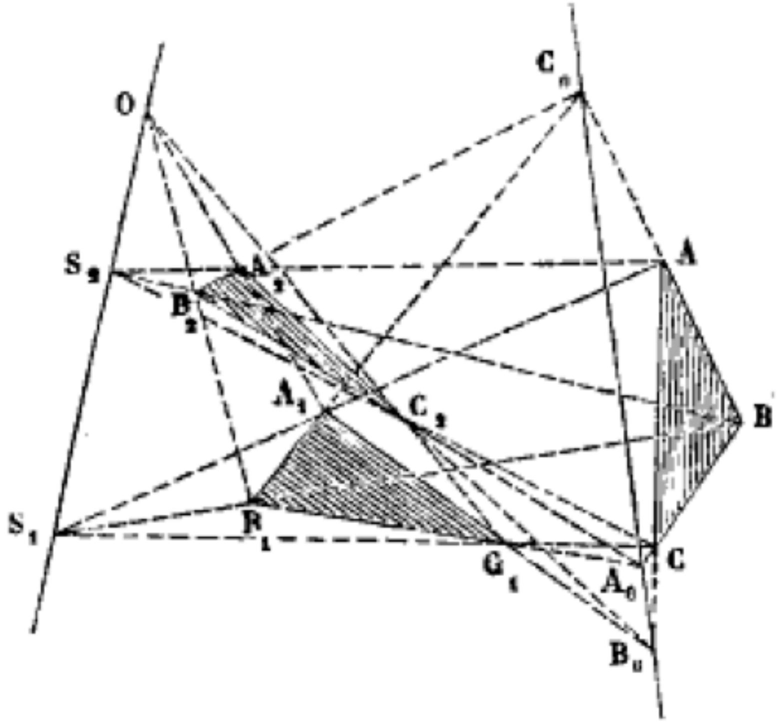


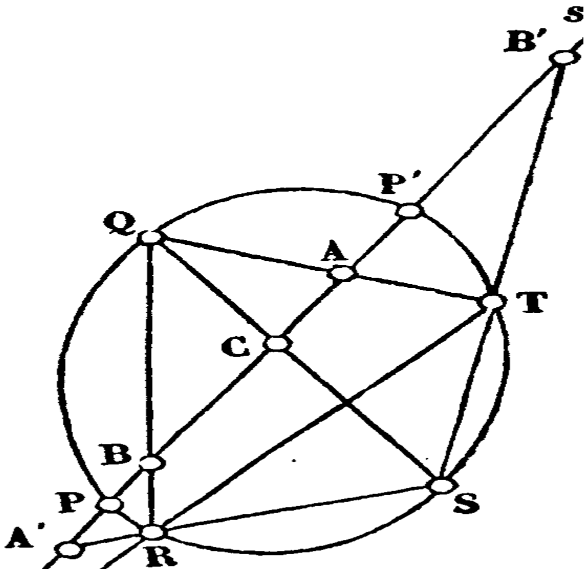
Fig. 3.



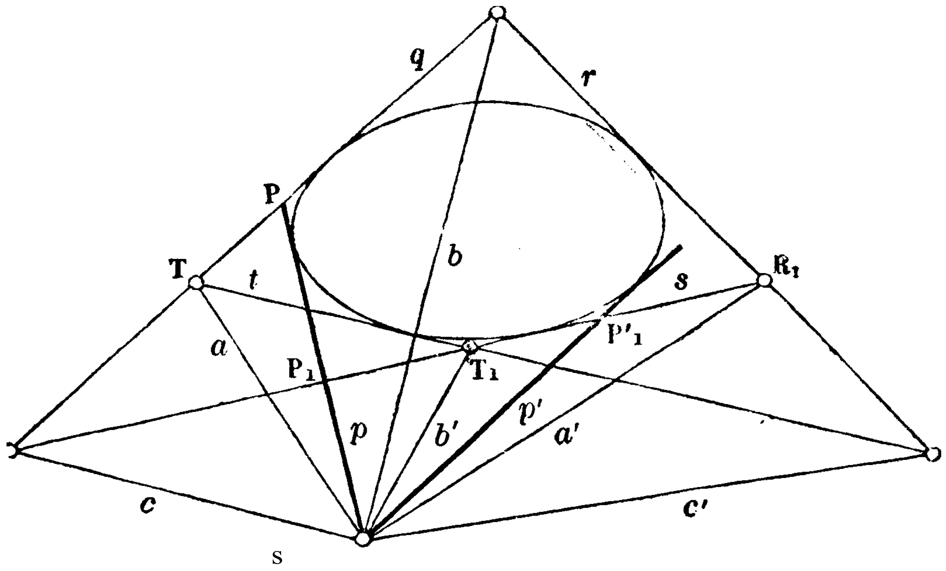
Then a specific chapter XVII on page 148. Presented dually, but …

THEOREM. *Any transversal meets a conic and the opposite sides of an inscribed quadrangle in three conjugate pairs of points of an involution.*

(This is known as DESARGUES theorem)



CORRELATIVE THEOREM. *The tangents from an arbitrary point to a conic and the straight lines which join the same point to the opposite vertices of any circumscribed quadrilateral form three conjugate pairs of rays of an involution.*



Pi_1891. **The same, the correlative theorem is presented as theorem of Sturm.**

Pascal (1640) – Brianchon (1806)

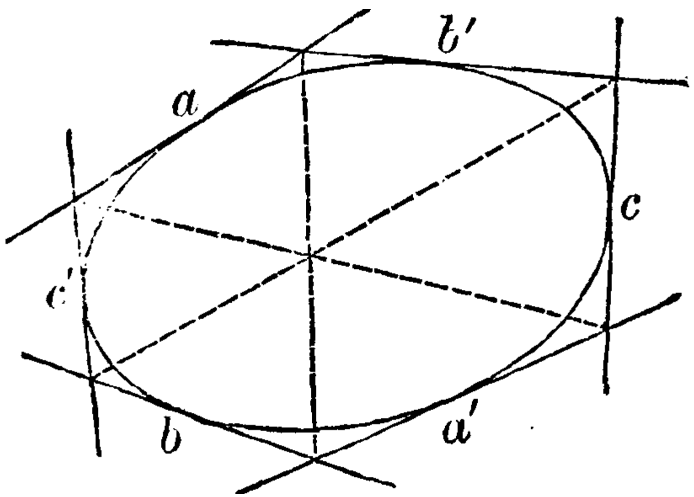
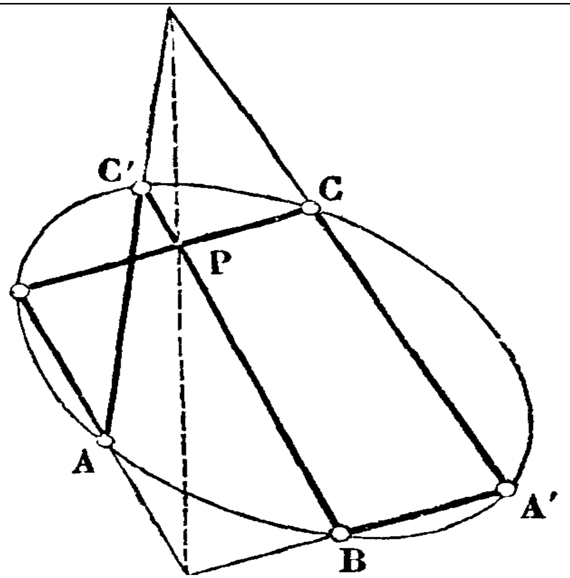
Pr_1871: theorems of Pascal, Brianchon, Desargues and their consequences. Construction of a conic section given five conditions (points or tangents)

Cre_1873, intro:

When only sixteen years old (in 1640) PASCAL discovered his celebrated theorem of the mystic hexagram, and in 1806 BRIANCHON deduced the correlative theorem (Art. 153) by means of the theory of pole and polar.

The properties of the quadrilateral formed by four tangents to a conic and of the quadrangle formed by their points of contact are to be found in the Latin appendix *(De linearum geometricarum proprietatibus generalibus tractatits)* to the *Algebra* of MACLAURIN, a posthumous work (London, 1748). He deduced from these properties methods for the construction of a conic by points or by tangents in several cases where five elements (points or tangents) are given. This problem, in its full generality, was solved at a later date by BRIANCHON.

Cre_1873. p. 123. **Theorems of Pascal and Brianchon**

Through five given points O, O', A , B, C in a plane, no three of which lie in a straight line, a conic can be described.	*Given five straight lines o, o', a, b, e in a plane, no three of which meet in a point, a conic can be described to touch them.*
If a hexagon is circumscribed to a conic, the straight lines joining the three pairs of opposite vertices are con-current. *This is known as BRIANCHON's theor.*	*If a hexagon is inscribed in a conic, the three pairs of opposite sides intersect one another in three collinear points.* *This is known as PASCAL's theorem.*
	

Pi_1891: **mentions the duality Pascal-Brianchon, but separated and different proofs (Brianchon proved as above, Pascal much earlier as transformation of a circle).**

Polar reciprocation (Poncelet, 1818)

Pr_1871: Poles and polars in a circle. Polar reciprocal figures.

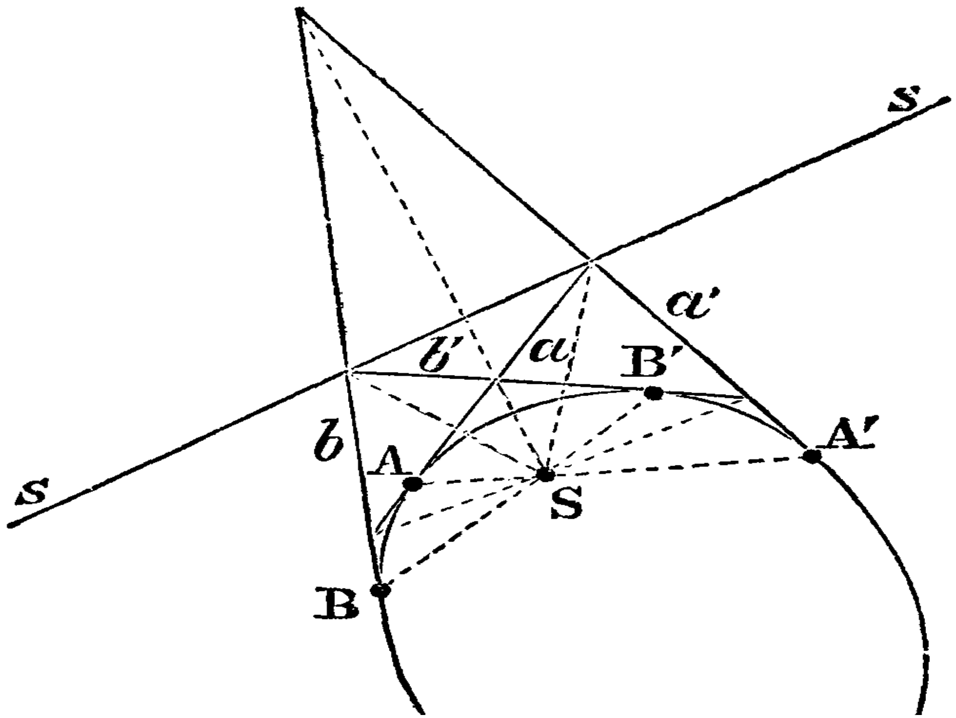
Cre_1873, intro:

The theory of pole and polar was already contained, under various names, in the works already quoted of DESARGUES and DE LA HIRE; it was perfected by MONGE, BRIANCHON, and PONCELET. The last-mentioned geometer derived from it the theory of polar reciprocation, which is essentially the same thing as the law of duality, called by him the '*principe de réciprocité polaire*.'

Cre_1873. Chapter XX, first definitions; XXII, p. 239

Polar reciprocal curves; polarity with regard to a conic.

Every point on the polar of a given point E has for its polar a straight line passing through E. Every straight line passing through the pole of a given straight line e has for its pole a point lying on e.



Consider now as polars all the tangents of a given curve C, or in other words suppose the polar to move, and to envelope the given curve. Its pole will describe another curve, which may be denoted by C'. Thus the points of C' are the poles of the tangents of C.

The polar reciprocal of a conic with respect to another conic is a conic.

…

Two figures which are polar reciprocals one of the other are correlative figures in accordance with the law of duality in plane Geometry (Art. 33); for to every point of the one corresponds a straight line of the other, and to every range in the one corresponds a pencil in the other. They lie moreover in the same plane; their positions in this plane are determinate, but may be interchanged, since every point in the one figure and the corresponding straight line in the other are connected by the relation that they are pole and polar with respect to a fixed conic. Thus two polar reciprocal figures are correlative figures which are coplanar, and which have a special relation to one another with respect to their positions in the plane in which they lie.

On the other hand, if two figures are merely correlative in accordance with the law of duality, there is no relation of any kind between them as regards their position.

Pi_1891. cap XV.

Two polar reciprocal figures with regard to a conic are dual or correlative in the ordinary sense (n. 117 [a property concerning position can be obtained from another exchanging the words point and line]).

This is a consequence of the fact that the points of each figure correspond to the lines of the other, and to two intersecting elements correspond intersecting elements.

Therefore, the theory of reciprocal polars teaches how to construct the dual figure of any given figure. Moreover it is clear that, given to polar reciprocal figures F and F', if a certain property is fulfilled for one of them, necessarily the correlative property is fulfilled for the other figure. So we can conclude that

The theory of polar forms with respect to a conic gives a direct proof of the principle of duality in the plane.

From now on we could retain this principle as a completely proven truth, and use it without the restrictions that were indicated in n. 119 [namely, that the properties deduced by duality need to be proven separately].

Then, theorems on poles and polars presented "dually" (in parallel)

SO_1871: **p. 218 Brief reference to duality with elementary examples**

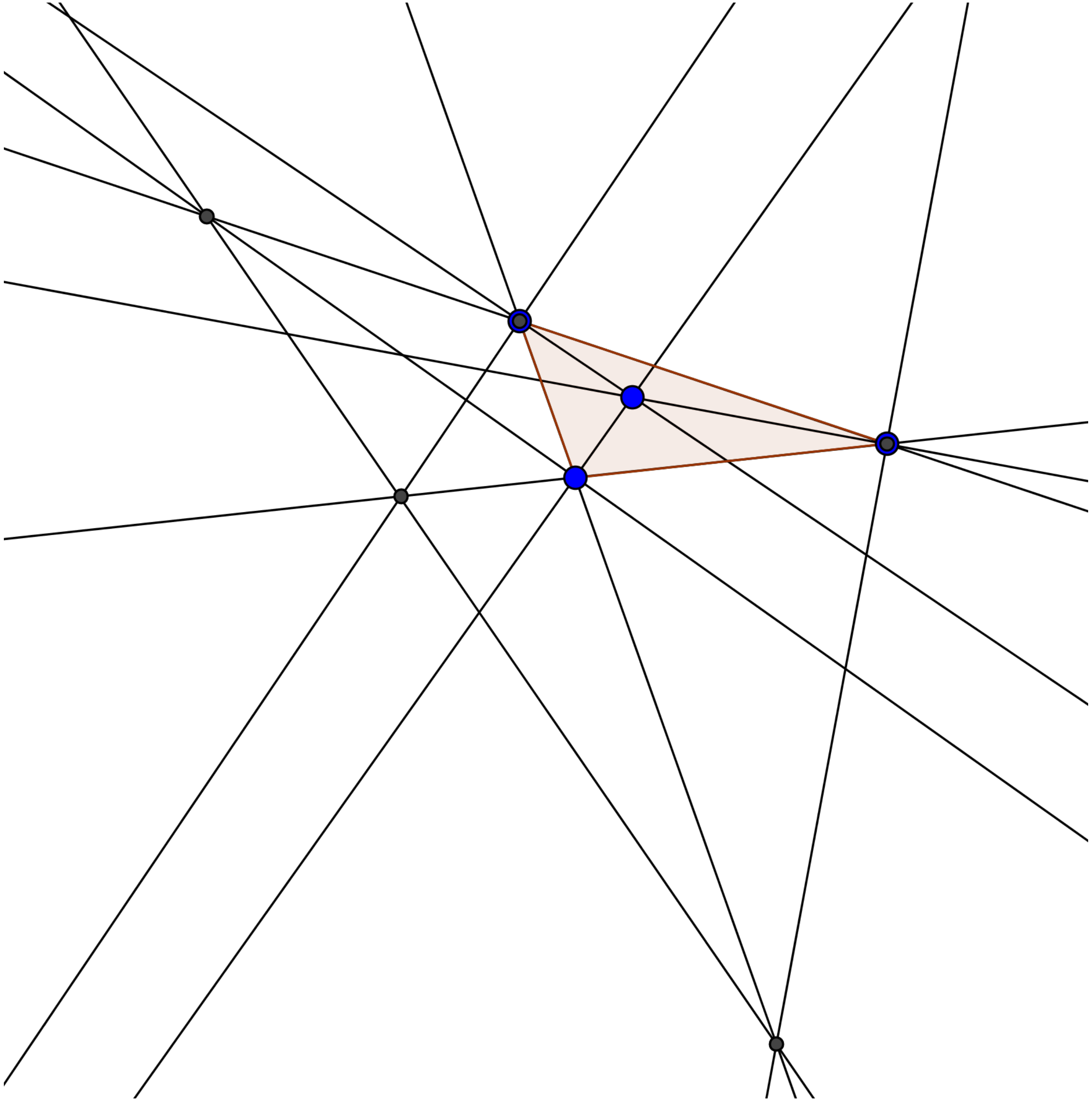
Harmonic ratio associated with four collinear points, polarity with respect to a circle, polar reciprocal figures.

From what we have seen before, we deduce that – if between some elements of a figure certain relations expressed by a theorem hold – it will be sufficient to substitute the reciprocal elements of the polar reciprocal figure to obtain a theorem concerning the latter.

This correspondence is a particular case of the *law of duality* (about which it is not the case to speak) and gives a method to deduce a theorem from another one.

Let us see some examples:

The three altitudes of a triangle are concurrent in a point. Therefore, if from a point we draw three lines to each vertex of a triangle, and then the perpendiculars to these, the intersecting points of these perpendiculars with the opposite sides are collinear.



Duality principles (Gergonne 1826, Poncelet 1827-29)

Pr_1871:

The duality principle in the plane;

Cre_1873, intro:

The law of duality, as an independent principle, was enunciated by GERGONNE; as a consequence of the theory of reciprocal polars (under the name *principe de reciprocité polaire)* it is due to PONCELET.

I have placed the law of duality quite at the beginning of the book, as being a logical fact which arises immediately and naturally from the possibility of constructing space by taking either the point or the plane as element.

Cre_1873.: **Everywhere (starts at page 26).**

THE PRINCIPLE OF DUALITY * (refers to von Staudt)

32. GEOMETRY (speaking generally) studies the generation and the properties of figures lying (1) in space of three dimensions, (2) in a plane, (3) in a sheaf. In each case, any figure considered is simply an assemblage of elements; or, what amounts to the same thing, it is the aggregate of the elements with which a moving or variable element coincides in its successive positions. The moving element which generates the figures may be, in the first case, the point or the plane; in the second case the point or the straight line; in the third case the plane or the straight line. There are therefore always two *correlative* or *reciprocal* methods by which figures may be generated and their properties deduced, and it is in this that geometric *Duality* consists. By this duality is meant the co-existence of figures (and consequently of their properties also) in pairs; two such co-existing (*correlative* or *reciprocal)* figures having the same genesis and only differing from one another in the nature of the generating element.

In the Geometry of space the range and the axial pencil, the plane of points and the sheaf of planes, the plane of lines and the sheaf of lines, are correlative forms. The flat pencil is a form which is correlative to itself.

In the Geometry of the plane the range and the flat pencil are correlative forms.
In the Geometry of the sheaf the axial pencil and the flat pencil are correlative forms.

The Geometry of the plane and the Geometry of the sheaf, considered in three-dimensional space, are correlative to each other. …

Two correlative propositions are deduced one from the other by interchanging the elements *point* and *plane*

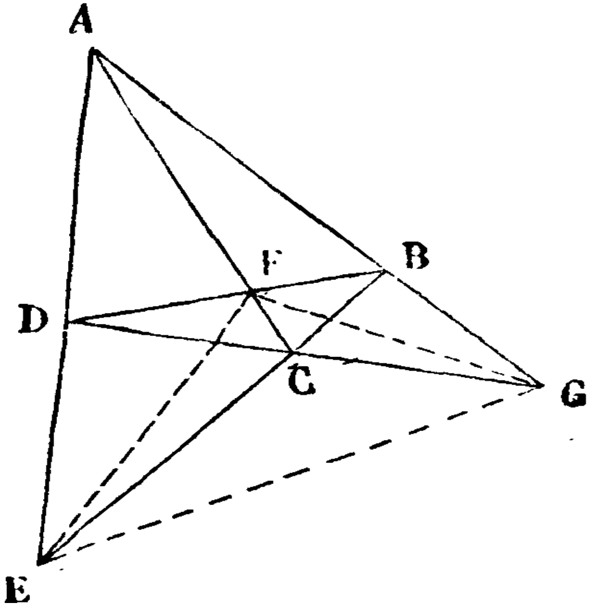
Examples

If two points A, B determine a straight line (viz. the straight line AB which passes through the given points) which contains an infinite number of other points.	Two planes α, β determine a straight line (viz. the straight line $\alpha\beta$, the intersection of the given planes), through which pass an infinite number of other planes.
A straight line a and a point B (not lying on the line) determine a plane, viz. the plane aB which connects the line with the point.	A straight line a and a plane β (not passing through the line) determine a point, viz. the point $a\beta$ where the line cuts the plane.

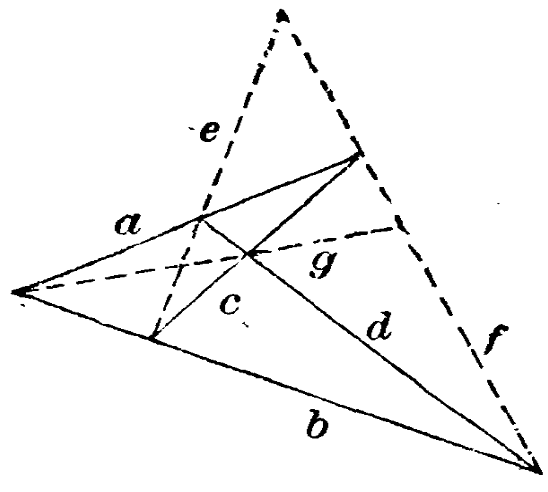
Three points A, B, C which are not collinear determine a plane, viz. the plane ABC which passes through the three points.	Three planes α, β, γ, which do not pass through the same line determine a point, viz. the point $\alpha\beta\gamma$, where the three planes meet each other.
Two straight lines which cut one another lie in the same plane.	Two straight lines which lie in the same plane intersect in a point.
Given four points A, B, C, D; if the straight lines AB, CD meet, the four points will lie in a plane, and consequently the straight lines BO and AD, CA and BD will also meet two and two.	Given four planes α, β, γ, δ, if the straight lines $\alpha\beta$, $\gamma\delta$ meet, the four planes will meet in a point, and consequently the straight lines $\beta\gamma$ and $\alpha\delta$, $\gamma\alpha$ and $\beta\delta$, will also meet two and two.
Given any number of straight lines; if each meets all the others, while the lines do not all pass through a point, then they must lie all in the same plane (and constitute a plane of lines).	Given any number of straight lines; if each meets all the others, while the lines do not all lie in the same plane, then they must pass all through the same point (and constitute a sheaf of lines).

In the Geometry of the plane, two correlative propositions are deduced one from the other by interchanging the words *point* and *line*, as in the following examples:

Four points *A, B, C, D* (Fig. 13), no three of which are collinear, form a figure called a *complete quadrangle*. The four points are called the *vertices*, and the six straight lines joining them in pairs are called the *sides* of the quadrangle.



Four straight lines *a, b, c, d* (Fig. 14), no three of which are concurrent, form a figure called a *complete quadrilateral*. The four straight lines are called the *sides* of the quadrilateral, and the six points in which the sides cut one another two and two are called the *vertices*.



Pi_1891. p. 180-186. **Duality Point-line and point-plane with some examples**.

Rarely used.

In plane geometry the following *duality principle* or *law* holds:

From every theorem of plane geometry which expresses a property of position, another can be deduced – generally different from the first one – by means of the substitution of the word point with the word straight line and vice versa, and of the idea of point *on* a line with the idea of line passing *through* a point, and vice versa.

(Plücker, 1831)